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A Numerical Method for Consumption-Portfolio Problems

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Abstract— **A numerical framework for continuous-time consumption-portfolio problems is set up by Markov chain approximation with the logarithmic transformation (We call it MCALT [1] algorithm). We show that the complexity of the algorithm is a polynomial. An example with and with prohibition of short-sale on risky securities is provided to demonstrate the proposed numerical method.**

Key words: absorbing boundary conditions, HJB equations, logarithmic transformation, Markov chain approximation, utilities with subsistence levels.

I. Introduction

Continuous-time consumption-portfolio problems have been extensively studied in the literature since the work of Merton (1969, 1971). However, two observations are worth mentioning: First, no explicit solution exists for general utility functions. Second, it is difficult, if not impossible, to obtain explicit solutions for constrained cases but there are various constraints which are important in applications, like prohibition of short sale on securities, etc. Therefore, effective numerical methods are called for.

The direct Markov chain approximation method has been used for the consumption-portfolio problems with an infinite horizon by Fitzpatrick and Fleming [3]. However, they imposed a restrictive condition on utility functions, namely, the constant relative risk aversion (CRRA for short) parameter is in the range of $(0, 1)$. This condition greatly limits the application of the Markov chain approximation in continuous-time economics, since the range of CRRA parameters is $(-\infty, 1)$. There are very few studies, if any at all, in literature on using the direct Markov chain approximation to consumption-portfolio problems with a finite horizon.

Recently, Monte Carlo methods have been applied to continuous-time consumption-portfolio problems by Detemple, Garcia, and Rindisbacher [2] and Cvitanić, Goukasian, and Zapatero [1]. their methods require that markets are complete as mentioned in Cvitanić, Goukasian, and Zapatero [1]. Therefore, their methods are inapplicable to constrained optimization problems due to the fact that constrained financial markets and/or constrained insurance markets are no longer complete since some contingent claims are not hedgeable in such markets.

The aim of this paper is two-fold: First, MCALT algorithm is used to set up the numerical framework for consumption-portfolio problems in the setting of the generalization of usual utility functions. MCALT algorithm resolves the deficiencies of the above two mentioned numerical methods for consumption-portfolio model. Second, we show that the complexity of the algorithm is a polynomial of the number of risky securities, the number of computed grid points, and the input size in the binary number system. Therefore MCALT algorithm is a powerful tool to study continuous-time (un)constrained portfolio problems.

This paper is organized as follows. In the next section we describe the consumption-portfolio problems and state the HJB equation with the boundary conditions. In Section 3 we introduce MCALT algorithm to consumption-portfolio problems, and apply it to an example with and without a nonnegative constraint on portfolio. We conclude with some remarks in Section 4.

II. Model Descriptions and HJB Equations with Boundary Conditions

Let $W(t) = (W_1(t), \cdots, W_N(t))'$, here $'$ is a transpose operator, be a standard N-dimensional Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t$ at time t be the P-augmentation of the filtration $\sigma\{W(s), s \le t\}, t \in [0, T]$, it represents the information at time t.

There is a riskless security and N risky securities in the financial market. The riskless security evolves according to

$$dS_0(t) = r(t)S_0(t)dt, \tag{1}$$

and N risky securities evolves according to

$$dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{N} \sigma_{ij}(t)dW_j(t)\right), \quad i = 1, \cdots, N. \tag{2}$$

Assume $r(t)$, $\mu(t) \triangleq (\mu_1(t), \cdots, \mu_N(t))'$, and $\sigma(t) \triangleq (\sigma_{ij}(t))_{N\times N}$, satisfy the usual assumptions such that the market is complete.

We define some processes describing the investor's input and decisions at time t:

- $i(t) \triangleq$ Income rate at time t.
- $c(t) \triangleq$ Consumption rate at time t.

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[1]The name 'MCALT' was first announced in my presentation at 2010 American Control Conference.

- $\theta_i(t) \triangleq$ Dollar amount in the risky security i at time t, $i = 1, \cdots, N$. $\theta(t) \triangleq (\theta_1(t), \cdots, \theta_N(t))'$.

The wealth process $X(t)$ satisfies the stochastic differential equation

$$\begin{aligned} dX(t) &= r(t)X(t)dt - c(t)dt + i(t)dt \\ &\quad +\theta'(t)[(\mu(t) - r(t)\bar{1})dt + \sigma(t)dW(t)]. \end{aligned} \quad (3)$$

Where $\bar{1}$ is the N-dimensional vector whose every component is 1.

