Adaptive $H_\infty$ Formation Control for Euler-Lagrange Systems by Utilizing Neural Network Approximators

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Abstract—Design methods of adaptive $H_\infty$ formation control of multi-agent systems composed of Euler-Lagrange systems by utilizing neural network approximators are introduced to estimate nonlinear parametric elements in the agents. The proposed control schemes are derived as solutions of certain $H_\infty$ control problems, where estimation errors of tuning parameters, artificial error terms in potential functions, and approximate and algorithmic errors in neural network estimation schemes are regarded as external disturbances to the process. It is shown that the resulting control systems are robust against uncertain system parameters and that the desirable formations are achieved asymptotically via adaptation schemes.

I. INTRODUCTION

Recently, formation control problems of multi-agent systems have attracted much attention, and several formation control schemes were proposed based on various strategies (for example, leader-follower [1], behavior-based [2], virtual structure [3], and potential function approaches [4], [5], [6]). Among those, the potential function approaches seem to be useful tools from the view points of flexibility of configurations of swarms, automatic avoidance of collisions of agents, and stability of maintaining formations. In those research works, adaptive control or sliding mode control methodologies were applied in order to deal with uncertainties of agents, and stability of control systems was assured via Lyapunov function analysis. Furthermore, robustness properties of the control schemes were also discussed in those works. However, so much attention has not been paid on control performance such as optimal property or transient performance in those approaches.

On the contrary, in recent decades, stable controller designs for nonlinear and adaptive control systems have been investigated from the view point of inverse optimality [7], [8]. In those research works, the resulting control systems are shown to be optimal to certain meaningful cost functionals, and stability of the overall systems is also assured. Those approaches are extended to the design of inverse optimal $H_\infty$ adaptive control systems, and various adaptive control systems are derived from those strategies together with additional control performances such as robustness to insufficient time-varying elements of system parameters [9], [10] and nonlinear parametric models [11], [12].

The purpose of the present paper is to present design methods of adaptive formation control of multi-agent systems composed of Euler-Lagrange systems based on the notion of inverse optimality. The neural network approximators are introduced to estimate nonlinear parametric elements in the agents. The proposed control schemes are derived as solutions of certain $H_\infty$ control problems, where estimation errors of tuning parameters, artificial error terms in potential functions, and approximate and algorithmic errors in neural network estimation schemes are regarded as external disturbances to the process. It is shown that the resulting control systems are robust against uncertain system parameters and that the desirable formations are achieved asymptotically via adaptation schemes.

II. PROBLEM STATEMENT

We consider a multi-agent system composed of $N$ fully actuated mobile robots which are described as a class of Euler-Lagrange systems [4], [5] written as follows:

$$M_i(y_i)\ddot{y}_i + C_i(y_i, \dot{y}_i)\dot{y}_i + F_i(y_i, \dot{y}_i) = \tau_i, \quad (i = 1, \ldots, N),$$

where $y_i \in \mathbb{R}^n$ is an output (a generalized coordinate), $\tau_i \in \mathbb{R}^n$ is a control input (a force vector), $M_i(y_i) \in \mathbb{R}^{n \times n}$ is an inertia matrix, and $C_i(y_i, \dot{y}_i) \in \mathbb{R}^{n \times n}$ is a matrix of Coriolis and centripetal forces. $F_i(y_i, \dot{y}_i)$ is a nonlinear term whose parametric structure is not specified in advance. Each component has the following properties as a Euler-Lagrange system.


1) $M_i(y_i)$ is a bounded, positive definite, and symmetric matrix.

2) $M_i(y_i) - 2C_i(y_i, \dot{y}_i)$ is a skew symmetric matrix.

3) A part of the left-hand side of (1) can be written into

$$M_i(y_i)\dot{a}_i + C_i(y_i, \dot{y}_i)\dot{b}_i = -Y_i(y_i, \dot{y}_i, a_i, b_i)\theta_i,$$

where $Y_i(y_i, \dot{y}_i, a_i, b_i)$ is a known function of $y_i$, $\dot{y}_i$, $a_i$, $b_i$ (a regressor matrix), and $\theta_i$ is an unknown system parameter vector.

