Abstract—A time-domain robust control design approach for minimizing error in transient responses of parametric uncertain systems is considered, as motivated by design and control of micro-electromechanical actuators. A quadratic cost function is formulated as the sum of error components over a finite time span, with the optimization problem of minimizing the least upper bound of the quadratic function represented in terms of the eigenvalue of a certain matrix. This further allows for a linear fractional transformation form by which the nominal and uncertain parameters are separated into P and Δ matrices, analogous to standard LFT representations for robust controller design. The structured singular value, μ, is replaced with the spectral radius of the LFT-expressed matrix in the P-Δ configuration. A bulk piezoelectrical actuator-driven micro-robotic flexure joint is considered as a test case, where the stacking process of placing a PZT ceramic actuator on top of a micro-machined silicon flexure is subject to substantial processing error.

I. INTRODUCTION

Fabrication processes associated with MEMS often have significant variation or error relative to the size of the device. For example, layers are subject to geometrical uncertainties due to fabrication variations, such as mask misalignment during photolithography process and variation of etching profiles during reactive ion etching. The structural properties of deposited film structures, such as elastic modulus and residual stress, depend not only on deposition parameters but also on the actual equipment and the history of that equipment used. Wafer-to-wafer bonding techniques entail significant misalignment errors which can have a major effect on structural dynamics. These variant microfabrication processes result in deviation of structure stiffness, total mass and possibly damping constant. Therefore, robust design technique of micro-scaled dynamic system is important needed for the system dynamics performance.

Previous works on robust design techniques for micro-scaled dynamic systems can be grouped into two approaches: robust open-loop design and robust feedback control design approaches. The former approach is based on multi-objective constrained optimization and open loop dynamics are directly optimized without feedback control implementation. For example, one optimization problem is to match the natural frequency of a system with a random vector noise factor to a predefined natural frequency when the effect of parameter uncertainty on perturbed finite duration, and in continuous rather than discrete time, searching for the best nominal parameters to minimize variation effects in finite step operation of a MEMS system.

II. PRELIMINARIES

A. Performance Index in Robust Design and Analysis

In this section, we propose a performance index to minimize the effect of parameter uncertainty on perturbed finite-time transient responses. It is similar to the objective function in finite-horizon LQ design. For an n-dimensional linear system where all states are assumed to be available during controller design, a system with perturbed and nominal parameters and a feedback signal with state feedback gain, $K_{FB}$, are shown in (1):

$$\begin{align*}
\dot{x}_{p}(t) &= (A_0 + DA)x_{p}(t) + (B_0 + DB)u(t) \\
\dot{x}_{n}(t) &= A_0x_{n}(t) + Bu(t) \\
y(t) &= x(t) + r + K_{FB}x(t)
\end{align*}$$

$$\begin{align*}
\dot{x}_{p}(t) &= (A_0 + DA)x_{p}(t) + B_0u(t) \\
\dot{x}_{n}(t) &= A_0x_{n}(t) + Bu(t) + B_0r
\end{align*}$$

where $x_{p}$ and $x_{n}$ are the perturbed and nominal states of a plant. $x_{p}$ is the nominal state of a closed loop system, $A_0$ and $B_0$ are nominal state and input matrices, $DA$ and $DB$ are
deviation of $A_0$ and $B_0$, respectively, and $r$ is the reference input.

Let the difference between the nominal, $x_{c0}$, and perturbed response, $x_{cl0}$, of the closed loop system be defined as follows, assuming the same initial conditions:

$$\mathbf{x}(0) = \exp((\mathbf{A} + \Delta \mathbf{A} + \Delta \mathbf{B} \mathbf{K}_n + \Delta \mathbf{B} \mathbf{K}_n) h) x_{c0}(0) - \exp((\mathbf{A} + \mathbf{B} \mathbf{K}_n) h) x_{cl0}(0)$$  \hspace{1cm} (2)

For a given specification of nominal, target transient behavior, the performance index, $J$, is proposed as an integration of the quadratic function of error between the nominal and perturbed output of the system over the finite duration time $t_p$ and is represented by the following minimization:

$$\min_{\Delta} \max_{\Delta} J(t_p) = \min_{\Delta} \max_{\Delta} \int_{0}^{t_p} \mathbf{e}^T(t)^2 \mathbf{R} \mathbf{e}(t) \, dt$$  \hspace{1cm} (3)

where $K$ is a set of optimization variables, $e_1$ and $e_2$ are the error of position and velocity components of output, $\Delta$ represents the parametric uncertainty, and $R$ is a positive definite diagonal matrix.

