Estimation of Benchmark Performance for Nonlinear Control Systems

Zhi Zhang, Li-Sheng Hu and Wen-Jun Shi

Abstract—During the last two decades, performance assessment of control systems has been receiving wide attention. However, estimation of the benchmark performance of nonlinear control systems still remains open. In this work, we consider an estimation problem of the benchmark performance when control loops are intervened by the complex non-differentiable nonlinearity. The considered nonlinearity includes control valve stiction of the control systems, as well as switching action involved by protection valves widely used in the safety-related control systems. Based on the idea of Lebesgue sampling, the paper proposes a novel threshold autoregressive model for unbiased estimation of the benchmark performance. Basically, the proposed method is based on the partitioning of the closed-loop routine operating data when it reaches certain thresholds. Modeling each partitioned regime as an autoregressive model, the prediction error variance as the benchmark performance can be obtained by the principle of pooled variance. A numerical example well shows the effectiveness of the proposed method.

I. INTRODUCTION

Research on control loops performance assessment has been increasing in the last two decades. There are many researches on performance assessment for linear control loops. Some literatures in this area were published, see [7], [8], [12], [13], [17], [16], [21], [27], and the references therein. The main motivation of this issue is to provide an online automated procedure for determining whether specified performance targets and response characteristics are being met by the controlled process variables or not. The key step for performance assessment procedure is to establish the appropriate benchmark performance in order to obtain reasonable assessment results. The popular benchmark performance are the minimum variance performance and the prediction error variance.

It is well-known that the most of industrial processes are nonlinear in nature. The nonlinearities may come from process plants, external disturbances, and sensors or actuators and so on. Because of nonlinearities, the system output may be nonlinear and non-Gaussian. Moreover, the process dynamics or disturbances models can not be well characterized by its impulse response, or equivalently by an time series model. These challenge both in modeling and parameter estimation, which may lead to biased estimations of benchmark performance for nonlinear systems. To obtain the appropriate benchmark performance of nonlinear systems, researchers developed a class of analytical nonlinear system model. These models are described by the nonlinear process plus linear output stochastic disturbance [9], [18], in which disturbance is the output of a time series model driven by the white noise. Under this kind of models, Harris and Yu [18] estimated the minimum variance performance bounds for nonlinear systems using the polynomial approximation of the nonlinear process.

In many industrial applications, however, the nonlinearity causing poor control performance is mostly control valve stiction. In terms of this problem, many researches focused on stiction detection and quantification methodologies, see [5], [10], [22], and the references therein. In the reports, valve stiction is non-differentiable and the loops with valve stiction exhibit limit cycle behavior. In this case, polynomial approximation may not sufficiently model the nonlinearity and will lead to the over-estimation of the benchmark performance for control loops performance assessment. Considering this issue, [32] proposed two indirect estimation methods for the benchmark performance. After removing the nonlinearity from the output by B-spline, the residuals between the output and B-spline are fitted by a linear time series model to estimate the benchmark performance. The other method is that the steady state periods caused by limit cycle are estimated firstly, then the benchmark performance is computed by linear models fitting in the given possible multiple segments of a linear time series.

However, the nonlinear actions of actuators located in the control loop may sometimes be controlled by other control system, e.g., the safety-related applications. According to [19], [20], safety-related systems and control systems are built separately for the same controlled variable [23], as shown in Figure 1. There are the control valve and the protection valve located in the control loop. The nonlinear action - switching action from the protection valve controlled by the safety-related control system will be introduced into the control system. This action is generally characterized by the non-differentiable or discontinuous function. For the nonlinearities involving the control valve stiction and protection valve action will make estimation of the benchmark performance more challenging. In terms of stochastic protection valve action, the correlation between the controlled variable and the persistent exciting control signal may be broken. Hence the estimation algorithm depending on the correlation may lead to a biased result.

Moreover, time series modeling based on equidistant sampled data or Riemann sampled data may be not easy to deal with non-differentiable nonlinear dynamics directly. For Riemann sampling, benchmark performance may only exist in some parts of the nonlinear dynamics or some dynamic
subsystems. In order to obtain benchmark performance, sufficient removing the non-differentiable nonlinearities is needed, sometimes even full identifications of process dynamics, disturbances and other external nonlinear dynamics, and more priori knowledge are needed [12].