Furthermore, it is assumed that $F_i(y_i, \dot{y}_i)$ is approximated by a three-layered neural network (a nonlinear parametric model) as follows:

$$F_i(y_i, \dot{y}_i) \equiv W_i^T S(V_i^T z_i) + \mu_{i1}(z_i).$$

where

$$z_i = [\dot{z}_i^T, 1]^T \in \mathbb{R}^{2n+1}, \quad \dot{z}_i = [\dot{y}_i^T, \dot{y}_i]^T \in \mathbb{R}^{2n},$$

$$W_{ij} = [w_{ij1}, \ldots, w_{ijm}]^T \in \mathbb{R}^n, \quad (1 \leq j \leq n),$$

$$V_{ij} = [v_{ij1}, \ldots, v_{ijm}] \in \mathbb{R}^{(2n+1) \times m},$$

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\( v_{ijk} \in \mathbb{R}^{2n+1}, \quad (1 \leq j \leq n, \quad 1 \leq k \leq m), \quad (6) \)

\( S(V_j^T \bar{z}_i) = [s(v_{ij1}^T \bar{z}_i), \ldots, s(v_{ijm}^T \bar{z}_i)]^T \in \mathbb{R}^m, \quad (7) \)

\( s(v^T \bar{z}) = \frac{1}{1 + \exp(-\gamma(v^T \bar{z}))}, \quad (\gamma > 0), \quad (8) \)

\( W_i = \begin{bmatrix} W_{i1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{mn \times n}, \quad (9) \)

\( S(V_j^T \bar{z}_i) = [S(V_{j1}^T \bar{z}_i)]^T, \quad \ldots, \quad S(V_{jm}^T \bar{z}_i)]^T \in \mathbb{R}^m, \quad (10) \)

\( \mu_{i1}(\bar{z}_i) = [\mu_{i11}(\bar{z}_i), \ldots, \mu_{i1n}(\bar{z}_i)]^T \in \mathbb{R}^m, \quad (11) \)

where \( V_j \) and \( W_i \) are layer weights of the j-th neural network for the i-th agent, and \( m \) is a number of cells of each neural network. \( s(v^T \bar{z}) \) is a sigmoid function, and \( \mu_{i1}(\bar{z}_i) \) is a vector of an approximation error for \( E_i(y_i, \bar{y}_i) \).

The control objective is to construct an adaptive formation control system for a swarm of mobile robots (1) in which desirable configurations are achieved asymptotically via adaptation schemes.

### III. NEURAL NETWORK APPROXIMATOR

Based on the fact that any continuous function over a compact set can be approximated by a three-layered neural network with an arbitrary small approximate error [14], the following assumption is introduced.

**Assumption 4** There exist layer weights \( V_j \) and \( W_i \) satisfying the following relations.

\( |\mu_{i1j}(\bar{z}_i)| \leq d_{i1j} \psi_{ij}(\bar{z}_i), \quad (1 \leq j \leq n), \quad (12) \)

where \( d_{i1j} \) are unknown positive constants, and \( \psi_{ij}(\bar{z}_i) \) are known positive functions.

It should be noted that (12) does not mean that the approximate errors \( \mu_{i1j}(\bar{z}_i) \) are small over \( \bar{z}_i \in \mathbb{R}^m \), but says that the magnitudes of those are evaluated from above utilizing known functions \( \psi_{ij}(\bar{z}_i) \). In fact, we can choose \( \psi_{ij}(\bar{z}_i) \) such that \( \psi_{ij}(\bar{z}_i) \to \infty \) as \( \|\bar{z}_i\| \to \infty \).

The estimates of the layer weights \( \hat{V}_j \) and \( \hat{W}_j \) are denoted by \( \hat{V}_j \) and \( \hat{W}_j \), respectively. Then, the neural network estimation error \( \hat{W}_j^T S(V_j^T \bar{z}_i) - \hat{V}_j^T S(V_j^T \bar{z}_i) \) is evaluated in the following lemma.

**Lemma 1** [15] For the three-layered neural network, the estimation error expression is described as follows:

\[
\hat{W}_j^T S(V_j^T \bar{z}_i) - \hat{V}_j^T S(V_j^T \bar{z}_i) = \hat{W}_j^T (\hat{S}_j - \hat{S}_j^T \hat{V}_j^T \bar{z}_i) + \hat{W}_j \hat{S}_j^T \hat{V}_j^T \bar{z}_i + \mu_{i2j},
\]

\[|\mu_{i2j}| \leq \|\hat{V}_j\| \cdot \|\hat{S}_j\| + \|\hat{W}_j\| \cdot \|\hat{S}_j^T \hat{V}_j^T \bar{z}_i\| + |W_{ij}|_1, \quad (13)\]

\[
\hat{W}_j = \hat{W}_j - W_j, \quad \hat{V}_j = \hat{V}_j - V_j, \quad (14)\]

\[
\hat{S}_j = S(V_j^T \bar{z}_i), \quad (15)\]

\[
\hat{S}_j = \text{diag}(\hat{s}_{ij1}, \ldots, \hat{s}_{ijm}), \quad (16)\]

\[
\hat{s}_{ijk} = s'(\hat{v}_{ijk}^T \bar{z}_i) = \left[ \frac{ds(z)}{dz} \right]_{z=\hat{v}_{ijk}^T \bar{z}_i}, \quad (17)\]