Then, the quadratic function of error will be shown to be bounded from above by the bound $J_{\Delta}(t_p)$. Differentiating the error, we obtain:

$$\dot{\mathbf{e}} = (\mathbf{A} + \mathbf{B} \mathbf{K}_n + \Delta \mathbf{A} + \Delta \mathbf{B} \mathbf{K}_n) \mathbf{e} + \mathbf{M}_0$$  \hspace{1cm} (4)

where

$$\mathbf{M}_0 = (\Delta \mathbf{A} + \Delta \mathbf{B} \mathbf{K}_n) \exp((\mathbf{A} + \mathbf{B} \mathbf{K}_n) h) x_{c0}(0).$$  \hspace{1cm} (5)

Note that the exponential term is no longer dependent on parameter variation and can be obtained only in terms of the nominal system during worst-case performance analysis. In addition, for small time response, $\mathbf{M}_0$ can be reduced to a sum of matrices by Picard series:

$$\mathbf{M}_e \cong (\Delta \mathbf{A} + \Delta \mathbf{B} \mathbf{K}_n) (1 + (\mathbf{A} + \mathbf{B} \mathbf{K}_n) t) x_{c0}(0) \quad \text{for} \quad t \ll 1$$  \hspace{1cm} (6)

By differentiating the performance index with respect to time, it can be shown that the ratio of the derivative of the performance index to itself is bounded by the minimum and maximum eigenvalues of a certain matrix, $Q$, by Rayleigh’s quotient as follows:

$$\dot{J}(t) = -\mathbf{e}^T \mathbf{M}_1 \mathbf{e} + \left(\mathbf{e}^T \mathbf{M}_2 + \mathbf{M}_2^T \mathbf{e}\right)$$

where

$$\mathbf{M}_1 = -\mathbf{R} (\mathbf{A} + \mathbf{B} \mathbf{K}_n + \Delta \mathbf{A} + \Delta \mathbf{B} \mathbf{K}_n) - (\mathbf{A} + \mathbf{B} \mathbf{K}_n + \Delta \mathbf{A} + \Delta \mathbf{B} \mathbf{K}_n)^T \mathbf{R}$$

and

$$\dot{\mathbf{e}} = \mathbf{e}^T \mathbf{M}_1 \mathbf{e} + \left(\mathbf{e}^T \mathbf{M}_2 + \mathbf{M}_2^T \mathbf{e}\right) \geq \lambda_1(Q) \quad \text{for} \quad t \ll 1$$  \hspace{1cm} (7)

Here, $M_1$ is a real symmetric matrix and $\lambda_1$ is the minimum eigenvalue of $Q$ (and the negative of the maximum eigenvalue of $Q$). To find the relation between $Q$ and the ratio of the derivative of the performance index to itself, as found in (8), an augmented vector, $\mathbf{1}_{m \times 1}$, whose elements are 1 in its first entry and 0 in the remaining, is introduced to produce a quadratic form in terms of error, and the equivalent performance index is defined in terms of $\mathbf{e}$:

$$\mathbf{e}^* := \left[\mathbf{e}^T \quad \mathbf{1}^T\right]^T$$

$$J(t) = \mathbf{e}^T \mathbf{M}_3 \mathbf{e} = -\mathbf{M}_1 \mathbf{e} + \mathbf{e}^T \mathbf{M}_3 \mathbf{e}$$

producing a relation in terms of $Q$:

$$\frac{J(t)}{J(t)} = \frac{e^T \mathbf{M}_1^T \mathbf{M}_3^{-1} e}{e^T \mathbf{R} e} \cong \lambda_1(Q) = \lambda_1(M^T R^{-1})$$  \hspace{1cm} (9)

where $M'$ is a real symmetric and $R'$ is a positive definite real symmetric matrix, so that the sufficient conditions for Rayleigh’s quotient are satisfied. Then, the lower bound of the ratio is the minimum eigenvalue of $Q$, and this becomes an optimal design variable that determines the upper bound of equivalent performance index $J'$ at a finite end time. From the definition, the maximum and minimum eigenvalues of $Q$ have the same magnitude but different signs, and therefore minimization of the proposed performance index is converted to the almost equivalent and easier to solve minimization of spectral radius of $Q$. $\lambda_1(Q)$:

$$\min_{\Delta} \lambda_1(Q)$$  \hspace{1cm} (10)

### B. Conversion to Linear Fractional Transformation

In this section, we obtain an upper linear fractional transformation (LFT) form of the eigenvalue minimization problem in terms of nominal and perturbed parameters where all nominal system parameters are shown in $P$ matrix while parametric uncertainty description is separated into the $\Delta$ matrix [1], [2]. This can be useful in the case of high-order systems with many sources of uncertainty for formulating the perturbed parameter model. For the sake of simplicity, it is shown that eigenvalues of the $Q$ matrix in (9) are equal to those of $Q'$ in (11) by the property of the determinant of a partitioned matrix [5], where the zeros of the determinant polynomial are eigenvalues of interest:

$$\det[(\lambda I - M' R^{-1})]
= \det\left(\begin{bmatrix} \lambda I & M_2 \\ M_2^T & I_{n(n-m)} \end{bmatrix}\right)$$ \hspace{1cm} (11)

Then, $Q'$ is represented in LFT as shown:

$$Q' = M_3 \mathbf{R}^{-1} \left[\begin{bmatrix} \lambda I \\ M_2 \\ I_{n(n-m)} \end{bmatrix}\right]$$ \hspace{1cm} (12)

where $P$ and $\Delta P$ are split into partitioned matrices:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad \Delta P = \text{blockdiag}(\Delta \mathbf{A}, \Delta \mathbf{A}', \Delta \mathbf{A})$$ \hspace{1cm} (13)

In (13), the partitioned matrices of $P$ are represented:
The structured singular value (SSV) is defined as \[ \mu_{\Delta_u}(M) = \frac{1}{\Delta_{\min}(\bar{\Delta}_u): \Delta_u \in \mathcal{B}_\Delta, \det(1-M\Delta_u) = 0} \] (16)

Then, minimizing the upper-bound of the eigenvalue of \( Q \) under uncertainties can be achieved by minimizing the structured singular value of \( P \) matrix with respect to an augmented uncertainty structure, which can reduce the complexity of setting up the optimization problem for high-order systems.

### C. Conversion to Structured Singular Value

The structured singular value (SSV) is defined as [1], where \( \mu_{\Delta_u} \) is the structured singular value of \( M \) with respect to \( \Delta_u \), and \( \bar{\sigma} \) is the maximum singular value:

\[
\mu_{\Delta_u}(M) = \frac{1}{\min_{\Delta_u}(\bar{\sigma}(\Delta_u): \Delta_u \in \mathcal{B}_\Delta, \det(1-M\Delta_u) = 0) \] (16)

It is known that from the Main Loop Theorem [1], when \( \beta > 0 \) is given,

\[
\Delta_{\text{aug}}(M) < \beta \iff \left\{ \begin{array}{l} \mu_{\Delta_u}(M_{11}) < \beta \\ \max_{\Delta_u} \mu_{\Delta_u}(F_u(M_{11}, \Delta_{\text{aug}})) < \beta \\ \mu_{\Delta_u}(M_{22}) < \beta \\ \max_{\Delta_u} \mu_{\Delta_u}(F_u(M_{22}, \Delta_{\text{aug}})) < \beta \end{array} \right\} \] (17)

where \( \Delta_u \) and \( \Delta_{\text{aug}} \) are the uncertainty blocks that are compatible in size of \( M_{11} \) and \( M_{22} \), respectively, and \( \Delta_{\text{aug}} \) is defined:

\[
\Delta_{\text{aug}} = \text{blockdiag}(\Delta_{11}, \Delta_{22}) \] (18)

