Mean-Square $H_\infty$ Filter Design: Application to a 2DOF Helicopter

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Abstract—This paper designs the central finite-dimensional $H_\infty$ filter for linear stochastic systems with integral-quadratically bounded deterministic disturbances, that is suboptimal for a given threshold $\gamma$ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. The original $H_\infty$ filtering problem for a linear stochastic system is reduced to the corresponding mean-square $H_2$ filtering problem, using the technique proposed in [1]. In the example, the designed filter is applied to estimation of the pitch and yaw angles of a two degrees of freedom (2DOF) helicopter.

I. INTRODUCTION

Over the past two decades, considerable attention has been paid to the $H_\infty$ estimation problem for deterministic and stochastic systems. The seminal papers on $H_\infty$ control ([1]) and estimation ([2], [3], [4]) established a background for consistent treatment of controller/filtering problems in the $H_\infty$ framework. The $H_\infty$ filter design implies that the resulting closed-loop filtering system is robustly stable and achieves a prescribed level of attenuation from the disturbance input to the output estimation error in $L_2/L_2$-norm. A large number of results on this subject have been reported for systems in the general situation (see, for example, [5]–[23] and references therein). Sufficient conditions for existence of an $H_\infty$ filter, where the filter gain matrices satisfy Riccati equations, were obtained for linear deterministic systems in [4] and linear systems with state delay in [24] or with measurement delay in [25]. However, the criteria of existence and suboptimality of solution for the central $H_\infty$ filtering problems based on the reduction of the original $H_\infty$ problem to the induced $H_2$ one, similar to those obtained in [1], [4] for linear systems, remain yet undeveloped for linear stochastic systems with integral-quadratically bounded deterministic disturbances.

This paper presents the central (see [1] for definition) finite-dimensional mean-square $H_\infty$ filter for linear stochastic systems, that is suboptimal with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with the opposite sign. In contrast to results previously obtained for linear systems [4], [24], [25], this paper reduces the original $H_\infty$ filtering problem to the corresponding mean-square $H_2$ filtering problem, using the technique proposed in [1].

II. MEAN-SQUARE $H_\infty$ FILTERING PROBLEM STATEMENT

Let $(\Omega,F,P)$ be a complete probability space with an increasing right-continuous family of $\sigma$-algebras $F_t$, $t \geq t_0$, and let $(W_1(t), F_t, t \geq t_0)$ and $(W_2(t), F_t, t \geq t_0)$ be independent Wiener processes. Consider the following linear stochastic time-varying system $\mathcal{S}_1$:

$$
\begin{align*}
    dx(t) &= (A(t)x(t) + B(t)u(t) + G(t)\omega(t))dt + b(t)dW_1(t), \\
    x(t_0) &= x_0, \\
    dy_1(t) &= C_1(t)x(t)dt + h(t)dW_2(t), \\
    y_2(t) &= C_2(t)x(t) + H(t)\omega(t), \\
    z(t) &= L(t)x(t),
\end{align*}
$$

Designing the central suboptimal mean-square $H_\infty$ filter for linear stochastic systems presents a significant advantage in filtering theory and practice, since (1) it enables one to address filtering problems for linear stochastic time-varying systems, where the linear matrix inequality technique is hardly applicable and the Hamilton-Jacobi-Bellman equation-based methods fail to provide a closed-form solution, (2) the obtained mean-square $H_\infty$ filter is suboptimal, that is, optimal for any fixed $\gamma$ with respect to the $H_\infty$ noise attenuation criterion, and (3) the obtained mean-square $H_\infty$ filter is finite-dimensional.

It should be commented that the proposed design of the central suboptimal mean-square $H_\infty$ filters for linear stochastic systems with integral-quadratically bounded disturbances naturally carries over from the design of the optimal mean-square $H_2$ filters for linear stochastic systems with unbounded disturbances (white noises). The entire design approach creates a complete filtering algorithm for handling the linear stochastic time-varying systems with unbounded or integral-quadratically bounded disturbances optimally for all thresholds $\gamma$ uniformly or for any fixed $\gamma$ separately. A similar algorithm for linear deterministic systems was developed in [4].

The designed filter is applied to estimation of the pitch and yaw angles of a two degrees of freedom (2DOF) helicopter. The simulation results show a reliable performance of the filter, in particular, the obtained attenuation level is five times less than a given threshold.