For convenience sake, \( \hat{W}_j^T (\hat{S}_j - \hat{S}_j^T \hat{V}_j^T \bar{z}_i) \) and \( \hat{W}_j^T \hat{S}_j \hat{V}_j^T \bar{z}_i \) in (13) are rewritten into the following regression forms.

\[
\hat{W}_j^T (\hat{S}_j - \hat{S}_j^T \hat{V}_j^T \bar{z}_i) = \hat{W}_j^T \omega_{ij0}, \quad (19)\]

\[
\hat{W}_j^T \hat{S}_j \hat{V}_j^T \bar{z}_i = \sum_{k=1}^{m} \hat{w}_{ijk} \hat{s}_{ijk} \hat{V}_j^T \bar{z}_i = \sum_{k=1}^{m} \hat{v}_{ijk}^T \omega_{ijk}, \quad (20)\]

\[
\omega_{ij0} = \hat{s}_{ij} - \hat{S}_j^T \hat{V}_j^T \bar{z}_i, \quad \omega_{ijk} = (\hat{w}_{ijk} \hat{s}_{ijk}) \bar{z}_i, \quad (21)\]

\[
\hat{v}_{ijk} = \hat{v}_{ijk} - v_{ijk}, \quad (22)\]

Then the overall representation for (13) is obtained such as

\[
\hat{W}_j^T S(V_j^T \bar{z}_i) - W_j^T S(V_j^T \bar{z}_i) = \hat{W}_j^T (\hat{S}_j - \hat{S}_j^T \hat{V}_j^T \bar{z}_i) + \hat{W}_j \hat{S}_j^T \hat{V}_j^T \bar{z}_i + \mu_{i2j}, \quad (23)\]

\[
\hat{W}_j = [\hat{W}_j^T, \hat{W}_j^T, \ldots, \hat{W}_j^T]^T, \quad (24)\]

\[
\hat{S}_j = [S(V_j^T \bar{z}_i), S(V_j^T \bar{z}_i), \ldots, S(V_j^T \bar{z}_i)]^T, \quad (25)\]

\[
\hat{S}_j' = \begin{bmatrix} \hat{S}_{i1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad (26)\]

\[
\hat{V}_j = [\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_n]. \quad (27)\]

Also, the right-hand sides of (12) and (14) are summarized into the following forms, respectively.

\[
\begin{bmatrix} d_{i11} \psi_{i1} \\ \vdots \\ d_{i1n} \psi_{in} \end{bmatrix} = \begin{bmatrix} \psi_{i1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} d_{i11} \\ \vdots \\ d_{i1n} \end{bmatrix} = \Psi_{i1} D_{i1}, \quad (28)\]

\[
\begin{bmatrix} \|V_{11}\| \cdot \|\hat{z}_i \hat{W}_j^T \hat{S}_{i1}\| + \|W_{11}\| \cdot \|\hat{S}_{i1} \hat{V}_j^T \bar{z}_i\| + |W_{11}|_1 \\ \vdots \\ \|V_{n1}\| \cdot \|\hat{z}_i \hat{W}_j^T \hat{S}_{in}\| + \|W_{n1}\| \cdot \|\hat{S}_{in} \hat{V}_j^T \bar{z}_i\| + |W_{n1}|_1 \end{bmatrix} = \begin{bmatrix} \Psi_{i21} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} D_{i21} \\ \vdots \\ D_{i2n} \end{bmatrix} = \Psi_{i2} D_{i2}, \quad (29)\]

Additionally, the next description is introduced to evaluate the term \( \bar{\phi}_j^T \Omega_{ij} \).

\[
\bar{\phi}_j^T \Omega_{ij} = \begin{bmatrix} \bar{\phi}_j^T \Omega_{11} \\ \vdots \\ \bar{\phi}_j^T \Omega_{in} \end{bmatrix}, \quad (30)\]

\[
\|\bar{\phi}_j^T \Omega_{ij}\| \leq \|\bar{\phi}_j^T\| \cdot \|\Omega_{ij}\|, \quad (31)\]

\[
\|\bar{\phi}_j^T \Omega_{ij}\| \leq \|\bar{\phi}_j^T\| \cdot \|\Omega_{ij}\|, \quad (32)\]
\[
\begin{bmatrix}
\|\Omega_i\| & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \|\Omega_{in}\|
\end{bmatrix}
\begin{bmatrix}
\|\Phi_i\| \\
\vdots \\
\|\Phi_{in}\|
\end{bmatrix}
\equiv \bar{\Omega}_i D_{i3}.
\]

(38)

IV. ADAPTIVE $H_\infty$ FORMATION CONTROL

First, we consider a formation control problem [4], [6] in which all agents continue to move with a desired velocity $\dot{y}_r$ (39) and with a desired relative configuration defined by (40).

\[
\dot{y}_i(t) = \dot{y}_r(t), \quad \|y_i(t) - y_j(t)\| = d_{ij}, \quad (d_{ij} = d_{ji}, i \neq j),
\]

(40)

where $y_r$ is a reference point (a virtual leader) of the agents.