Suppose that the investor's preference structure is given by (U_1, U_2). $U_1(\cdot, t)$ is a utility function for the consumption with the subsistence consumption $\bar{c}(t) : [0, T] \mapsto [-\infty, \infty)$, and $U_2(\cdot)$ is a utility function for the terminal wealth with the subsistence terminal wealth $\bar{X} \in [-\infty, \infty)$.

Remark 2.1:
- Here we loose the nonnegative requirements on the subsistence levels $\bar{c}(\cdot)$, and $\bar{X}$ (see Definition 3.4.1, Karatzas and Shreve [4]). Moreover they can possibly take $-\infty$, in this case, it means no subsistence level. Examples are exponential utility functions which has no subsistence level.
- There are nonnegative constraints on consumption and the terminal wealth. Together with subsistence levels, we define

$$\begin{aligned} \bar{c}_0(t) &\triangleq \max\{\bar{c}(t), 0\}, \quad t \in [0, T], \\ \bar{X}_0 &\triangleq \max\{\bar{X}, 0\}, \end{aligned}$$

which we call the essential levels for consumption and the terminal wealth, respectively.

Given an wealth x at time $t \leq T$ and a pair of investor's decisions, (c, θ), his expected utility at time t with the wealth x is

$$\begin{aligned} V^{(c,\theta)}(t, x) &\triangleq E\bigg[\int_t^T U_1(c(s), s)ds \\ &\quad +U_2(X(T))\Big|\mathcal{F}_t\bigg]. \end{aligned} \quad (4)$$

The investor's problem at time t with the wealth x is to choose consumption c^* and portfolio strategies θ^* to maximize his expected utility (4). Denote by $V(t, x)$ the maximized expected utility at time t with the wealth x . By the dynamic programming technique (see Merton [7], [8]), we have

$$V_t(t, x) + \sup_{(c \geq \bar{c}_0(t), \theta)} \Psi(t, x; c, \theta) = 0 \quad (5)$$

on the domain $D \triangleq \{(t, x) \in [0, T] \times (-\infty, +\infty), x > \bar{b}_0(t) - b(t)\}$. Where

$$\begin{aligned} &\Psi(t, x; c, \theta) \\ &\triangleq \frac{1}{2}\theta'\sigma(t)\sigma'(t)\theta V_{xx}(t, x) + (\theta'(\mu(t) - r(t)\bar{1}) \\ &\quad +r(t)x + i(t) - c)V_x(t, x) + U_1(c, t), \end{aligned} \quad (6)$$

In addition, V satisfies a boundary condition

$$V(T, x) = U_2(x), \quad (7)$$

and an absorbing boundary condition

$$\begin{aligned} &V(t, \bar{b}_0(t) - b(t)) \\ &= \int_t^T U_1(\bar{c}_0(s), s)ds + U_2(\bar{X}_0). \end{aligned} \quad (8)$$

Remark 2.2: In the absorbing boundary condition (8), $b(t)$ is define as

$$b(t) = \int_t^T i(s) \exp\left\{-\int_t^s r(v)dv\right\} ds, \quad (9)$$

and $\bar{b}_0(t)$ is defined as

$$\begin{aligned} \bar{b}_0(t) &= \int_t^T \bar{c}_0(s) \exp\left\{-\int_t^s r(v)dv\right\} ds \\ &\quad +\bar{X}_0 \exp\left\{-\int_t^T r(s)ds\right\}. \end{aligned} \quad (10)$$

The absorbing boundary condition (8) is the generalization of the one in Theorem 3.8.11, Karatzas and Shreve [4]).

We do the following transformations on (5)

$$\begin{aligned} w &\triangleq x + b(t) - \bar{b}_0(t), && (11) \\ \widetilde{V}(t, w) &\triangleq V(t, x), && (12) \\ \widetilde{c}(t) &\triangleq c(t) - \bar{c}_0(t). && (13) \end{aligned}$$

We have

$$\widetilde{V}_t(t, w) + \sup_{(\widetilde{c} \geq 0, \theta)} \widetilde{\Psi}(t, w; \widetilde{c}, \theta) = 0 \quad (14)$$

on the domain $\widetilde{D} \triangleq \{(t, w) \in [0, T] \times (0, +\infty)\}$ with the boundary conditions

$$\widetilde{V}(T, w) = U_2(w + \bar{X}_0), \quad (15)$$

$$\widetilde{V}(t, 0) = \int_t^T U_1(\bar{c}_0(s), s)ds + U_2(\bar{X}_0). \quad (16)$$