![Diagram of control system and safety system](image)

Fig. 1. Integration example of control system and safety system

Interestingly, Lebesgue sampling or event-based sampling - an alternative to Riemann sampling can be considered. Lebesgue sampling, as a class of non-union sampling, is a technique for time independent event-based sampling [2], [26]. Actually, the idea of Lebesgue sampling or event-based triggered sampling is closely related to control systems, e.g., relay feedback control system [1], variable structure control system [31] and multi-rate sampled-data control system [4], [14], [15] which can be regarded as special cases of Lebesgue sampling. From the application point of view, Lebesgue sampling is a natural method to describe functionality of devices used in industrial control systems. For example, A/D converters of sigma delta modulator [2]. Moreover, some nonlinear phenomena can also be viewed as the responses of some events. For example, the valve stiction and the valve switching can be considered as abnormal events occurring in control loops.

For modeling the complex nonlinear time series data based on the Lebesgue sampling, threshold autoregressive (TAR) model can be considered, see [28], [29], [30], and the references therein. The time series data are partitioned according to the variation of the data value, i.e., thresholds are imposed on the value. For the data over each partitioned regime, a time series model is identified.

In this paper, we will explore a new method inspired by the idea of the Lebesgue sampling to estimate benchmark performance of nonlinear control systems. Considering the nonlinearities including the control valve stiction and the protection valve action, the prediction error variance of TAR fitting from routine operating data is used to estimate benchmark performance directly. In this paper, we assume that the control system is controlled by computer, and the stiction does not exist in the protection valve. In terms of control valve stiction, the two-parameter empirical model developed in [6] is used in the simulation of this work. A major contribution of this paper is the application of the idea of Lebesgue sampling and its usage to estimate the benchmark performance considering the nonlinear action of the safety-related control system are introduced into the nonlinear control system. The estimation of the benchmark performance by the proposed method is better than that obtained by the method under Riemann sampling. The result of estimation can be considered as an unbiased benchmark to establish a better quantification index of performance and avoid over-estimation or under-estimation.

The remainder of this paper is organized as follows. Problem formulation on estimation of benchmark performance for a class of nonlinear system is presented in Section II. Section III presents the method based on Lebesgue sampling to estimate benchmark performance of nonlinear control systems. In Section IV, numerical examples show the effectiveness of the proposed method. Finally, the conclusion remarks are given in Section V.

II. PROBLEM FORMULATION

Consider a control system assisted with the safety-related control system. As shown in Figure 2, if the controller of safety system receives the fault information, the higher priority commands will be sent to protection valve located in the control loop to guarantee the safety of the control system.

![Schematic of system hardware](image)

Fig. 2. Schematic of system hardware

Remark 1: In applications, for safety systems, there are different requirements of the state of the valve on different occasions. For example, when a steam heating facility encounters a electrical or air failure, the steam should be cut off by closing the valve. When water cooling of system encounters a fault, the water can not be cut off. However, sometimes the valve is needed to remain in the original position before failure. For example, when crystallization process encounters the air failure, the coolant should maintain a fixed flow rate. Hence, the protection valve can be used to implement the fail close, fail open and fail lock [24].

Suppose that this system can be modeled by

\[ y_t = q^{-b}T(q^{-1})\Psi(u_t) + N(q^{-1})\alpha_t, \]

as shown in Figure 3, where \( q^{-1} \) is the backward shift operator, \( b \) is the process time delay, \( u_t \) denotes controller output and \( y_t \) denotes measured process output, \( \alpha_t \) denotes white noise sequence with zero mean and variance \( \sigma^2 \). \( T(q^{-1}) \) is the delay-free polynomial function in \( q^{-1} \) for the plant. \( N(q^{-1}) \) is the polynomial function in \( q^{-1} \) for the stochastic load disturbance occurring in the control loop, which is driven by white noise sequence \( \alpha_t \). Such a disturbance representation can handle the non-stationary random process, as well as stationary one [3], [25]. \( N \) can be described as \( N = \frac{\nabla}{\phi} \), where \( \nabla = (1 - q^{-1}) \) is the...
difference operator and $h$ is a nonnegative integer less than 2. $\theta$ and $\phi$ are polynomials in $q^{-1}$, and monic and stable. $\Psi(\cdot)$ is the nonlinear function representing the relationship between the controller output and the plant input of the control system. In the Figure 3, $S(u_t)$ represents nonlinear actions of actuators in the control system, e.g., control valve stiction, and $V(s_t, z_t)$ denotes the nonlinear actions of actuators located in this control system, but controlled by the safety-related control system, e.g., protection valve. $z_t$ denotes the control commands from the safety-related control system. To simplify the sequel development, $Ψ$ denotes the control commands from the safety-related control system, e.g., protection valve.

