Abstract—This paper addresses the mean-module filtering problem for a stochastic polynomial system with Gaussian white noises. The obtained solution contains a sliding mode term, signum of the innovations process. It is shown that the designed sliding mode filter generates the mean-module estimate, which yields a better value of the mean-module criterion in comparison to the mean-square polynomial filter. The theoretical result is complemented with an illustrative example verifying performance of the designed filter, which is compared to the mean-square polynomial filter. The simulation results confirm an advantage in favor of the designed sliding mode filter.

I. INTRODUCTION

Since the sliding mode control was invented in the beginning of 1970s (see a historical review in [1], [2], [3]), it has been applied to solve several classes of problems. For instance, the sliding mode control methodology has been used in stabilization [4], [5], tracking [6], [7], observer design [8], [9], frequency domain analysis [10], and other control problems. Further modifications of the original sliding mode concept, such as integral sliding mode [11] and higher order sliding modes [12], [3], have been developed. The sliding mode optimal regulators have been recently designed for linear systems with non-quadratic Bolza-Meyer criteria [13], [14]. Application of the sliding mode method is extended to stochastic systems [15], [16], [17], [18] and stochastic filtering problems [19], [20], [21], [22], [23]. The last two papers present mean-square and mean-module sliding mode filters for stochastic linear systems.

This paper presents the solution to the mean-module filtering problem for a stochastic polynomial system, which contains a sliding mode term, signum of the innovations process. The mean-module sliding mode filter is obtained in a closed form, which includes the equations for a mean-module estimate and a filter gain matrix. It is shown that the designed sliding mode filter generates the mean-module estimate, which yields a better value of the mean-module criterion in comparison to the mean-square polynomial filter [24]. To the best of our knowledge, this is the first designed sliding mode filter for stochastic polynomial systems that is optimal with respect to the mean-module criterion. The theoretical result is complemented with an illustrative example verifying performance of the designed filter, which is compared to the conventional mean-square polynomial filter. The simulation results confirm an advantage in favor of the designed sliding mode filter.

The paper is organized as follows. Section 2 states the mean-module filtering problem for stochastic polynomial systems with Gaussian white noises. The sliding mode solution to the mean-module filtering problem is given in Section 3. The proof of the obtained results is given in Appendix. Section 4 contains an illustrative example.

II. MEAN-MODULE FILTERING PROBLEM STATEMENT

Let \((\Omega,F,P)\) be a complete probability space with an increasing right-continuous family of \(\sigma\)-algebras \(F_t, t \geq 0\), and let \((W_1(t), F_{t}, t \geq t_0)\) and \((W_2(t), F_{t}, t \geq t_0)\) be independent standard Wiener processes. The \(F_t\)-measurable random process \((x(t), y(t))\) is described by a nonlinear differential equation with a polynomial drift term for the system state

\[
dx(t) = f(x(t), t) dt + b(t) dW_1(t), \quad x(t_0) = x_0,
\]

and a linear differential equation for the observation process

\[
dy(t) = (A_0(t) + A(t) x(t)) dt + B(t) dW_2(t).
\]

Here, \(x(t) \in \mathbb{R}^p\) is the state vector and \(y(t) \in \mathbb{R}^m\) is the linear observation vector, \(m \leq n\). The initial condition \(x_0 \in \mathbb{R}^n\) is a Gaussian vector such that \(x_0, W_1(t) \in \mathbb{R}^p\), and \(W_2(t) \in \mathbb{R}^q\) are independent. The observation matrix \(A(t) \in \mathbb{R}^{m \times n}\) is not supposed to be invertible or square. It is assumed that \(B(t) B^T(t)\) is a positive definite matrix, therefore, \(m \leq q\). All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.