From (16), the spectral radius, \( \rho \), of \( M \) is intimately related to SSV when the uncertainty block satisfies the condition:

If \( \Delta_u [\delta I : \delta \in \mathbb{C}] \), then \( \mu_{\Delta_u}(M) = \rho(M) \) (19)

Thus, the relation of spectral radius to SSV in (19) is considered for minimizing the proposed performance index, and minimizing the transient response over finite duration is converted to minimization of the upper bound of structured singular value with respect to uncertainty structure, \( \Delta_{\text{aug}} \):

\[
\mu_{\text{aug}}(P) < \beta \iff \left\{ \begin{array}{l} \mu_{\Delta_v}(P_{11}) < \beta \\ \max_{\Delta_v} \rho(F_v(P, \Delta_v)) = \max_{\Delta_v} \rho(Q) < \beta \end{array} \right\} \] (20)

In turn, a design simulation of robust finite-duration transient response based on SSV is equivalent to the conventional robust performance analysis available in MATLAB [12]. More details of conversion to upper bound minimization of SSV are shown in the Appendix.

### III. SIMULATION STUDY

The specific robust design problem motivating this work is a bulk-PZT ceramic actuator driven micro-robot which consists of a base silicon understructure and flexure joint that are fabricated on an SOI wafer with a Cr/Au deposition, and bulk PZT strips that are diced and bonded on top of the base silicon understructure with epoxy resin as shown in Figure 1. The design procedures will seek target dimensions of PZT ceramic actuator and of silicon flexure joint for the open-loop case, and the state feedback gain coefficient in addition to the structural dimensions for the closed-loop case so that transient motions of the joint are minimally sensitive to unknown dimensions or misalignment.

To perform a simulation study of leg joint parameter optimization, the dynamic equations of lateral and vertical motion are obtained in (21) and (25), where \( \theta \) and \( x \) are the in-plane lateral angular displacement and out-of-plane vertical linear displacement, respectively. A set of nominal parameters is shown in Table 1. Our design parameters are the width of PZT ceramic actuator, \( w \), and of silicon flexure joint, \( w_i \), the length of silicon base, \( L_i \), and of silicon flexure joint, \( L_s \), and the state feedback gain coefficient, \( K_{FB} \).

The dynamic model for the in plane system dynamics is

\[
J_{\text{leg}} \ddot{\theta} + b_\theta \dot{\theta} + k_\theta \theta = F_\theta \] (21)

where \( b_\theta \) is the lateral damping constant, \( F_\theta \) is the applied force by the piezoelectric actuator, and \( J_{\text{leg}} \) is approximated by using lumped equivalent mass parameters, such as a micro-robotic foot, \( m_{\text{foot}} \), the base silicon structure assembled with a PZT actuator, \( m_{\text{Act}+\text{Si}} \), the base silicon structure without a PZT actuator, \( m_{\text{Act}+\text{Si}} \), the silicon flexure joint, \( m_{\text{flex}} \), and \( L_{CM} \) is the length of the center of mass of an entire leg:

\[
J_{\text{leg}} \cong \left( m_{\text{foot}} + \frac{m_{\text{Act}+\text{Si}}}{3} + \frac{m_{\text{Act}+\text{Si}}}{3} + \frac{m_{\text{flex}}}{3} \right) \frac{L_{CM}^2}{2} \] (22)