Problem 1: Considering the complex nonlinearity, estimate the benchmark performance of the control loop based on the idea of Lebesgue sampling.

Problem 2: Verify the effectiveness of the proposed method by examples.

III. ESTIMATION OF BENCHMARK PERFORMANCE

If $Ψ$ only includes $S$ in a nonlinear process described as (1), there exists feedback control invariant or the $b$-step ahead prediction error variance as the benchmark performance [18], [32]. In this paper, when $Ψ$ is composed of $S$ and $V$, the benchmark performance is obtained.

$Na_t$ in (1) can be denoted as $d_t = Na_t$, and the impulse coefficients $f_i$, for $i = 0, 1, \ldots, \infty$, of the $N$ can be obtained as

$$ N = 1 + f_1 q^{-1} + \cdots + f_b q^{-b} + \cdots $$

Thus,

$$ y_{t+b} = TΨ(u_t) + d_{t+b} $$
$$ = TΨ(u_t) + f_b d_t $$
$$ + (1 + f_1 q^{-1} + \cdots + f_{b-1} q^{-b+1})a_{t+b}, $$

where $d_{t+b} = (1 + f_1 q^{-1} + \cdots + f_{b-1} q^{-b+1})a_{t+b} + f_b d_t$ can be obtained by solving Diophantine equations. The conditional expectation of $y_{t+b}$ can be obtained [18], [32].

$$ \hat{y}_{t+b} = E\{y_{t+b} | I_t\} = TΨ(u_t) + f_b d_t, $$

where $I_t$ is the information set [18]. The prediction error is

$$ y_{t+b} - \hat{y}_{t+b} = (1 + f_1 q^{-1} + \cdots + f_{b-1} q^{-b+1})a_{t+b}, $$

Hence, under minimum variance control, the $\hat{y}_{t+b}$ equals zero, the process output $y^ MV_{t+b}$ will only depend on the recent $b$ past disturbances,

$$ y^MV_{t+b} = (1 + f_1 q^{-1} + \cdots + f_{b-1} q^{-b+1})a_{t+b}. $$

The minimum variance performance benchmark, $\sigma^2 MV$, exists and equals the prediction error variance,

$$ \sigma^2_{MV} = \text{Var}\{y^ MV_{t+b}\} = (1 + f_1^2 + \cdots + f_{b-1}^2)\sigma^2_a. $$

Although the benchmark performance analytically exists, it would be estimated from the close-loop routine operating data. In order to obtain a unbiased estimation of the benchmark performance as good as possible, we model the nonlinear close-loop data on the basis of the idea of Lebesgue sampling, and then obtain the prediction error variance as the benchmark performance.

Consider the following self-exciting TAR (SETAR) model which is a basic and popular class of TAR model. The $k$-regime SETAR model denoted by $\text{SETAR}(k; b; p_1, p_2, \ldots, p_k)$ has the following form,

$$ y_t = \begin{cases} 
\phi_{0,1} + \sum_{j=1}^{p_1} \phi_{j,1} y_{t-j} + e_{t,1}, & \text{if } y_{t-d} \leq r_1; \\
\phi_{0,2} + \sum_{j=1}^{p_2} \phi_{j,2} y_{t-j} + e_{t,2}, & \text{if } r_1 < y_{t-d} \leq r_2; \\
\vdots & \\
\phi_{0,k} + \sum_{j=1}^{p_k} \phi_{j,k} y_{t-j} + e_{t,k}, & \text{if } r_{k-1} < y_{t-d},
\end{cases} $$

where $y_t$ is measured output. $\phi_{p_i,i}$ is autoregressive parameter, $p_i$, for $i = 1, 2, \ldots, k$, is the autoregressive order in the $i$-th regime of the model and $k$ is the number of regimes in the model. $d$ is the length of threshold. $r_i$, for $i = 1, 2, \ldots, k$, is the threshold value, satisfying $-\infty = r_0 < r_1 < \cdots < r_k = \infty$. $e_{t,i}$, for $i = 1, 2, \ldots, k$, is the prediction error in the $i$-th regime.