The nonlinear function \(f(x,t)\) is considered polynomial of \(n\) variables, components of the state vector \(x(t) \in \mathbb{R}^n\), with time-dependent coefficients. Since \(x(t) \in \mathbb{R}^n\) is a vector, this requires a special definition of the polynomial for \(n > 1\). In accordance with [24], a \(p\)-degree polynomial of a vector \(x(t) \in \mathbb{R}^n\) is regarded as a \(p\)-linear form of \(n\) components of \(x(t)\)

\[
f(x,t) = a_0(t) + a_1(t) x + a_2(t) x^T + \ldots + a_p(t) x \ldots \times x,
\]

where \(a_0(t)\) is a vector of dimension \(n\), \(a_1\) is a matrix of dimension \(n \times n\), \(a_2\) is a 3D tensor of dimension \(n \times n \times n\), \(a_p\) is an \((p+1)\)D tensor of dimension \(n \times \ldots \times (p+1)\) times \(n\), and \(x \times \ldots \times x\) is a \(p\)D tensor of dimension \(n \times \ldots \times n\) obtained by \(p\) times spatial multiplication of the vector \(x(t)\) by itself. Such a polynomial can also be expressed in the summation form

\[
f(x,t) = a_0 + \sum_1 a_1 k_1(t) x_1(t) + \sum_{ij} a_2 k_{ij}(t) x_i(t) x_j(t) + \ldots
\]
Here, \( x(t) \in \mathbb{R}^n \) is the state vector and \( y(t) \in \mathbb{R}^m \), \( m \leq n \), is the observation process. The initial condition \( x_0 \in \mathbb{R}^n \) is a Gaussian vector such that \( x_0, W_1(t), \) and \( W_2(t) \) are independent. It is assumed that \( B(t)B^T(t) \) is a positive definite matrix. All coefficients in (1)–(2) are deterministic functions of time of appropriate dimensions.

The state and observation equations can also be written in an alternative form
\[
\dot{x}(t) = f(x(t))dt + b(t)\psi_1(t), \quad x(t_0) = x_0, \quad (1*)
\]
\[
y(t) = A(t)x(t) + B(t)\psi_2(t), \quad (2*)
\]
where \( y(t) = Y(t) \), and \( \psi_1(t) \) and \( \psi_2(t) \) are white Gaussian noises, which are the weak mean-square derivatives of standard Wiener process \( W_1(t) \), and \( W_2(t) \) (see [25]). The representations (1), (2) and (1*), (2*) are equivalent ([26]). The equations (1*), (2*) present the conventional form for the equations (1), (2), which is actually used in practice.

The mean-square filtering problem is to find the estimate \( \hat{x}(t) \) of the system state \( x(t) \), based on the observation process \( Y(t) = \{y(s), t_0 \leq s \leq t \} \), that minimizes the mean-square norm
\[
J = E[(x(t) - \hat{x}(t))^T(x(t) - \hat{x}(t))] | F^t_Y
\]
at every time moment \( t \). Here, \( E[z(t) | F^t_Y] \) means the conditional expectation of a stochastic process \( z(t) = (x(t) - \hat{x}(t))^T(x(t) - \hat{x}(t)) \) with respect to the \( \sigma \) - algebra \( F^t_Y \) generated by the observation process \( Y(t) \) in the interval \([t_0, t]\). The solution to this filtering problem for polynomial systems is given by the mean-square polynomial filter [24] generalizing the optimal Kalman-Bucy filter [27] for linear systems.

This paper addresses the mean-module filtering problem to find the estimate \( \hat{x}(t) \) of the system state \( x(t) \), based on the observation process \( Y(t) = \{y(s), t_0 \leq s \leq t \} \), that minimizes the mean-module norm
\[
J = E[|x(t) - \hat{x}(t)|] | F^t_Y
\]
at every time moment \( t \). Here, \( |x| = [|x_1|, \ldots, |x_n|] \in \mathbb{R}^n \) is defined as the vector of absolute values of the components of the vector \( x \in \mathbb{R}^n \).

The solution to the stated filtering problem, involving the sliding mode term, is given in the next section and then proved in Appendix. As demonstrated, the obtained sliding mode filter is optimal with respect to the criterion (3).

III. SLIDING MODE MEAN-MODULE FILTER DESIGN

The solution to the mean-module filtering problem for the linear system (1) and the criterion (3) is given as follows.