The rotational spring stiffness, \( k_\theta \), is calculated from the system parameters according to:

\[
k_\theta^{-1} = \frac{C}{c} + \left[ \left( \frac{1}{c} \right) D - \left( \frac{Q}{c} \right) d \right] \] (23)
where the parameters $c$ and $C$ are nominal parameter terms, $d$ and $D$ are the perturbed parameter terms, $Q$ is defined in (28), and $E_1$ and $E_2$ are elastic modulus of PZT and silicon:

$$ C = \beta_l w t_1, \quad \beta_l = 6 L_s w d_1 E_1 $$

$$ c = \beta_l t_1 w + \beta_l t w + E_1 w d_1 t_1, \quad \beta_l = 3 L_s w E_1 Call to action, \quad \beta_l = 3 L_s w E_1 R $$

$$ \Delta L = (\pm \beta_l w L) \Delta w - (\beta_l w) \Delta L $$

$$ d = (\pm \beta_l t_1 w + \beta_l t w + E_1 w d_1 t_1) \Delta t_1 + \epsilon \Delta w - (E_1 w d_1 t_1) \Delta L $$

$$ \epsilon = \pm \beta_l t w \pm \beta_l t_2 w + 3E_1 w d_1 t_2 $$

(24)

The out-of-plane dynamics are

$$ m_{o-p} \ddot{z} + b_o z + k_o z = F_z $$

(25)

where $b_o$ is the vertical damping constant, $F_z$ is the vertical force by the actuator, $m_{o-p}$ is the total mass of a leg including a foot, two joint flexures, an assembled PZT-Si layer, and a single Si layer, as shown in Figure 1:

$$ m_{o-p} \approx m_{act} + m_{act+Si} + m_{noAct+Si} + m_{flex} $$

(26)

The vertical spring stiffness, $k_{o-p}$, is calculated from the individual spring stiffness and its deviation:

$$ k_{o-p} = \frac{Q+S}{q+s} $$

(27)

where $Q$ and $q$ are nominal parameters and $S$ and $s$ are perturbed parameter terms:

$$ Q = (k_{o-p} + k_{so})(k_{o-p} + k_{so}) $$

$$ S = (k_{a} + k_{so})(\Delta k_{a} + \Delta k_{o-p}) + (k_{o-p} + k_{so})(\Delta k_{o-p} + \Delta k_{a}) $$

$$ = (k_{o-p} + k_{so}) \Delta k_{a} + (k_{o-p} + k_{so} + 2k_{so}) \Delta k_{o-p} + (k_{a} + k_{so}) \Delta k_{a} $$

$$ q = k_{o-p} + k_{so} + 2k_{so}\quad s = \Delta k_{a} + \Delta k_{o-p} + 2 \Delta k_{a} $$

The individual nominal spring stiffness, $k_{a}$, $k_{so}$, and $k_{o-p}$, with perturbed terms, $\Delta k_{a}$, $\Delta k_{so}$, and $\Delta k_{o-p}$, are the vertical stiffness of a single base silicon understructure, flexure joint, and an assembled PZT-Si layer, respectively:

$$ k_{a} = \frac{E_1 w d_1 t_1}{4(L_1 + L_2)} $$

$$ k_{so} = \frac{E_1 w d_1 t_2}{4L_1} $$

$$ k_{o-p} = \frac{E_1 w d_1 t_3}{3L_1^2 L_2 + 2L_1 L_2} $$

(29)

$$ \Delta k_{a} = \left( \frac{3E_1 t_1}{4(L_1 + L_2)} \right) \delta t_1 + \left( \frac{E_1 w d_1 t_1}{4(L_1 + L_2)} \right) \Delta w $$

$$ \Delta k_{so} = \left( \frac{3E_1 t_1}{4L_1} \right) \delta t_1 + \left( \frac{E_1 w d_1 t_1}{4L_1} \right) \Delta w $$

$$ \Delta k_{o-p} = \left( \frac{6}{3L_1^2 L_2 + 2L_1 L_2} \right) \left( \sum_{j=1}^{2} (\alpha_{e1j} + \alpha_{e2j}) \delta t_1 + (\alpha_{e1w} + \alpha_{e2w}) \delta w \right) $$

$$ - \left( \frac{E_1}{2} \left( \frac{E_1 w d_1 t_1}{3L_1^2 L_2 + 2L_1 L_2} \right) \right) \left( \sum_{j=1}^{2} (\alpha_{e1j} + \alpha_{e2j}) \delta t_1 + (\alpha_{e1w} + \alpha_{e2w}) \delta w \right) $$

$$ \left( 3L_1^2 L_2 + 2L_1 L_2 \right) \delta w $$

where

$$ \alpha_{e1j} = \pm 3w t_1^3 \pm 3w t_2^3, \quad \alpha_{e2j} = \pm 3w t_1^3 \pm 3w t_2^3 $$

$$ \alpha_{e1w} = 2w t_1^2, \quad \alpha_{e2w} = 2w t_1^2 $$

(31)