The paper is organized as follows. Section 2 presents the mean-square $H_\infty$ filter problem statement for linear stochastic time-varying systems. The central suboptimal mean-square $H_\infty$ filter is designed in Section 3. In Section 4, the designed filter is applied to estimation of the pitch and yaw angles of a two degrees of freedom (2DOF) helicopter. Conclusions are given in Section 5.
where \( x(t) \in \mathbb{R}^n \) is the unmeasured state, \( u(t) \in \mathbb{R}^l \) is a known input signal, \( y_1(t) \in \mathbb{R}^{m_1} \) and \( y_2(t) \in \mathbb{R}^{m_2} \) are the measured observations, \( z(t) \in \mathbb{R}^{q} \) is the output to be estimated, \( \omega(t) \in \mathcal{L}_2^2[0, \infty) \) is the deterministic disturbance input, \( A(t), B(t), G(t), b(t), C_1(t), h(t), C_2(t), H(t) \), and \( L(t) \) are known deterministic continuous time-varying functions of appropriate dimension. The initial condition \( x_0 \in \mathbb{R}^n \) is a Gaussian random variable such that \( x_0, W_1(t) \in \mathbb{R}^{p_1} \), and \( W_2(t) \in \mathbb{R}^{p_2} \) are independent. It is assumed that \( h(t)h^T(t) \) is a positive definite matrix.

For the system given by (1)–(4), the following assumptions are made over the time interval \([t_0, t]\):

- \( (A(t), b(t)) \) is stabilizable and \( (C_1(t), A(t)) \) is detectable; \( (\mathscr{E}_1) \)
- \( (A(t), G(t)) \) is stabilizable and \( (C_2(t), A(t)) \) is detectable; \( (\mathscr{E}_2) \)
- \( (A(t), B(t)) \) is stabilizable and \( (L(t), A(t)) \) is detectable, and \( (\mathscr{E}_3) \)
- \( H(t)G^T(t) = 0 \) and \( H(t)H^T(t) \) is a positive definite matrix. \( (\mathscr{E}_4) \)

As usual, the first two assumptions ensure that the estimation error, provided by the designed filter, converge to zero [26]. The noise orthogonality condition \( H(t)G^T(t) = 0 \) is technical and represents the independence between the state and measurement deterministic disturbances. Extensive comments on the assumption \( (\mathscr{E}_4) \) can be found in [1].

The filtering problem to be addressed is as follows: develop a central suboptimal mean-square \( H_\infty \) filter for the linear stochastic system \((\mathcal{J}_1)\) as a linear filter based on the observations \( \{y_1(s), t_0 \leq s \leq t\} \) and \( \{y_2(s), t_0 \leq s \leq t\} \) such that the following three requirements are satisfied.

1. The resulting dynamics of the estimation error \( E(x(t)) - m(t) \), where \( x(t) \) is the state of \((\mathcal{J}_1)\) and \( m(t) \) is the mean-square \( H_\infty \) estimate produced by the designed filter, is asymptotically stable in the absence of disturbances, \( \omega(t) \equiv 0 \). Here, \( E(x(t)) \) denotes the expectation of stochastic process \( x(t) \).

2. The variance of the mean-square \( H_\infty \) estimate \( m(t) \) of the system state \( x(t) \), based on the observation process \( Y(t) = \{y_1(s), 0 \leq s \leq t\} \), is equal to the minimum estimation error variance (27)

\[
E[(x(t) - E(x(t) | F_1^T))(x(t) - E(x(t) | F_1^T))^T | F_1^T] \tag{5}
\]

at every time moment \( t \). Here, \( E[\xi(t) | F_1^T] \) means the conditional expectation of a stochastic matrix process \( \xi(t) = (x(t) - E(x(t) | F_1^T))(x(t) - E(x(t) | F_1^T))^T \) with respect to the \( \sigma \)-algebra \( F_1^T \) generated by the observation process \( Y(t) \) in the interval \([t_0, t]\).

3. Given a noise attenuation level \( \gamma \), the \( H_\infty \) noise attenuation condition (6) is ensured. More specifically, for any nonzero disturbance input \( \omega(t) \in \mathcal{L}_2^2[0, \infty) \), the inequality

\[
\|z(t) - L(t)m(t)\|_2^2 < \gamma^2 \{\|\omega(t)\|_2^2 + E(x^T(t_0))RE(x(t_0))\} \tag{6}
\]

holds, where \( \|f(t)\|_2^2 := \int_{t_0}^{t} f^T(t) f(t) dt \), \( t_1 \) is the selected filter horizon, \( R \) is a symmetric positive definite matrix, and \( \gamma \) is a given real positive scalar.