The mean-module estimate satisfies the differential equation with the sliding mode term (the proof is given in Appendix)
\[
\dot{m}(t) = E(f(x, t) | F^t_Y)dt + \dot{Q}(t)A^T(t)(B(t)B^T(t))^{-1}x(t) - m(t), \quad (4)
\]
\[
A(t)\text{Sign}[A^T(t)(A(t)A^T(t))^{-1}x(t) - m(t)],
\]
with the initial condition \( m(t_0) = E|x(t_0)| | F^t_Y \).

The matrix function \( Q(t) \) satisfies the matrix equation with time-varying coefficients
\[
\dot{Q}(t) = B(t)B^T(t) + E(f(x, t)(x(t) - m(t))^T | F^t_Y), \quad (5)
\]
with the initial condition \( Q(t_0) = E[(x(t_0) - m(t_0))^T(A(t_0)A^T(t_0))(A(t_0)A^T(t_0))^{-1}A(t_0)x(t_0) - m(t_0))^T | F^t_Y] \).

Note that the equations (4) and (5) do not form a closed system of equations due to the presence of polynomial terms depending on \( x, E(f(x, t) | F^t_Y), \) and \( E(f(x, t)(x(t) - m(t))^T | F^t_Y) \), which are not expressed yet as functions of the filter variables, \( m(t) \) and \( Q(t) \) (or \( P(t) \)). However, as shown in [29], the closed system of the filtering equations can be obtained for any polynomial state (1) over linear observations (2), using the technique of representing of superior moments of the conditionally Gaussian random variable \( x(t) = m(t) \) as functions of only two its lower conditional moments, \( m(t) \) and \( P(t) \) (see [29] for more details of this technique). Apparently, the polynomial dependence of \( f(x, t) \) and \( f(x, t)(x(t) - m(t))^T \) on \( x \) is the key point making this representation possible.

Next, a closed form of the filtering equations is obtained from (4) and (5) for a third-order function \( f(x, t) \) in the equation (1), as follows. It should be noted, however, that application of the same procedure would result in designing a closed system of the filtering equations for any polynomial function \( f(x, t) \) in (1).

Let the function
\[
f(x, t) = a_0(t) + a_1(t)x + a_2(t)xx^T + a_3(t)xxx^T \quad \text{(6)}
\]
be a third-order polynomial, where \( x \) is an \( n \)-dimensional vector, \( a_0(t) \) is an \( n \)-dimensional vector, \( a_1(t) \) is a \( n \times n \)-dimensional matrix, \( a_2(t) \) is a 3D tensor of dimension \( n \	imes n \times n \), and \( a_3(t) \) is a 4D tensor of dimension \( n 	imes n 	imes n 	imes n \). In this case, the following filtering equations for the optimal estimate \( m(t) \) and the filter gain matrix \( Q(t) \) are obtained
\[
m(t) = a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_3(t)xxx^T \quad \text{(7)}
\]
\[
a_2(t)Q(t)^* | A^T(t)(A(t)A^T(t))^{-1}y(t) - m(t) | +
3a_3(t)Q(t)^* | A^T(t)(A(t)A^T(t))^{-1}y(t) - m(t) | +
3a_5(t)Q(t)^* | A^T(t)(A(t)A^T(t))^{-1}y(t) - m(t) |
\]
\[
A(t)\text{Sign}[A^T(t)(A(t)A^T(t))^{-1}y(t) - m(t)],
\]
\[
m(t_0) = E|x(t_0) | F^t_Y \),
\]
\[
\dot{Q}(t) = a_1(t)Q(t) + 2a_2(t)m(t)Q(t) + a_3(t)[Q(t)Q(t)^* | A^T(t)(A(t)A^T(t))^{-1}y(t) - m(t) | +
3m(t)^T(t)Q(t)] + b(t)b^T(t),
\]
\[
Q(t_0) = E[(x(t_0) - m(t_0))^T | F^t_Y\).
\]
Therefore, this result is formulated in the following theorem and proved in Appendix.

**Theorem 1.** The mean-module filter for the third degree polynomial system state (6) over the linear observations (2) is given by the equation (7) for the estimate \( \hat{m}(t) \) and the equation (8) for the filter gain matrix \( Q(t) \).

**IV. Example**

This section presents an illustrative example of designing the mean-module sliding mode filter for a second degree polynomial state (6) over linear observations (2), using the filtering equations (7),(8).