Where

$$\begin{aligned} &\widetilde{\Psi}(t, w; \widetilde{c}, \theta) \\ &\triangleq \frac{1}{2}\theta'\sigma(t)\sigma'(t)\theta\widetilde{V}_{ww}(t, w) + (\theta'(\mu(t) - r(t)\bar{1}) \\ &\quad +r(t)w - \widetilde{c})\widetilde{V}_w(t, w) + U_1(\widetilde{c} + \bar{c}_0(t), t). \end{aligned} \quad (17)$$

Now the term i is disappeared in (14) compared to (5), and the absorbing boundary is no more a moving boundary after the transformations.

For CRRA utility functions with nonnegative subsistence levels, if any one of CRRA parameters for U_1 and U_2 is negative or zero, the absorbing boundary condition (16) is singular. At this point, if we introduce the numerical scheme to (14), the scheme will be greatly limited since the possibly singular boundary condition (16) is involved in it.

III. MCALT Algorithm

To resolve the difficulty mentioned in the previous section, we will do the logarithmic transformation on (14) as follows

$$u \triangleq \ln w, \tag{18}$$
$$\widehat{V}(t,u) \triangleq \widetilde{V}(t,w), \tag{19}$$
$$\widehat{c} \triangleq \frac{\widetilde{c}}{w} = e^{-u}\widetilde{c}, \tag{20}$$
$$\widehat{\theta} \triangleq \frac{\theta}{w} = e^{-u}\theta. \tag{21}$$

Here $\widehat{c}$ and $\widehat{\theta}$ are consumption proportion and portfolio proportions.

We have

$$\widehat{V}_t(t,u) + \sup_{(\widehat{c}\geq 0,\widehat{\theta})} \widehat{\Psi}(t,u;\widehat{c},\widehat{\theta}) = 0 \tag{22}$$

on the domain $\widehat{D} \triangleq \{(t,u) \in [0,T] \times (-\infty,+\infty)\}$ with the boundary conditions

$$\widehat{V}(T,u) = U_2(e^u + \bar{X}_0), \tag{23}$$
$$\widehat{V}(t,-\infty) = \int_t^T U_1(\bar{c}_0(s),s)ds + U_2(\bar{X}_0). \tag{24}$$

Where

$$\begin{aligned}
&\widehat{\Psi}(t,u;\widehat{c},\widehat{\theta}) \\
&\triangleq \frac{1}{2}\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta}(\widehat{V}_{uu}(t,u) - \widehat{V}_u(t,u)) \\
&\quad +(\widehat{\theta}'(\mu(t) - r(t)\bar{1}) + r(t) - \widehat{c})\widehat{V}_u(t,u) \\
&\quad +U_1(e^u\widehat{c} + \bar{c}_0(t),t).
\end{aligned} \tag{25}$$

The absorbing boundary condition moves to $-\infty$ under the logarithmic transformation, the domain of the above HJB equation becomes $[0,T] \times (-\infty,\infty)$. Clearly the absorbing boundary condition won't be involved in numerical methods since it occurs at $-\infty$.

Now we introduce the following approximating rules (Kushner and Dupuis [6]):

$$\widehat{V}(t,u) \to \widehat{V}^{h,\delta}(t,u). \tag{26}$$
$$\widehat{V}_t(t,u) \to \frac{\widehat{V}^{h,\delta}(t+\delta,u) - \widehat{V}^{h,\delta}(t,u)}{\delta}. \tag{27}$$

If a coefficient of $\widehat{V}_u$ is positive,

$$\widehat{V}_u(t,u) \to \frac{\widehat{V}^{h,\delta}(t+\delta,u+h) - \widehat{V}^{h,\delta}(t+\delta,u)}{h}. \tag{28}$$

If a coefficient of $\widehat{V}_u$ is negative

$$\widehat{V}_u(t,u) \to \frac{\widehat{V}^{h,\delta}(t+\delta,u) - \widehat{V}^{h,\delta}(t+\delta,u-h)}{h}. \tag{29}$$

$$\begin{aligned}
&\widehat{V}_{uu}(t,u) \\
&\to \frac{\widehat{V}^{h,\delta}(t+\delta,u+h)}{h^2} - 2\frac{\widehat{V}^{h,\delta}(t+\delta,u)}{h^2} \\
&\quad + \frac{\widehat{V}^{h,\delta}(t+\delta,u-h)}{h^2}.
\end{aligned} \tag{30}$$