III. CENTRAL SUBOPTIMAL MEAN-SQUARE \( H_\infty \) FILTER DESIGN

The proposed design of the suboptimal mean-square \( H_\infty \) filter for linear stochastic systems is based on the general result (see Theorem 3 in [1]) reducing the \( H_\infty \) controller problem to the corresponding optimal \( H_2 \) controller problem. In this paper, only the filtering part of this result, valid for the entire controller problem, is used. Then, the optimal mean-square Kalman-Bucy filter for linear stochastic systems [28] and the \( H_\infty \) filter for linear systems (Theorem 4 in [4]) are employed to obtain the desired result, which is given by the following theorem.

**Theorem 1.** The central suboptimal mean-square \( H_\infty \) filter for the linear stochastic system (1)–(4), ensuring the minimum of the mean-square criterion (5) and the \( H_\infty \) noise attenuation condition (6), is given by the equation for the mean-square \( H_\infty \) estimate \( m(t) \)

\[
dm(t) = (A(t)m(t) + B(t)u(t))dt + \tag{7}
\]
\[
P(t)C_1^T(t)(h(t)h^T(t))^{-1}[dy_1(t) - C_1(t)m(t)]dt + \tag{8}
\]
\[
S(t)C_2^T(t)(H(t)H^T(t))^{-1}[y_2(t) - C_2(t)m(t)]dt,
\]

with initial condition \( m(t_0) = E(x(t_0)) | F_{t_0}^1 \), where the matrix function \( P(t) \) (minimum estimation error variance) is the solution of the differential Riccati equation

\[
P(t) = A(t)P(t) + P(t)A^T(t) - \tag{9}
\]
\[
b(t)b^T(t) - \gamma^2L^T(t)L(t),
\]

with initial condition \( P(t_0) = E[(x(t_0) - m(t_0))(x(t_0) - m(t_0))^T | F_{t_0}^1] \), and the symmetric matrix function \( S(t) \) is the solution of the differential Riccati equation

\[
S(t) = A(t)S(t) + S(t)A^T(t) + G(t)G^T(t) - \gamma^2L^T(t)L(t),
\]

with initial condition \( S(t_0) = R^{-1} \).

**Proof.** First, let us design the estimate \( \hat{x}(t) \) satisfying the minimum variance condition (5) of Section 2. As known [27], this mean-square estimate is given by the conditional expectation \( \hat{x}(t) = E(x(t) | F_1^T) \) of the system state \( x(t) \) with respect to the \( \sigma \)-algebra \( F_1^T \), generated by the observations (2) in the interval \([t_0, t]\), and is produced by the Kalman-Bucy filter [28] applied to the linear stochastic system (1) over the linear observations (2) in the presence of Gaussian disturbances (Wiener processes) \( W_1(t) \) and \( W_2(t) \). The corresponding filtering equations for the estimate \( \hat{x}(t) \) and the estimation error variance \( \tilde{P}(t) \) take the form

\[
d\tilde{x}(t) = (A(t)\hat{x}(t) + B(t)u(t) + G(t)\omega(t))dt + \tag{10}
\]
\[
P(t)C_1^T(t)(h(t)h^T(t))^{-1}[dy_1(t) - C_1(t)\tilde{x}(t)]dt,
\]

with the initial condition \( \tilde{x}(t_0) = E(x(t_0)) | F_{t_0}^1 \), and

\[
\tilde{P}(t) = A(t)\tilde{P}(t) + \tilde{P}(t)A^T(t) + b(t)b^T(t) - \tag{11}
\]

This completes the proof.
\( \hat{P}(t)C(t)(h(t)h^T(t))^{-1}C^T(t)\hat{P}(t), \)

with the initial condition
\( \bar{P}(t_0) = E((x(t_0) - \bar{x}(t_0))(x(t_0) - \bar{x}(t_0))^T | P^0). \)

Note that the latter equation coincides with (8). Now, applying the central suboptimal \( H^\infty \) filter for linear systems [4] to the estimate \( \bar{x}(t) \) governed by equations (10), (11) yields the central suboptimal mean-square \( H^\infty \) estimate equation (7), where the matrix function \( \hat{P}(t) \) satisfies equation (8), and the matrix \( S(t) \) in (7) satisfies equation (9).