Consider a scalar linear unmeasured state
\[
\dot{x}(t) = 0.1x^2(t) + \psi_1(t), \quad x(0) = x_0,
\]
and the scalar linear observation process
\[
y(t) = x(t) + \psi_2(t),
\]
with the initial conditions \( \hat{m}(0) = E(x(0) | y(0)) = m_0 \),
\[
\dot{Q}(t) = 0.2m(t)Q(t) + 1,
\]
where \( \psi_1(t) \) and \( \psi_2(t) \) are white Gaussian noises, which are the weak mean-square derivatives of standard Wiener processes (see [25]). The equations (6),(7) correspond to the alternative conventional form \((1^*),(2^*)\) for the equations (1),(2).

The filtering problem is to find the mean-module estimate for the second degree polynomial state (9), using linear observations (10) confused with independent and identically distributed disturbances modeled as white Gaussian noises.

The filtering equations (7),(8) take the following particular form for the system (9),(10)
\[
\hat{m}(t) = 0.1m^2(t) + 0.1Q(t) | y(t) - m(t)| + Q(t) \text{sign}(y(t) - m(t)),
\]
with the initial condition \( \hat{m}(0) = E(x(0) | y(0)) = m_0 \),
\[
\dot{Q}(t) = 0.2m(t)Q(t) + 1,
\]
with the initial condition \( Q(0) = E((x(0) - m(0))(x(0) - m(0))^T | y(0)) \).

The estimates obtained upon solving the equations (11),(12) are also compared to the estimates satisfying the mean-square filtering equations [24] for the second degree polynomial system (9),(10)
\[
\hat{m}_p(t) = 0.1m^2_p(t) + 0.1P(t) + Q(t) | y(t) - m_p(t)|,
\]
with the initial condition \( m(0) = E(x(0) | y(0)) = m_0 \),
\[
\dot{P}(t) = 1 + 0.4m_p(t)P(t) - P^2(t),
\]
with the initial condition \( P(0) = E((x(0) - m(0))(x(0) - m(0))^T | y(0)) \).

Numerical simulation results are obtained solving the systems of filtering equations (11),(12) and (13),(14). The obtained values of the estimates \( \hat{m}(t) \) and \( \hat{m}_p(t) \) satisfying the equations (11) and (13), respectively, are compared to the real values of the state variables \( x(t) \) in (9).

For each of the two filters (11),(12) and (13),(14) and the reference system (9),(10), involved in simulation, the following initial values are assigned: \( x_0 = 1 \), \( m_0 = 4 \), \( P(0) = Q(0) = 100 \). The filtering horizon is set to \( T = 0.4 \). Gaussian disturbances \( \psi_1(t) \) and \( \psi_2(t) \) in (9),(10) are realized using the built-in MatLab white noise function.

Note that the initial conditions \( P(0) \) and \( Q(0) \) are assigned equal for simulation purposes, since the results should be compared with respect to the mean-module criterion (3). If the initial value for \( Q \) is assigned as \( Q(0) = 10 \), the mean-square polynomial filter of [24] would yield a better result as the mean-square polynomial filter.

The following graphs are obtained: graphs of the reference state \( x(t) \), satisfying the equation (9), the mean-module sliding mode filter estimate \( m(t) \), satisfying the equations (11), and the mean-square polynomial filter estimate \( m_p(t) \), satisfying the equation (13), are shown in the entire simulation interval \([0,0.4]\) in Fig. 1.

It can be observed that the mean-module sliding mode filter (11),(12) yields a certainly better value of the mean-module criterion (3) in comparison to the mean-square polynomial filter (13),(14).

The comparison of the designed mean-module sliding mode filter (11),(12) to the mean-square polynomial filter (13),(14) with respect to the criterion (3) is conducted for illustration purposes, since the filter (11),(12) should theoretically yield a better result, as follows from Theorem 1.