As shown in Figure 4, nonlinear time series are partitioned by SETAR model. In each regime, a linear autoregressive process captures the dynamical behavior of the time series.

Remark 2: SETAR model is piece-wise linearization via the idea of Lebesgue sampling, which can be used to decompose a complex stochastic system into simpler subsystems. Switching between different linear subsystems depends only on the levels of the threshold variable. Since the threshold variable of SETAR model stems from the lagged value of the process itself, SETAR model appears self-exciting. If the number of regimes, $k = 1$, the SETAR model is equivalent to the AR model, and can be used to model the linear time series model directly.

Remark 3: SETAR model is able to model some nonlinear phenomena, e.g., jump resonance, amplitude-frequency dependency, limit cycles, subharmonics, higher harmonics and so on [30]. Since the control loop with valve stiction exhibit limit cycle behavior, SETAR model can be used to model measurements. If the complex nonlinearity composed of the control valve stiction and the protection valve action is introduced into the control loop, which can be considered that
the limit-cycle signal caused by the control valve stiction is multiplied by the subharmonic signal with varying frequency generated by the protection valve, SETAR modeling is also available.

Fig. 4. Partition for nonlinear time series with Lebesgue sampling

On SETAR modeling, there is a iterative and heuristic optimization strategy for determining $k$, $d$ and $r_i$, for $i = 1, 2, \ldots, k - 1$ and estimating $\phi_{p,i}$, for $i = 1, 2, \ldots, k$. In each regime, $b$ is a priori knowledge. The modeling procedure is summarize as following [28], [29], [30].

Step 1. Give the initial value of $k$, $d$, and $r_i$, for $i = 1, 2, \ldots, k - 1$. Determine the maximum $P$ of $p_i$, for $i = 1, 2, \ldots, k$, the maximum $D$ of $d$, and the maximum $K$ of $k$. The length of data is $N$.

Step 2. Set an initial autoregressive order $p_0 = \max\{P, D\}$.

Step 3. According to the $r_i$, for $i = 1, 2, \ldots, k - 1$, rearrange the time series data into each regime,

\[-\infty < \{y_{1,1}, y_{2,1}, \ldots, y_{N_i,1}\} \leq r_1,
\]

\[r_1 < \{y_{1,2}, y_{2,2}, \ldots, y_{N_i,2}\} \leq r_2,
\]

\[
\vdots
\]

\[r_{k-1} < \{y_{1,k}, y_{2,k}, \ldots, y_{N_i,k}\} \leq +\infty,
\]

where $y_{t,i}$, for $i = 1, 2, \ldots, k$, is the $t$-th measured output in the $i$-th regime. $N_i$, for $i = 1, 2, \ldots, k$ is the length of data in the $i$-th regime, and $\sum_{i=1}^{k} N_i = N - p_0$.

Step 4. Use the least-square method to estimate autoregressive parameter, $\phi_{p,i}$, from $1^{st}$ to $k$-th regime. In each regime, Akaike’s information criterion (AIC) is used to determine the order of AR model which takes as the following form,

\[AIC = N_i \ln \sigma^2_i + 2(p_i + 1),\]

for $i = 1, 2, \ldots, k$, where $\sigma^2_i$ is the variance of residuals from the fitted model in each regime. Thus the AIC of valid AR models in each regime is,

\[AIC(p_i) = \min_{1 \leq p_i \leq p_k} \{N_i \ln \sigma^2_i + 2(p_i + 1)\},\]

for $i = 1, 2, \ldots, k$. The model order $p_i$, for $i = 1, 2, \ldots, k$, is increased until the $AIC$ value shows no appreciable change.