Note that filter (7)–(9) yields, in view of Theorems 3 and 4 in [1], the asymptotic stability of the mean value \( E(\bar{x}(t)) \) of the estimate (10) in the absence of disturbances and the prescribed attenuation level \( \gamma \) for this variable: \( \|E(\bar{x}(t))\|^2 < \gamma^2 \|\omega(t)\|^2 + E(x(t_0))^TRE(x(t_0)) \). Since \( E(\bar{x}(t)) = E(x(t)) \) and the variances of the estimated errors produced by the estimates \( \bar{x}(t) \) and \( m(t) \) are equal, the conditions 1–3 of Section 2 hold. The theorem is proved.

**Remark 1.** The convergence of the designed mean-square \( H^\infty \) state estimate \( m(t) \) to the real state value \( x(t) \) is assured by the conditions (6') and (6'') in view of the results of Theorem 7.4 and Section 7.7 in [26]. Note that boundedness of the noise-output \( H^\infty \) norm for the system (7'), controlled by filter (7)–(9), i.e., admissibility of the mean-square \( H^\infty \) filter (7)–(9), is determined by the conditions I–III of Theorem 3 in [1].

**Remark 2.** According to the comments in Subsection V.G in [1], the obtained central mean-square \( H^\infty \) filter (7)–(9) presents a natural choice for \( H^\infty \) filter design among all admissible \( H^\infty \) filters satisfying the inequality (6) for a given threshold \( \gamma \), since it does not involve any additional actuator loop (i.e., any additional external state variable) in constructing the filter gain matrix. Moreover, the obtained central mean-square \( H^\infty \) filter has the suboptimality property, i.e., it minimizes the criterion
\[
J = \|z(t) - L(t)m(t)\|^2 - \gamma^2 \left( \|\omega(t)\|^2 + E(x(0))^TRE(x(0)) \right)
\]

**Remark 3.** Following the discussion in Subsection V.G in [1], note that the complementarity condition always holds for the obtained filter (7)–(9), since the positive definiteness of the initial condition matrix \( R \) implies the positive definiteness of the filter gain matrix \( S(t) \) as the solution of (9).

**IV. Example**

This section presents the design of the central suboptimal mean-square \( H^\infty \) filter to estimate the pitch and yaw angles for a 2DOF helicopter, ensuring the minimum of the mean-square criterion (5) and the \( H^\infty \) noise attenuation condition (6) holds for \( \gamma = 1.1 \).

Let the 2DOF helicopter system with the state space representation
\[
\begin{align*}
x(t) &= Ax(t) + Bu(t) + G\omega(t) + b\psi_1(t), x(t_0) = x_0 \quad (12) \\
y(t) &= C_1x(t) + h\psi_2(t) \quad (13) \\
y_2(t) &= C_2x(t) + H\omega(t) \quad (14)
\end{align*}
\]

where the state vector is: \( x = [\Theta, \Psi, \dot{\Theta}, \dot{\Psi}]^T \), in which \( \Theta \) and \( \Psi \) are pitch and yaw angles respectively, \( \dot{\Theta} \) and \( \dot{\Psi} \) are pitch and yaw rates respectively. The matrices are:
\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -9.2751 & 0 \\
0 & 0 & 0 & -3.4955
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0.2367 & 0.0790 \\
0.2410 & 0.7913
\end{bmatrix}, \quad b = \begin{bmatrix}
0.9024 & 0.0876 \\
0.0919 & 0.8772
\end{bmatrix}
\]
\[
G = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.9024 & 0.0876 & 0 & 0 \\
0.0919 & 0.8772 & 0 & 0
\end{bmatrix}
\]
\[
C_1 = C_2 = L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad h = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
\[
H = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Here, \( u(t) \) is the motor voltage input, \( \omega(t) \) is an \( L_2 \) disturbance input, \( \psi_1(t) \) and \( \psi_2(t) \) are Gaussian white noises, which are the weak mean square derivatives of standard Wiener processes \( W_1(t) \) and \( W_2(t) \) (see [27]), respectively. The Wiener processes are considered independent of each other and of a Gaussian random variable \( x_0 \) serving as the initial condition in (12). Equations (12) and (13) present the conventional form for equations (1) and (2), which is actually used in practice [29]. It can be easily verified that the noise orthogonality condition holds for the system (12)–(15).