The discretized version of (22) is

$$\widehat{V}^{h,\delta}(t,u) = \sup_{(\widehat{c}\geq 0,\widehat{\theta})} \widehat{\Psi}^{h,\delta}(t,u;\widehat{c},\widehat{\theta}) \tag{31}$$

with the boundary condition

$$\widehat{V}^{h,\delta}(T,u) = U_2(e^u + \bar{X}_0). \tag{32}$$

Where

$$\begin{aligned}
&\widehat{\Psi}^{h,\delta}(t,u;\widehat{c},\widehat{\theta}) \\
&\triangleq \widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u+h)\widehat{V}^{h,\delta}(t+\delta,u+h) \\
&\quad +\widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u)\widehat{V}^{h,\delta}(t+\delta,u) \\
&\quad +\widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u-h)\widehat{V}^{h,\delta}(t+\delta,u-h) \\
&\quad +\delta U_1(e^u\widehat{c} + \bar{c}_0(t),t),
\end{aligned} \tag{33}$$

here

$$\begin{aligned}
&\widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u+h) \\
&\triangleq \frac{\delta}{2h^2}\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} \\
&\quad +\frac{\delta}{h}(\widehat{\theta}'(\mu(t) - r(t)\bar{1})\mathbf{1}_{\widehat{\theta}'(\mu(\mathbf{t})-\mathbf{r}(\mathbf{t})\bar{\mathbf{1}})\geq \mathbf{0}} + \mathbf{r}(\mathbf{t})),
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u-h) \\
&\triangleq (\frac{\delta}{2h^2} + \frac{\delta}{2h})\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} \\
&\quad -\frac{\delta}{h}(\widehat{\theta}'(\mu(t) - r(t)\bar{1})\mathbf{1}_{\widehat{\theta}'(\mu(\mathbf{t})-\mathbf{r}(\mathbf{t})\bar{\mathbf{1}})\leq \mathbf{0}} - \widehat{\mathbf{c}}),
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u) \\
&\triangleq 1 - \widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u+h) - \widehat{P}_{(\widehat{c},\widehat{\theta})}(t,u;u-h).
\end{aligned} \tag{36}$$

Here (31) is the finite difference equation which needs to be solved backward in time. $\widehat{P}_{(\widehat{c},\theta)}(t,u;u+h)$, $\widehat{P}_{(\widehat{c},\theta)}(t,u;u)$, and $\widehat{P}_{(\widehat{c},\theta)}(t,u;u-h)$ are interpreted as transition probabilities of a Markov chain at time t since the sum of them is 1, and they are associated with three states $u+h$, $u-h$, u at time $t+\delta$, respectively. Hence the key assumption is that the right hand sides of (34) – (36) are nonnegative. The right hand sides of (34) – (36) do not depend on the state variable u explicitly. In addition, the proportions $\widehat{c}$ and $\widehat{\theta}$ are small in practice. So we can choose some appropriate values of δ and h such that the right hand sides of (34) – (36) are uniformly nonnegative.

Remark 3.1: In this remark, we are going to verify the local consistency (see Equation 12.1.5 in Kushner and Dupuis [6]) of the approximation scheme (31) . Let

$$U(t) \triangleq \ln(X(t) + b(t) - \bar{b}_0(t)),$$
$$\widehat{c}(t) \triangleq \frac{c(t) - \bar{c}_0(t)}{X(t) + b(t) - \bar{b}_0(t)},$$
$$\widehat{\theta}(t) \triangleq \frac{\theta(t)}{X(t) + b(t) - \bar{b}_0(t)}.$$

According to (4), by Itô lemma, we have

$$dU(t) = \bigg(r(t) - \frac{1}{2}\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} - \widehat{c}(t) + \widehat{\theta}'(t)(\mu(t) - r(t)\bar{1})\bigg)dt + \widehat{\theta}'(t)\sigma(t)dW(t)$$

Which is the dynamics corresponding to HJB equation (22).