Step 5. SETAR models are composed of the valid AR model in each regime as (2) described. The AIC value of this SETAR models is the sum of the $AIC$ value in each regime,

\[AIC(k; b; d; r_1, r_2, \ldots, r_{k-1}) = \sum_{i=1}^{k} AIC(p_i).\]

Step 6. Fixing $d$ and $k$, using steepest descent method, optimize $r_i$ under the following object function,

\[AIC(k; b; d; \hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{k-1}) = \min_{r_i, i=1, 2, \ldots, k-1} \{AIC(k; b; d; r_1, r_2, \ldots, r_{k-1})\}.\]

for $i = 1, 2, \ldots, k - 1$.

Repeat step 3 to step 5 to obtain the minimum $AIC$ and the corresponding optimal $\hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{k-1}$.

Step 7. Fixing $k$, repeat step 2 to step 6 to search the optimal $\hat{d}$ corresponding to the minimum $AIC$ from $d = d + 1$ to $d = D$. The object function is

\[\text{ATC}(k; b; d; \hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{k-1}) = \min_{1 \leq d \leq D} \{AIC(k; b; d; r_1, r_2, \ldots, r_{k-1})\},\]

where $\text{ATC}$ is the normalized $AIC$, and $\text{ATC} = AIC/(N - p_0)$.

Step 8. Repeat step 2 to step 7 to search the optimal $\hat{k}$ corresponding to the minimum $AIC$ from $k = k - 1$ to $k = K$. The object function is

\[\text{ATC}(k; b; d; \hat{r}_1, \hat{r}_2, \ldots, \hat{r}_{k-1}) = \min_{2 \leq k \leq K} \{\text{ATC}(k; b; d; r_1, r_2, \ldots, r_{k-1})\}.\]

Remark 4: In actual application, the length of threshold and the number of regimes can be determined by empirical value and process knowledge. Thus the optimization procedure can be simplified only to determine the threshold value, the autoregressive parameter and the model order of each regime. Furthermore, if the number of regimes and the threshold value are determined, the local regimes are independent with each other. Thus the modeling in the local regimes can be accomplished by the parallel computing strategy similar to the parallel multi-way signals processing. It can be conveniently realized in the computer control system for industrial process, and significantly improve the computation efficiency.

Hence, the $b$-step prediction error variance can be estimated in each regime after the nonlinear time series appropriately modeled by the SETAR model,

\[\hat{\sigma}^2_{ie|\text{mv}} = E\{e_{te,i}|\text{mv}\},\]

for $i = 1, 2, \ldots, k$.

However, the global SETAR model is the collection of every regime over a range of values for the sampling instant. Reasonable estimation of the $b$-step prediction error variance of the global SETAR model can be determined by using the principle of pooled variance [11], [30]. Pooled variance is a method for estimating variance given several different samples taken in different domains where the true variance is equivalently assumed to remain the same. Thus the $b$-step
prediction error variance of the global SETAR model can be calculated by

\[ \hat{\sigma}^2_{mv} = \left( \frac{\sum_{i=1}^{k} (N_i - 1) \hat{\sigma}^2_{mv}}{\sum_{i=1}^{k} N_i - k} \right) . \]

By now, the global benchmark performance of the nonlinear control system based on the idea of Lebesgue sampling is obtained.

**IV. NUMERICAL EXAMPLES**

Consider a class of complex nonlinearity described as (1). We present an example to demonstrate the effectiveness of the proposed method. The nonlinear autoregressive model based on Riemann sampling, e.g., Polynomial-AR (PAR) or Polynomial-ARX (PARX) [18], will also be applied to show the advantageous properties of the proposed method based on Lebesgue sampling.

**A. Example 1**

Consider a control system assisted with the safety system. As shown in Figure 3, \( S(u_t) \) denotes the asymmetric stiction of control valve, i.e., different amounts of stiction in the upward and downward directions of the valve [6], which is non-differentiable. The levels of valve stiction for slip parameter \( SU = 5 \) and \( SD = 3 \), where \( SU \) denotes stiction in the upward direction (opening) of valve travel and \( SD \) denotes stiction in the downward direction (closing) of valve travel. The magnitude of slip-jump parameter \( J = 4 \). \( V(s_t, z_t) \) denotes the protection valve action.