A. $H_\infty$ FORMATION CONTROL

We introduce a positive potential function $J(y) \in \mathbb{R}$ ($y = [y_1^T, \ldots, y_N^T]^T \in \mathbb{R}^{nN}$) in order to handle the desired configuration (40), where the minimal point of $J(y)$ such as

\[
J(y) \to \min, \quad \left( \frac{\partial J(y)}{\partial y_i} = 0, \quad (1 \leq i \leq N) \right),
\]

(41)

corresponds to the relative configuration (40). It is assumed that $J(y)$ is twice differentiable.

Define a control error $s_i$ by

\[
\begin{align*}
\Delta y_i &= y_i - y_r, \\
g_i(y) &= \frac{\partial J(y)}{\partial y_i},
\end{align*}
\]

(43)

Then, we obtain the next relation.

\[
\begin{align*}
\dot{s}_i &= \Delta y_i + g_i(y), \\
\dot{\Delta} y_i &= \dot{y}_i - \dot{y}_r, \\
g_i(y) &= \frac{\partial J(y)}{\partial y_i}.
\end{align*}
\]

(44)

We determine the control law as follows:

\[
\begin{align*}
\tau_i &= Y_i(y_i, \dot{y}_i, a_i, b_i)\dot{\theta}_i + \hat{W}_i^T S(\hat{\nu}_i^T z_i) - k_g g_i + v_i,
\end{align*}
\]

(47)

where $v_i$ is a stabilizing signal to be determined later based on an $H_\infty$ criterion, and $k_g$ is a positive constant. We consider the following positive function $V_0$.

\[
V_0 = \frac{1}{2} \sum_{i=1}^N s_i^T M_i s_i + (k_g + \delta) J(y),
\]

(48)

where $\delta (> 0)$ is an artificial error added to $J(y)$. We take the time derivative of $V_0$ along the trajectories of $s_i$ and $y_i$.

\[
\dot{V}_0 = \sum_{i=1}^N \left\{ s_i^T \left( Y_i(y_i, \dot{y}_i, a_i, b_i)\dot{\theta}_i + \hat{W}_i^T S(\hat{\nu}_i^T z_i) - k_g g_i + v_i \right) \\
+ (k_g + \delta) g_i^T (s_i - g_i) + (k_g + \delta) g_i^T \dot{y}_r \right\},
\]

(49)

\[
\dot{Y}_i \equiv Y_i(y_i, \dot{y}_i, a_i, b_i), \\
\dot{\theta}_i \equiv \dot{\theta}_i - \theta_i.
\]

(50)

Here, we assume that

\[
\sum_{i=1}^N g_i = 0.
\]

(52)

It should be noted that the potential function $J(y)$ satisfying (52), is easily realized by choosing $d_{ij} = d_{ji}$ ($i \neq j$) and by adjusting other parameters. Then, the following relation holds.

\[
\dot{V}_0(t) = \sum_{i=1}^N \left\{ s_i^T \left( Y_i(y_i, \dot{y}_i, a_i, b_i)\dot{\theta}_i + \hat{W}_i^T S(\hat{\nu}_i^T z_i) - k_g g_i + \delta g_i^T s_i \right) \right\}.
\]

(53)

From the evaluation of $\dot{V}_0$, we introduce the following virtual system.

\[
\dot{s}_i = f_i + g_i \dot{\theta}_i + g_i \dot{D}_{i3} + g_i \dot{\Omega}_{i3} + g_i D_{i2} + g_i \delta + g_i v_i,
\]

(54)

\[
f_i = 0, \quad g_i = \Omega_i, \quad g_i = \Psi_i, \quad g_i = 1.
\]

(55)

We are to stabilize the virtual system via a control input $v_i$ by utilizing $H_\infty$ criterion, where $\dot{\theta}_i$, $D_{i3}$, $D_{i2}$ and $\delta$ are regarded as external disturbances to the process [9], [10]. For that purpose, we introduce the following Hamilton-Jacobi-Isaacs (HJI) equation and its solution $V_{0i}$.