Given the dynamic system defined by (21) and (25), with parametric uncertainties defined by (24) and (28), the design parameters shown in Table 2 were optimized to minimize finite-duration error in the leg response. Table 2 shows the structural dimensions of a pre-existing reference design, which were selected prior to the work in this paper to optimize weight-bearing capacity while reaching a target joint angle. For a closed-loop reference design, a constant gain negative state feedback controller was considered where gain matrix, $K_{FB}$, was designed with desired poles at $[-500+164.3j; -500-164.3j; -400; -1]$, which were chosen for faster response and less overshoot. The weighting matrix, $R$, in (3), is chosen as $diag(50, 100, 25, 50)$ in both open- and closed-loop cases. The results of the optimization show that increasing robustness of the design significantly reduces variation in the response due to parameter variation, even without changing the controller dramatically. While this can be useful for improving robustness of a controller even using simple controller designs, it should be noted that non-zero order robust controllers, such as $H_{\infty}$ or mixed $H_{2}/H_{\infty}$, should be compared in place of the state feedback controller in (1).
This would allow comparison of designing for robustness in just the controller or just the physical design, but that comparison has not been completed at this time.

Table 3 shows the comparison of the sum of errors in a finite time interval and the relative errors at the finite time. The results show that the newly designed values tend to give less deviation both over a time interval and at a finite final time, which is further reduced by applying the design technique proposed in this paper with a controller in place. In this optimization, dimensions of the structure in the closed-loop case show significant changes, while state feedback gain coefficients vary only slightly. This appears to be due to the fact that, as shown in (15), the uncertainties result from parametric deviation of structural dimensions. As the robust design method presented in this paper only considers the optimization with respect to parametric uncertainties of structure, the controller will tend to have the same affect on the various plants that have reduced variation between themselves, and less change may be expected in the controller design. For this test case, designs were constrained by minimal or maximal dimensions that might be fabricated, which are set as bounds in constrained optimization, though weight-bearing capacity, for the time being, was not explicitly included. As some of the optimized parameter are at the dimension constraints, this result shows that true local optimum, at least for purely minimizing dynamic variation, was not present within the design spaces considered physically possible to build.

IV. CONCLUSION

Conversions from a finite-duration quadratic cost function for error due to parameter variation in a dynamic system to an eigenvalue minimization problem, to an LFT form, and to a structured singular value problem are shown in this paper. The LFT is derived from a proposed performance index which is different from conventional robust performance analysis technique, where system matrices are concerned with asymptotic behavior of systems in LFT form. The interior-point algorithm of nonlinear optimization is applied to find the local minima of design parameters within the bounds by minimizing the upper bound of the structured singular value in MATLAB. In addition to limitations noted above, this optimization procedure has not considered the closed loop stability while searching for optimal parameters, and different reference designs or a weighting matrix, R, could result in unstable system over the finite time although this could be solved by constraints on the feedback gain matrix. Nonetheless, the optimization algorithm significantly reduces variation in transient responses in open- and closed-loop scenarios. The design is also somewhat analogous to the design of finite-horizon LQ regulators where closed loop stability may not be conserved. For future works, comparison of nominal and robust design of closed loop dynamics with conventional robust controllers should be done. Experimental verification with the resulting robust bulk PZT actuated micro-robot prototype will be performed, after further including some constraints on weight-bearing capacity of the final design.