From (34) – (35),

$$\begin{aligned}
&E[U(t+\delta) - u \mid U(t) = u] \\
&\triangleq \widehat{P}_{(\widehat{c},\widehat{\theta})}(t, u; u+h) * h - \widehat{P}_{(\widehat{c},\widehat{\theta})}(t, u; u-h) * h \\
&= \delta\bigg(r(t) - \frac{1}{2}\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} - \widehat{c}(t) \\
&\qquad + \widehat{\theta}'(t)(\mu(t) - r(t)\bar{1})\bigg)
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
&\widehat{P}_{(\widehat{c},\widehat{\theta})}(t, u; u+h) * h^2 + \widehat{P}_{(\widehat{c},\widehat{\theta})}(t, u; u-h) * h^2 \\
&\quad - (E[U(t+\delta) - u \mid U(t) = u])^2 \\
&= \delta\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} + O(h\delta).
\end{aligned} \tag{38}$$

Therefore, the approximation scheme (31) is locally consistent.

We are going to discuss the optimal decisions in the difference equation (31). From (34) – (36), after some algebra, we have

$$\begin{aligned}
&\sup_{(\widehat{c}\geq 0,\widehat{\theta})} \widehat{\Psi}^{h,\delta}(t, u; \widehat{c}, \widehat{\theta}) \\
&= \delta \sup_{\widehat{\theta}} \bigg\{ \frac{1}{2}(\widehat{V}^{h,\delta}_{uu}(t, u) - \widehat{V}^{h,\delta}_{u,-}(t, u))\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} \\
&\quad + \widehat{V}^{h,\delta}_{u,+}(t, u)\widehat{\theta}'(\mu(t) - r(t)\bar{1})\mathbf{1}_{\widehat{\theta}'(\mu(\mathbf{t}) - \mathbf{r}(\mathbf{t})\bar{\mathbf{1}}) \geq \mathbf{0}} \\
&\quad + \widehat{V}^{h,\delta}_{u,-}(t, u)\widehat{\theta}'(\mu(t) - r(t)\bar{1})\mathbf{1}_{\widehat{\theta}'(\mu(\mathbf{t}) - \mathbf{r}(\mathbf{t})\bar{\mathbf{1}}) \leq \mathbf{0}} \bigg\} \\
&\quad + \delta \sup_{\widehat{c}\geq 0} \Big\{ U_1(e^u\widehat{c} + \bar{c}_0(t), t) - \widehat{V}^{h,\delta}_{u,-}(t, u)\widehat{c} \Big\} \\
&\quad + \delta r(t)\widehat{V}^{h,\delta}_{u,+}(t, u) + \widehat{V}^{h,\delta}(t+\delta, u). \\
&= \delta \max\bigg[\\
&\sup_{\widehat{\theta}'(\mu(t) - r(t)\bar{1}) \geq 0} \bigg\{ \frac{1}{2}(\widehat{V}^{h,\delta}_{uu}(t, u) - \widehat{V}^{h,\delta}_{u,-}(t, u))\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} \\
&\quad + \widehat{V}^{h,\delta}_{u,+}(t, u)\widehat{\theta}'(\mu(t) - r(t)\bar{1}) \bigg\}, \\
&\sup_{\widehat{\theta}'(\mu(t) - r(t)\bar{1}) \leq 0} \bigg\{ \frac{1}{2}(\widehat{V}^{h,\delta}_{uu}(t, u) - \widehat{V}^{h,\delta}_{u,-}(t, u))\widehat{\theta}'\sigma(t)\sigma'(t)\widehat{\theta} \\
&\quad + \widehat{V}^{h,\delta}_{u,-}(t, u)\widehat{\theta}'(\mu(t) - r(t)\bar{1}) \bigg\} \bigg] \\
&\quad + \delta \sup_{\widehat{c}\geq 0} \Big\{ U_1(e^u\widehat{c} + \bar{c}_0(t), t) - \widehat{V}^{h,\delta}_{u,-}(t, u)\widehat{c} \Big\} \\
&\quad + \delta r(t)\widehat{V}^{h,\delta}_{u,+}(t, u) + \widehat{V}^{h,\delta}(t+\delta, u).
\end{aligned} \tag{39}$$

Where

$$\begin{aligned}
&\widehat{V}^{h,\delta}_{u,+}(t, u) \\
&\triangleq \frac{\widehat{V}^{h,\delta}(t+\delta, u+h) - \widehat{V}^{h,\delta}(t+\delta, u)}{h},
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\widehat{V}^{h,\delta}_{u,-}(t, u) \\
&\triangleq \frac{\widehat{V}^{h,\delta}(t+\delta, u) - \widehat{V}^{h,\delta}(t+\delta, u-h)}{h},
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\widehat{V}^{h,\delta}_{uu}(t, u) \\
&\triangleq \frac{\widehat{V}^{h,\delta}(t+\delta, u+h)}{h^2} - 2\frac{\widehat{V}^{h,\delta}(t+\delta, u)}{h^2} \\
&\quad + \frac{\widehat{V}^{h,\delta}(t+\delta, u-h)}{h^2}.
\end{aligned} \tag{42}$$

Note, the first term of the right hand side of (39) is a combination of two quadratic programming problems with linear constraints, and the second term of it is a single-variable convex programming problem with a nonnegative constraint.