\[
\mathcal{L}_f V_{0i} + \frac{1}{4} \sum_{j=1}^N \left\{ \frac{\|L_{g_{i2}} V_{0i}\|^2}{\gamma_{i2}^2} - (L_{g_{i0}} V_{0i}) R_{i}^{-1} (L_{g_{i0}} V_{0i})^T \right\} + q_i = 0,
\]

(56)

\[
V_{0i} = \frac{1}{2} \|s_i\|^2,
\]

(57)

where $q_i$ and $R_i$ are a positive function and a positive definite matrix, respectively, and those are derived from HJI equation based on inverse optimality [7], [8], [9], [10] for the given solution $V_{0i}$ and the positive constants $\gamma_{i1} \sim \gamma_{i5}$. The substitution of the solution $V_{0i}$ (57) into HJI equation (56) yields

\[
\left\{ \frac{1}{4} \left( \frac{s_i^T Y_i^T Y_i^T s_i}{\gamma_{i1}^2} + \frac{s_i^T \Omega_i^T \Omega_i^T s_i}{\gamma_{i2}^2} + \frac{s_i^T \Psi_i^T \Psi_i^T s_i}{\gamma_{i3}^2} \right) + \frac{s_i^T \Psi_i^T \Psi_i^T s_i}{\gamma_{i4}^2} + \frac{s_i^T \Psi_i^T \Psi_i^T s_i}{\gamma_{i5}^2} - s_i^T R_{i}^{-1} s_i \right\} + q_i = 0.
\]

(58)

Then, $q_i$ and $R_i$ are given as follows:

\[
q_i = \frac{1}{4} s_i^T K_i s_i,
\]

(59)

\[
R_i = \left( \frac{Y_i^T Y_i}{\gamma_{i1}^2} + \frac{\Omega_i^T \Omega_i}{\gamma_{i2}^2} + \frac{\Psi_i^T \Psi_i}{\gamma_{i3}^2} \right) + \left( \frac{\Psi_i^T \Psi_i}{\gamma_{i4}^2} + \frac{g_i^T g_i}{\gamma_{i5}^2} + K_i \right)^{-1},
\]

(60)

where $K_i$ is a free parameter. By utilizing $R_i$, $v_i$ is deduced as a solution for the corresponding $H_\infty$ control problem.
\[
\dot{v}_i = -\frac{1}{2} R_i^{-1} L g_i V_0 i = -\frac{1}{2} R_i^{-1} s_i
\]
\[
= -\frac{1}{2} \left( \frac{Y_i^T \Omega_i}{\gamma_{i1}} + \frac{\Omega_i^T \gamma_i^1}{2 \gamma_{i2}} + \frac{\Psi_{i1} \Psi_i^T}{2 \gamma_{i3}} + \frac{\Psi_{i2} \Psi_i^T}{2 \gamma_{i4}} + g_i \frac{s_i}{2 \gamma_{i5}} + K_i \right) s_i. \tag{62}
\]

Then, we obtain the following theorem for the multi-agent system (1).

**Theorem 1** It is assumed that \( J(y) \) satisfies the condition (52). Then, the \( H_{\infty} \) formation control system composed of (1), (47) and (62) is uniformly bounded for arbitrary bounded design parameters \( \theta_i \) and \( \Phi_i \). Furthermore, \( v_i \) is a sub-optimal control solution which minimizes the upper bound of the following cost functional \( J_{\text{cost}} \).

\[
J_{\text{cost}} = \sup_{\dot{\theta}_i, \Phi_i, D_{1i}, D_{2i}, \delta, \in \mathbb{C}^2} \left[ \sum_{i=1}^{N} \int_{0}^{t} (q_i + v_i^T R_i v_i) d\tau + V_0(t) \right.
\]
\[
- \sum_{i=1}^{N} \left\{ \gamma_{i1}^2 \int_{0}^{t} \| \dot{\theta}_i \|^2 d\tau + \gamma_{i2}^2 \int_{0}^{t} \| \Phi_i \|^2 d\tau + \gamma_{i3}^2 \int_{0}^{t} \| D_{1i} \|^2 d\tau + \gamma_{i4}^2 \int_{0}^{t} \| D_{2i} \|^2 d\tau + \gamma_{i5}^2 \int_{0}^{t} \delta^2 d\tau \right\}.
\] \tag{63}

Additionally, the next inequality holds for any finite \( t (> 0) \).