**V. Appendix**

**Proof of Theorem 1.** According to the general filtering theory based on the innovations process [25], the optimal estimate is a linear function of the minimized residual criterion. For instance, the mean-square polynomial estimate linearly depends on the integral of \( x(t) - E(x(t) | F_t^\lambda) \), which is the derivative of the minimized mean-square residue \((1/2)(x(t) - E(x(t) | F_t^\lambda))^T (x(t) - E(x(t) | F_t^\lambda))\), given that the right-side of the mean-square polynomial filter estimate equation linearly includes the derivative term \( x(t) - E(x(t) | F_t^\lambda) \) (see [24]). Similarly, the mean-module estimate equation linearly includes the derivative \( \text{sign}(x(t) - E(x(t) | F_t^\lambda)) \) of the minimized mean-module residue \( |x(t) - E(x(t) | F_t^\lambda)| \) in the criterion (3). Therefore, the mean-module estimate can be represented by the equation
\[
\hat{m}(t) = a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_3(t)P(t) + 3a_3(t)m(t)P(t) + a_3(t)m(t)m^T(t) + a_3(t)P(t) + \ldots
\]
\[
+ Q(t)A^T(t)(B(t)B^T(t))^{-1} \text{sign}(A^T(t)(A(t)A^T(t))^{-1}y(t) - m(t)),
\]
with the initial condition \( m(t_0) = E(x(t_0) | F_{t_0}^\lambda) \). Here, the gain matrix \( Q(t) \) should be selected to minimize the conditional variance of the estimation error produced by the estimate \( m(t) \). According to the Ito formula (see, for example, [25]), the equation for the estimation error conditional variance \( P(t) = E[(x(t) - m(t))^2(x(t) - m(t))^T | F_t^\lambda] \), produced by the estimate \( m(t) \), takes the form
\[
\dot{P}(t) = (a_1(t)P(t) + P(t)a_1^T(t) +
\]
\[2a_2(t)m(t)P(t) + 2(a_2(t)m(t)P(t))^T + 3(a_3(t)[P(t)P(t) + m(t)m^T(t)P(t)]) + \]
\[3(a_3(t)[P(t)P(t) + m(t)m^T(t)P(t)])^T + + b(t)b^T(t) - Q(t)A^T(t)(B(t)B^T(t))^{-1}A(t) \times \]
\[E(\text{Sign}(A^T(t)(A(t)A^T(t))^{-1}A(t)x(t) - m(t)))(x(t) - m(t)) \times \]
\[\times (x(t) - m(t))^T | F^T \) - E((x(t) - m(t)) \times \]
\[\times (\text{Sign}(A^T(t)(A(t)A^T(t))^{-1}A(t)x(t) - m(t))^T | F^T) \times \]
\[A^T(t)(B(t)B^T(t))^{-1}A(t)Q^T(t) + \]
\[Q(t)A^T(t)(B(t)B^T(t))^{-1}A(t)Q^T(t). \]

As follows from the preceding equation, the variable \( P(t) \) is minimized, if the gain matrix \( Q(t) \) is assigned as \( Q(t) = E((x(t) - m(t))\text{Sign}(A^T(t)(A(t)A^T(t))^{-1}A(t)x(t) - m(t))(x(t) - m(t))^T | F^T) \). In view of the definition of \( Q(t) \), the equation for \( m(t) \) is represented as (7) and, in view of the Ito formula [25], the equation for \( Q(t) \) is given by (8), with the initial condition \( Q(t_0) = E((x(t_0) - m(t_0))\text{Sign}(A^T(t_0)(A(t_0)A^T(t_0))^{-1}A(t_0)x(t_0) - m(t_0))(x(t_0) - m(t_0))^T | F^T) \). The theorem is proved. 

VI. CONCLUSIONS

This paper presents a mean-module filtering problem and designs, as a solution, a filter based on a sliding mode gain. The mean-module filtering problem is considered for a stochastic polynomial system with Gaussian white noises. It is shown that the designed sliding mode filter generates the mean-module estimate, which yields a better value of the mean-module criterion in comparison to the mean-square polynomial filter. This conclusion is theoretically proved and numerically verified in an illustrative example. The proposed approach based on involving a sliding mode innovations term is expected to be applicable to other non-mean-square filtering problems for nonlinear systems, where the conventional mean-square polynomial filter would not work.

REFERENCES