There are various constraints on consumption proportion $\widehat{c}$ and portfolio proportions $\widehat{\theta}$ due to various requirements from the reality besides the nonnegativity of essential levels, for example, prohibition of short sale on securities, e.g., $\widehat{\theta} \geq 0$. Such constraints on $\widehat{c}$ and $\widehat{\theta}$ are added to the right hand side of (39) to succeed in mathematical modeling. Adding the constraints to $\widehat{c}$ in the second term of (39), it is still a single-variable convex programming which can be solved if we know the forms of U_1 by the elementary calculus. Adding the constraints to $\widehat{\theta}$ in the first term of (39), it is still a quadratic programming. If $\widehat{\theta}$ subjects to one or more linear inequality and/or equality constraints like

$$A\widehat{\theta} \leq d, \tag{43}$$

$$B\widehat{\theta} = z, \tag{44}$$

according to Kozlov, Tarasov, and Khachiyan (1979), the complexity of quadratic programming in (39) is polynomial solvability under the assumption $\widehat{V}^{h,\delta}_{uu}(t, u) < \widehat{V}^{h,\delta}_{u,-}(t, u)$.

We describe the procedure of the algorithm as follows. The time step δ and the spatial step h divide the space (t, u) into grid points. Assuming the time step δ divide the time interval $[0, T]$ into L time rows.

- Step 1: Compute the utility V for each grid point at the L-th time row using the terminal condition (32). Set the variable $l = L - 1$.
- Step 2: Solve the optimization problem (39) (possibly with additional constraints) for each grid point at the l-th time row, and store the utility value for each grid point at the l-th time row by the equation (31).
- Step 3: If $l > 0$, decrease l by 1 and go to Step 2. Otherwise, the algorithm succeeds.

By the above descriptions, we have the following complexity theorem:

Theorem 3.2: Assuming $\widehat{\theta}$ subjects to one or more constraints in the forms of (43) – (44). If $\widehat{V}^{h,\delta}_{uu}(t,u) < \widehat{V}^{h,\delta}_{u,-}(t,u)$ holds for each grid point to be computed, The complexity of the algorithm is bounded by a polynomial of the number of risky securities, the number of computed grid points, and the input size in the binary number system.

Now we are in a position to implement the numerical method. Let's consider an investor to invest in stock 1 and stock 2 in the financial market for one year. His CRRA parameter $\gamma = -3$, the utility discounted rate $\rho = 0.03$, utility weights for the consumption and the terminal wealth are 1, and the subsistence levels are zero. Suppose the annual interest rate $r = 4\%$. The drift term for stock 1 and 2 is given by $(\mu_1, \mu_2)' = (6\%, 7\%)'$, and the volatility is given by

$$\sigma = \begin{bmatrix} 12\% & 9\% \\ 9\% & 15\% \end{bmatrix}$$

We consider the example with and without a nonnegative constraint on portfolio. In both cases, the time step δ is taken as 0.01 and the state step h is taken as 0.02. All transition probabilities are nonnegative as the computer programs check. We use the values of u between -8 and 8 to plot the figures for both cases. The values of u between -8 and 8 are equivalent to the total available wealth between $e^{-8} \approx 0$ and $e^{8} \approx 2981$ thousand dollars. As we note from the numerical method, in order to compute the numerical solutions for the discretized $u \in [-10, 10]$ at time $t = 0$, it is necessary to involve the terminal condition for every discretized $u \in [-12, 12]$ at time $T = 1$.

We plot Figures 1–3 for the example without the prohibition of short sale on securities, while plot Figures 4 – 6 for the example with the prohibition of short sale on securities. Figure 2 shows that optimal portfolio proportion on stock 1 is a constant and close to -0.3 which is calculated from explicit solutions, meaning that the investor short sells stock 1, while Figure 5 shows that the optimal portfolio proportion on stock 1 is zero. Figure 3 shows that optimal portfolio proportion on stock 2 is a constant close to the number 0.5 which is calculated from explicit solutions, while Figure 6 shows that the optimal portfolio proportion on stock 2 is a constant around 0.25.