\[
\sum_{i=1}^{N} \int_{0}^{t} (q_i + v_i^T R_i v_i) d\tau + V_0(t)
\]
\[
\leq \sum_{i=1}^{N} \left\{ \gamma_{i1}^2 \int_{0}^{t} \| \dot{\theta}_i \|^2 d\tau + \gamma_{i2}^2 \int_{0}^{t} \| \Phi_i \|^2 d\tau + \gamma_{i3}^2 \int_{0}^{t} \| D_{1i} \|^2 d\tau + \gamma_{i4}^2 \int_{0}^{t} \| D_{2i} \|^2 d\tau + \gamma_{i5}^2 \int_{0}^{t} \delta^2 d\tau \right\} + V_0(0). \tag{64}
\]

**Proof:** By considering HJ equation, we take the time derivative of \( V_0(t) \) (48) along the trajectories of the multi-agent system (1) and the \( H_{\infty} \) formation control scheme.

\[
\dot{V}_0 \leq \sum_{i=1}^{N} \left( v_i + \frac{1}{2} R_i^{-1} s_i \right)^T R_i \left( v_i + \frac{1}{2} R_i^{-1} s_i \right)
\]
\[
- \sum_{i=1}^{N} q_i^T R_i v_i - \sum_{i=1}^{N} q_i - \left( \sum_{i=1}^{N} (k_g + \delta) g_i^T g_i \right)
\]
\[
- \sum_{i=1}^{N} \gamma_{i1}^2 \left\| \dot{\theta}_i - \frac{Y_i^T \gamma_i^1}{2 \gamma_{i2}} s_i \right\|^2 + \sum_{i=1}^{N} \gamma_{i2}^2 \left\| \Phi_i \right\|^2
\]
\[
- \sum_{i=1}^{N} \frac{\gamma_{i3}^2}{2 \gamma_{i4}} \sum_{j=1}^{n} \left( d_{ij} - \frac{\psi_{ij} \gamma_{i4}^1}{2 \gamma_{i5}} s_{ij} \right)^2 + \sum_{i=1}^{N} \frac{\gamma_{i2}^2}{2 \gamma_{i3}^2} \sum_{j=1}^{n} d_{ij}^2
\]
\[
- \sum_{i=1}^{N} \frac{\gamma_{i4}^2}{2 \gamma_{i5}^2} \sum_{j=1}^{n} \left( \frac{\psi_{ij} \gamma_{i5}^1}{2 \gamma_{i6}} s_{ij} \right)^2 + \sum_{i=1}^{N} \frac{\gamma_{i2}^2}{2 \gamma_{i3}^2} \sum_{j=1}^{n} d_{ij}^2.
\]

\[
- \sum_{i=1}^{N} \gamma_{i1}^2 \sum_{j=1}^{n} \left( \frac{\psi_{ij} \gamma_{i1}^1}{2 \gamma_{i2}} s_{ij} \right)^2 + \sum_{i=1}^{N} \frac{\gamma_{i2}^2}{2 \gamma_{i3}^2} \sum_{j=1}^{n} d_{ij}^2
\]
\[
- \sum_{i=1}^{N} \frac{\gamma_{i4}^2}{2 \gamma_{i5}^2} \sum_{j=1}^{n} \left( \frac{\psi_{ij} \gamma_{i5}^1}{2 \gamma_{i6}} s_{ij} \right)^2 + \sum_{i=1}^{N} \frac{\gamma_{i2}^2}{2 \gamma_{i3}^2} \sum_{j=1}^{n} d_{ij}^2.
\] \tag{65}

where

\[
s_i(t) = \left[ s_{i1}(t), \ldots, s_{in}(t) \right]^T. \tag{66}
\]

Then, Theorem 1 is derived from the evaluations of \( \dot{V}_0(t) \) (65).

**B. Adaptive \( H_{\infty} \) Formation Control I**

Next, we determine the adaptation scheme of \( \dot{\theta}_i \) and \( \Phi_{ij} \). We assume that the upper bounds of \( \| \theta_i \| \) and \( \| \Phi_{ij} \| \) are known a priori. Then, \( \theta_i \) and \( \Phi_{ij} \) are tuned by the following adaptive laws.

\[
\dot{\hat{\theta}}_i(t) = \text{Pr}\{-\Gamma_{1i} \hat{\gamma}_i t \} s_i(t), \tag{67}
\]
\[
\dot{\Phi}_{ij}(t) = \text{Pr}\{-\Gamma_{2ij} \Omega_{ij}(t) s_j(t)\}, \tag{68}
\]
\[
(\Gamma_{1i} = \Gamma_{i1}^T > 0, \Gamma_{2ij} = \Gamma_{ij}^T > 0),
\]
\[
(i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, n),
\]

where \( \text{Pr}(\cdot) \) are projection operations in which tuning parameters are constrained to bounded regions deduced from upper bounds of those parameters [16]. Then, the tuning parameters \( \dot{\theta}_i \) and \( \Phi_{ij} \) are made uniformly bounded by the projection-type adaptive laws, and we obtain the following theorem for the multi-agent system (1).