APPENDIX

The following derivation shows the conversion from the eigenvalue problem (10) to an upper LFT representation. The first term of the determinant polynomial in (11) is:

\[-M_R R^{-1} = R \left[ (A + BK_{tn} + \Delta A + \Delta BK_{yn}) \right] R^{-1} + \left[ (A + BK_{yn} + \Delta A + \Delta BK_{yn}) \right]^T \]  

(32)

Deviation terms, \( \Delta A \) and \( \Delta B \) which include uncertain parameters in A and B matrices, are represented in \( \Delta \), and the upper linear fractional transformation is obtained:

\[ A + BK_{tn} + \Delta A + \Delta BK_{yn} = F_n (F(M, K_{sl}), A) = F_n (M, A, K_{sl}) \]

where

\[ M := \begin{bmatrix} 0 & 0 \\ I_{mx} & 1 \\ I_{mx} & A \\ 0 & I_{mx} \end{bmatrix} \]

(33)

TABLE 1
REFERENCE DESIGN DIMENSION OF BULK PZT ACTUATOR LEG JOINT

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Quantity</th>
<th>Reference Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_t )</td>
<td>Elastic modulus of PZT</td>
<td>100 GPa</td>
</tr>
<tr>
<td>( E_s )</td>
<td>Elastic modulus of Si</td>
<td>170 GPa</td>
</tr>
<tr>
<td>( d_{33} )</td>
<td>Effective piezoelectric stress coefficient</td>
<td>210</td>
</tr>
<tr>
<td>( V )</td>
<td>Applied voltage</td>
<td>30 V</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>Length of bonded PZT actuator on Silicon</td>
<td>2125 μm</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>Length of un-bonded PZT actuator Silicon</td>
<td>375 μm</td>
</tr>
<tr>
<td>( L_3 )</td>
<td>Length of Si flexure joint</td>
<td>544 μm</td>
</tr>
<tr>
<td>( L_{act} )</td>
<td>Length between two Si flexure joints</td>
<td>5 μm</td>
</tr>
<tr>
<td>( w )</td>
<td>Width of PZT actuator</td>
<td>10 μm</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>Thickness of PZT actuator</td>
<td>150 μm</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>Thickness of base Si</td>
<td>100 μm</td>
</tr>
</tbody>
</table>

TABLE 2
COMPARISON OF REFERENCE AND ROBUST DESIGN DIMENSIONS

<table>
<thead>
<tr>
<th>Design Parameters</th>
<th>Reference Design</th>
<th>Robust Design Closed loop</th>
<th>Robust Design Open loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width of PZT stripe</td>
<td>458 μm</td>
<td>1832 μm</td>
<td>115 μm</td>
</tr>
<tr>
<td>Width of Si flexure</td>
<td>10 μm</td>
<td>5 μm</td>
<td>5 μm</td>
</tr>
<tr>
<td>Length of PZT actuator on base Si</td>
<td>2125 μm</td>
<td>10000 μm</td>
<td>2338 μm</td>
</tr>
<tr>
<td>Length of Si flexure</td>
<td>544 μm</td>
<td>624 μm</td>
<td>598 μm</td>
</tr>
<tr>
<td>( K_{tn} )</td>
<td>[ \begin{bmatrix} -0.6491 \ -0.9048 \ -0.0001 \ 0.0001 \end{bmatrix} ]</td>
<td>[ \begin{bmatrix} -0.6491 \ -0.9048 \ -0.0001 \ 0.0001 \end{bmatrix} ]</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 3
INDICES COMPARISON OF REFERENCE AND ROBUST DESIGN

<table>
<thead>
<tr>
<th>Sum of Error (Angle/Force)</th>
<th>Relative Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closed loop Ref. Design (Fig. 3)</td>
<td>0.3688</td>
</tr>
<tr>
<td>Closed loop Robust Design (Fig.3)</td>
<td>0.1929</td>
</tr>
<tr>
<td>Improvement</td>
<td>48%</td>
</tr>
<tr>
<td>Open loop Ref. Design (Fig. 2)</td>
<td>0.486</td>
</tr>
<tr>
<td>Open loop Robust Design (Fig. 2)</td>
<td>0.425</td>
</tr>
<tr>
<td>Improvement</td>
<td>12%</td>
</tr>
</tbody>
</table>