IV. Discussion

We set up the numerical framework for the consumption-portfolio problems by MCALT algorithm. MCALT algorithm consists of two steps: First, do the logarithmic transformation to push the possibly singular absorbing boundary condition to $-\infty$ and transform the HJB equation. Second, apply Markov chain approximation to approximate the resulting HJB equation from the first step. The polynomial complexity of the algorithm shows that it is a powerful tool to study continuous-time (un)constrained portfolio problems.

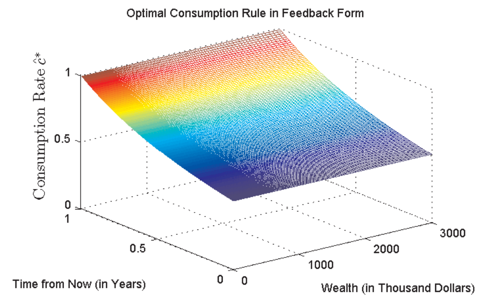

Fig. 1. Optimal Consumption Proportion without the Prohibition of Short Sale on Securities

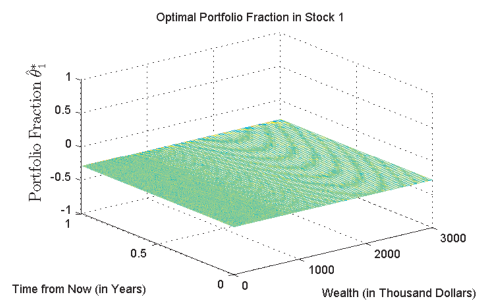

Fig. 2. Optimal Portfolio Proportion on Stock 1 without the Prohibition of Short Sale on Securities

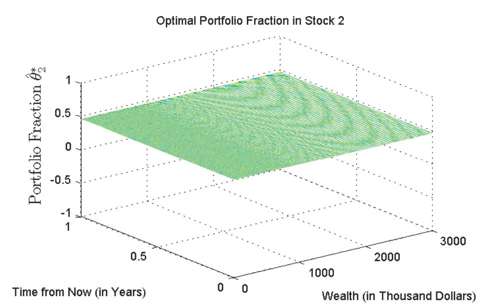

Fig. 3. Optimal Portfolio Proportion on Stock 2 without the Prohibition of Short Sale on Securities

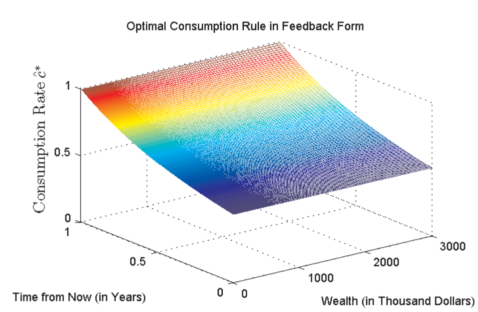

Fig. 4. Optimal Consumption Proportion with the Prohibition of Short Sale on Securities

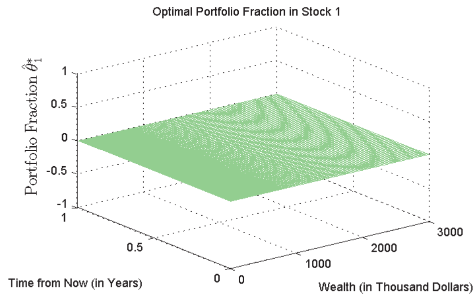

Fig. 5. Optimal Portfolio Proportion on Stock 1 with the Prohibition of Short Sale on Securities

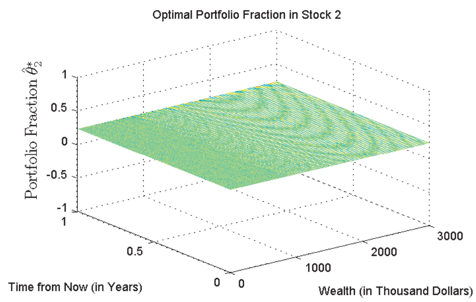

Fig. 6. Optimal Portfolio Proportion on Stock 2 with the Prohibition of Short Sale on Securities

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