The filtering problem to be addressed is the same as described at Section 2. The filtering horizon is set from \( t_0 = 0 \) to \( t_1 = 80 \) s.

The central suboptimal mean-square \( H^\infty \) filter takes the following form for the system (12)–(15)
\[
\dot{m}(t) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -9.2751 & 0 \\
0 & 0 & 0 & -3.4955
\end{bmatrix} m(t) \quad (16)
\]
\[
+ \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix} \left( y(t) - \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} m(t) \right)
\]
\[
+ \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix} \left( y_2(t) - \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix} m(t) \right)
\]
with $m(0) = m_0 = E(x_0 \mid F_0^T)$, where $S(t)$ and $P(t)$ are the solutions to the differential Riccati equations

$$
\dot{S}(t) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & -9.2751 & 0
\end{bmatrix} S(t) + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} P(t) \tag{17}
$$

$$
+ S(t) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & -9.2751
\end{bmatrix}
$$

$$
- S(t) = \begin{bmatrix}
0.1736 & 0 & 0 \\
0 & 0.1736 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

$$
\dot{P}(t) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -9.2751 & 0
\end{bmatrix} P(t) + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} P(t)
$$

$$
+ P(t) = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & -9.2751
\end{bmatrix}
$$

$$
- P(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} P(t) + \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0.8220 & 0.1597
\end{bmatrix}
$$

$$
P(0) = E[(x_0 - m_0)(x_0 - m_0)^T \mid F_0^T] = P_0,
$$

respectively.

Numerical simulations results are obtained solving the equations (12)–(15), and the following initial values:

$$
x_0 = [-0.7069, 0.0, 0.0]^T, \quad P_0 = diag(10, 10, 15, 5), \quad R = [0.8, 0.5, 0.2, 0.1; 0.5, 0.8, 0.7, 0.2; 0.2, 0.7, 0.9, 0.4; 0.1, 0.2, 0.4, 0.9], \quad \text{and} \quad m_0 = [0.1745; -0.5236; -0.15; -0.4]^T.
$$

The motor voltage $u(t)$ is $[12.495, -3.835]^T$, the $L_2$ disturbance $\omega(t) = [\omega_1(t), \omega_2(t), \omega_3(t), \omega_4(t)]^T$ is realized as $\omega_1(t) = 1/(1 + t), \omega_2(t) = 0.1(1 - e^{-0.3t}), \omega_3(t) = 0.1745/(1 + t)$, and $\omega_4(t) = 0.0873(1 - e^{-0.3t})$. The attenuation level value is set to $\gamma = 1.1$. The disturbances $\eta_1(t)$ and $\eta_2(t)$ in (12),(13) are realized using the built-in MatLab white noise function.

As a result of the numerical simulation, the following graphs are presented: graph of the noise-output $H_\infty$ norm (Figure 1); graphs of the pitch angle and the corresponding estimation error (Figure 2); graph of the yaw angle and the corresponding estimation error (Figure 3).

Note that the maximum value of the noise-output $H_\infty$ norm $T = \|z(t) - L(t)m(t)\|_2/(\|\omega(t)\|_2^2 + E(x_0)R E(x_0))^{1/2}$ is 0.208 in the considered simulation interval, which is five times less than the given $H_\infty$ attenuation level, $\gamma = 1.1$. In addition, the estimation errors converge to zero.

V. CONCLUSIONS

This paper designs the central finite-dimensional $H_\infty$ filter for linear stochastic systems with integral-quadratically bounded deterministic disturbances, that is suboptimal for a given threshold $\gamma$ with respect to a modified Bolza-Meyer quadratic criterion including the attenuation control term with opposite sign. The designed filter is applied to estimation of the pitch and yaw angles of a two degrees of freedom (2DOF) helicopter. The simulation results show a reliable performance of the filter, in particular, the obtained attenuation level is five times less than a given threshold. This significant improvement is obtained due to the more reasonable selection of the filter gain matrix in the designed filter. Although this conclusion follows from the developed theory, the numerical simulation serves as a convincing illustration. The presented approach would be applied in the future to obtain the central suboptimal mean-square $H_\infty$ filters for nonlinear polynomial stochastic systems.

REFERENCES


Fig. 1. Noise-output $H_{\infty}$ norm $T$.

Fig. 2. **Above.** Pitch angle. **Below.** Estimation error.

Fig. 3. **Above.** Yaw angle. **Below.** Estimation error.