**Theorem 2** It is assumed that \( J(y) \) satisfies the condition (52). Then the adaptive \( H_{\infty} \) formation control system composed of (1), (47), (62) and the adaptation laws (67), (68) is uniformly bounded, and the following relation holds

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \sum_{i=1}^{N} \| \Delta \dot{y}_i(t) \|^2 dt \leq \text{const.} \sum_{i=1}^{N} (\gamma_{i3}^2 + \gamma_{i4}^2), \tag{69}
\]

and the desirable relative configuration (39), (40) is achieved approximately by the accuracy proportional to \( \sum_{i=1}^{N} \sqrt{\gamma_{i3}^2 + \gamma_{i4}^2} \), that is, \( \sum_{i=1}^{N} (\| \Delta \dot{y}_i \| + \| \dot{y}_i \|) \sim \sum_{i=1}^{N} \sqrt{\gamma_{i3}^2 + \gamma_{i4}^2} \).

**Proof:** From Theorem 1 and the property of the projection-type adaptive laws, it is shown that the adaptive control system is uniformly bounded. For further stability analysis, a positive function \( V \) is defined by

\[
V = \frac{1}{2} \sum_{i=1}^{N} s_i^T M_i s_i + k_g J(y)
\]
\[
+ \frac{1}{2} \sum_{i=1}^{N} \left( \hat{\theta}_i^T \Gamma_{i1}^{-1} \hat{\theta}_i + \sum_{j=1}^{n} \Phi_{ij}^T \Gamma_{ij2}^{-1} \Phi_{ij} \right). \tag{71}
\]

We take the time derivative of \( V \) along its trajectories, and obtain
\[ V \leq -\frac{1}{2} \sum_{i=1}^{N} s_i^T K_i s_i - k_g \sum_{i=1}^{N} g_i^T g_i + \sum_{i=1}^{N} \gamma_3^2 \|D_{i1}\|^2 + \sum_{i=1}^{N} \gamma_4^2 \|D_{i2}\|^2. \]  

(72)

Then, we derive the following inequality
\[
\frac{1}{T} \sum_{i=1}^{N} \left( \frac{1}{2} \int_{0}^{T} s_i^T K_i s_i dt + k_g \int_{0}^{T} g_i^T g_i dt \right) + V(T) \leq -\frac{1}{2} \sum_{i=1}^{N} s_i^T K_i s_i - k_g \sum_{i=1}^{N} g_i^T g_i + \sum_{i=1}^{N} \gamma_3^2 \|D_{i1}\|^2 + \sum_{i=1}^{N} \gamma_4^2 \|D_{i2}\|^2 + \frac{V(0)}{T}. \]

(73)

Since \( V(0) \) and \( V(T) \) are uniformly bounded for \( \forall T > 0 \), we deduce the relations \( (69), (70) \) in Theorem 2.

**Remark 1** In the proposed adaptive control system, it is also shown that \( J(y) \) is uniformly bounded. Therefore, the collision of agents \( y_i = y_j \ (i \neq j) \) is avoided automatically, if we choose \( J(y) \) with the property such that \( J(y) \to -\infty \) as \( y_i \to y_j \ (i \neq j) \) \([4], [5], [6]\).

**V. Adaptive \( H_\infty \) Formation Control II**

Next, we generalize the previous formation control I, and consider a formation control problem of the leader-follower type \([5]\), where all agents continue to move with a desired velocity \( \dot{y}_r \)
\[ \dot{y}_r(t) = \dot{y}_r(t), \]
and also satisfy the formation constraints on the maximum distance from the reference point \( y_r \) and on the minimum relative distance from other agents written as below:
\[ \|y_i - y_r\| \leq r_i, \quad (r_i > 0, 1 \leq i \leq N), \]
\[ \|y_i - y_j\| \geq d_{ij}, \quad (d_{ij} = d_{ji} > 0, 1 \leq i \neq j \leq N). \]

(75)

(76)

Instead of (76), the relative configuration \( (40) \) can be also adopted as a specified case of the constraint on relative distances from other agents.