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\[ M_{MK} := F_\nu(M, K_u) = \begin{bmatrix} 0_{nss} & 0_{nss} & I_{ssn} \\ I_{ssn} & 0_{ss} & K_{ssn} \end{bmatrix} \] (34)

Then, the first and second term in (32) is represented as LFT:
\[ R^{(F_u(M_{MK}, \Delta))} R^{-1} \Delta F_u(M_u, \Delta) \]
where \[ M_u := \begin{bmatrix} 0_{nss} & 0_{nss} & I_{ssn} \\ I_{ssn} & 0_{ss} & K_{ssn} \end{bmatrix} \]
\[ F_u(M_{MK}, \Delta)^T \Delta F_u(M_u, \Delta) \]
where \[ M_u := \begin{bmatrix} 0_{nss} & 0_{nss} & I_{ssn} \\ I_{ssn} & 0_{ss} & K_{ssn} \end{bmatrix} \]
\[ R^{(M_{MK}, \Delta)} R^{-1} \Delta R^{(M_{MK}, \Delta)} \]
Therefore, LFT of \[-M_1 R^{-1} \] is obtained as parallel connection:
\[ M_2 \] and \[ 1_{n \times (n-1)} \] are represented in upper LFT, respectively:
\[ \begin{bmatrix} M_{MK1} & M_{MK2} \end{bmatrix} \]
\[ \begin{bmatrix} R^{(M_{MK}, \Delta)} R^{-1} \Delta R^{(M_{MK}, \Delta)} \end{bmatrix} \]
\[ \Delta_{par} := \text{blockdiag}(\Delta, \Delta) \]

Now the second term of the determinant polynomial is:
\[ \begin{bmatrix} M_2 & 1_{n \times (n-1)} \end{bmatrix} = R^{(M_{MK}, \Delta)} R^{-1} \Delta R^{(M_{MK}, \Delta)} \]
\[ \Delta_{par} := \text{blockdiag}(\Delta, \Delta) \]
(38)

\[ M_2 \] and \[ 1_{n \times (n-1)} \] are represented in upper LFT, respectively:
\[ (R M_2) \]
\[ = F_u(N, \Delta^T) \]
where
\[ N := \begin{bmatrix} 0_n & 0_{n \times 1} \\ \theta_{(1+(A+BK_{FB})^T)} x_{(0)}^T & 0_{1 \times n} \end{bmatrix} \]
\[ \Delta_{par} := \text{blockdiag}(\Delta, \Delta) \]
(40)

Note that \( \Delta \) is a complex uncertainty block which helps avoid convergence difficulty in SSV computation. Thus, an LFT of \[ (M_2 1_{(n-1) \times n}) \] is obtained by row concatenation of two LFTs in (39) and (40):
\[ \begin{bmatrix} M_2 \end{bmatrix}_{(n-1) \times n} = F_u(N, \Delta^T) = F_u(N, \Delta^T) \]
\[ \Delta_{par} := \text{blockdiag}(\Delta, \Delta) \]
(41)

\[ \Phi_{\Delta}^T := (1+(A+BK_{FB})^T) x_{(0)}^T \]
\[ \Delta_{par} := \text{blockdiag}(\Delta, \Delta) \]
\[ M_{\Delta} := \begin{bmatrix} 0_{n \times (n-1)} & I_n \\ I_n & 0_{n \times (n-1)} \end{bmatrix} \]
\[ \Phi_{\Delta}^T := (1+(A+BK_{FB})^T) x_{(0)}^T \]
\[ \Delta_{par} := \text{blockdiag}(\Delta, \Delta) \]
(42)

Therefore, \( Q' \) in (10) is represented to LFT by cascade connection of two LFTs in (37) and (41):
\[ M_1 R^{-1} M_2 = F_u(M_{par}, \Delta_{par}) F_u(M_{\Delta}, \Delta_{par}) = F_u(P, \Delta_{par}) \]