Considering the process transfer function is

\[ \hat{T} = q^{-5} \frac{1.45}{1-0.8q^{-1}}, \]

the PI controller is

\[ Q = \frac{0.3 - 0.15q^{-1}}{1-0.8q^{-1}}. \]

disturbance transfer function is \( N = 1 \), and an additive white noise with zero mean and variance 0.05. Thus the true value of the minimum variance performance benchmark, \( \sigma^2_{mv} \), is 0.05. The sampling interval is 1s.

In this example, \( V(s_t, z_t) \) denotes the process of protection valve action. \( V(s_t, z_t) \) can be described as follows,

\[ u^v_t = \begin{cases} 
    s_t, & \text{if } 0 < z_t \leq 225; \\
    s_{t-1}, & \text{if } 225 < z_t \leq 300; \\
    s_t, & \text{if } 300 < z_t \leq 500.
\end{cases} \]

When \( 0 < t \leq 225s \), the control system is in the normal operation mode; when \( 225s < t \leq 300s \), the protection valve is in action and the control system is in the safety mode, the controller output and the control valve output are held as the value at \( t = 225s \); when \( 300s < t \leq 500s \), protection valve action is off, the control system returns to the normal operation mode. \( z_t \) is the sampling instant, and the sampling interval is 1s.

In this example, \( \Psi \) includes \( S(u_t) \) and \( V(s_t, z_t) \). The time trend of \( u_t \) and \( y_t \), the \( y_t - u_t \) plot, time trend of \( u_t \) and \( u^v_t \), and the \( u^v_t - u_t \) plot are shown in Figure 5. When \( 225s < t \leq 300s \), the controller output is be held as \( u_t = -0.0022 \) at \( t = 225s \) and the position of control valve is held as \( u^v_t = -0.8491 \) at \( t = 225s \).

The estimations of the \( \sigma^2_{mv} \) using three models are shown in Table I and Table II. The comparative box plots of the quality estimations are shown in Figure 6. In order to show the partitions for close-loop routine operating data, we consider the 6-Regime \( SETAR(6; 5; 1; 16, 12, 21, 28, 30, 35) \) and the threshold values with \( (3.82, 2.06, 0.23, -1.23; -2.55) \), as shown in Figure 7.

**TABLE I**

<table>
<thead>
<tr>
<th>Linear(y)</th>
<th>Quadratic</th>
<th>Linear(u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PAR</td>
<td>16</td>
<td>136</td>
</tr>
<tr>
<td>PARX</td>
<td>15</td>
<td>120</td>
</tr>
<tr>
<td>Total</td>
<td>152</td>
<td>0.6531</td>
</tr>
<tr>
<td>PAR</td>
<td>150</td>
<td>0.3335</td>
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</tbody>
</table>

**TABLE II**

<table>
<thead>
<tr>
<th>Linear (regime1)</th>
<th>Linear (regime2)</th>
<th>Linear (regime3)</th>
<th>Linear (regime4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-Regime</td>
<td>16</td>
<td>12</td>
<td>21</td>
</tr>
<tr>
<td>Linear (regime5)</td>
<td>Linear (regime6)</td>
<td>Total</td>
<td>( \sigma^2_{mv} )</td>
</tr>
<tr>
<td>6-Regime</td>
<td>30</td>
<td>35</td>
<td>142</td>
</tr>
</tbody>
</table>

From Table I, Table II and Figure 6, we can see that \( SETAR \) model has more accurate result than that of the PAR or PARX model at the less expenses of total terms for estimating minimum variance performance. However, the results of both PAR and PARX are also over-estimated.

**V. CONCLUSIONS**

In this paper, a class of complex nonlinearity composed of the control valve stiction and the protection valve action is considered on the control loop performance assessment. In terms of this nonlinearity, a new technology for estimating...
benchmark performance based on the idea of Lebesgue sampling is proposed. The proposed method is implemented by the SETAR model, and the prediction error variance as the benchmark performance can be estimated directly from close-loop routine operating data. The result of estimation is better than that obtained by the method under Riemann sampling, and can be used to establish a better unbiased quantification index to avoid over-estimation or under-estimation. The numerical example well illustrates the effectiveness and advantages of the proposed method.

REFERENCES