**A. \( H_\infty \) Formation Control II**

We introduce a positive potential function \( J_G(\Delta y) \in \mathbb{R} \)
\[ (\Delta y = [\Delta y_1^T, \cdots, \Delta y_N^T]^T) \] in order to handle the formation constraint on the maximum distance from the reference point \( y_r \) \((75)\), and introduce another positive potential function \( J_L(y) \) to handle the formation constraint on the minimum relative distance from other agents \((76)\). It is assumed that \( J_G(\Delta y) \) and \( J_L(y) \) are twice differentiable, and that the desired total configurations \((75), \ (76)\) correspond to the minimal points of \( J_G(\Delta y) \) and \( J_L(y) \) such as
\[ J_G(\Delta y) \to \min, \quad \frac{\partial J_G(\Delta y)}{\partial \Delta y_i} = 0 \ (1 \leq i \leq N), \]
\[ J_L(y) \to \min, \quad \frac{\partial J_L(y)}{\partial y_i} = 0 \ (1 \leq i \leq N). \]

(77)

(78)

Or equivalently, \( (77), (78) \) hold uniformly in the appropriate region defined by \((75), (76)\).

Define a control error \( s_i \) by \((42), (43) \) and \( g_i \) is newly defined by
\[ g_i(y) = \xi_i + \rho_i, \]
\[ \xi_i(y) = \frac{\partial J_G(\Delta y)}{\partial \Delta y_i}, \]
\[ \rho_i(y) = \frac{\partial J_L(y)}{\partial y_i}. \]

(79)

(80)

(81)

Then, we obtain the same relation as \((45), \) where \( a_i \) and \( b_i \) are defined by \((46), \) but the definition of \( g_i \) \((79)\) is different from the previous case \((44)\). Furthermore, the control law is the same form as the previous one \((47)\) with the new definition of \( g_i \) \((79)\). We consider the following positive function \( V_0 \)
\[ V_0 = \frac{1}{2} \sum_{i=1}^{N} s_i^T M_i s_i + (k_g + \delta) J_G(\Delta y) + (k_g + \delta) J_L(y), \]

(82)

where \( \delta(>0) \) is an artificial error added to \( J_G(\Delta y) \) and \( J_L(y) \). We take the time derivative of \( V_0 \) along the trajectories of \( s_i, \Delta y_i \) and \( y \).
\[ \dot{V}_0 = \sum_{i=1}^{N} \left\{ s_i^T (v_i + Y_i \dot{\theta}_i + \dot{\Phi}_i^T \Omega_i - \mu_{i1} + \mu_{i2} - k_g g_i) \right. \]
\[ + (k_g + \delta)(\dot{\theta}_i^T y_r - (\dot{\theta}_i^T y_r) + (k_g + \delta) \rho_i^T \dot{y}_r) \right\}. \]

(83)

Here, we assume that
\[ \sum_{i=1}^{N} \rho_i = 0. \]

(84)

Similarly to the previous case, the potential function \( J_L(y) \) satisfying \((84)\), is easily realized by choosing \( d_{ij} = d_{ji} \) and by adjusting other parameters. Then, the following relation holds,
\[ \dot{V}_0 = \sum_{i=1}^{N} \left\{ s_i^T (v_i + Y_i \dot{\theta}_i + \dot{\Phi}_i^T \Omega_i - \mu_{i1} + \mu_{i2} - k_g g_i) \right. \]
\[ - (k_g + \delta)(\dot{\theta}_i^T y_i - (\dot{\theta}_i^T y_i) + (k_g + \delta) \rho_i^T \dot{y}_r) \right\}. \]

(85)

From the evaluation of \( \dot{V}_0 \), we introduce the virtual system \((54), (55)\) in which \( g_i \) is defined by \((79)\), and are to stabilize the virtual system via a control input \( v_i \) by utilizing \( H_\infty \) criterion, where the same terms \( \dot{\theta}_i, D_{i1}, D_{i2}, D_{i3}, \delta \) are regarded as external disturbances to the process \([9], [10]\). Then, by repeating the same discussion with the new definition of \( g_i \) \((79)\), for \( g_i \) and \( R_i \) defined by \((59), (60), (61) \) and for \( v_i \) determined as such as \((62) \) we obtain the following theorem.

**Theorem 3** It is assumed that \( J_L(y) \) satisfies the condition \((84)\). Then, the \( H_\infty \) formation control system composed of \((1), (47), (62) \) with the new definition of \( g_i \) \((79)\), is uniformly bounded for arbitrary bounded design parameters \( \theta_i \) and \( \dot{\Phi}_i \). Furthermore, \( v_i \) is a sub-optimal control solution which minimizes the upper bound of the cost functional \( J_{\text{cost}} \) \((63)\). Additionally, the same inequality as \((64) \) holds for any finite \( t(>0) \).

**Proof:** The proof is carried out by the similar procedure to Theorem 1, where \( V_0 \) is defined by \((82), g_i \) is defined by \((79)\).
Next, we determine the adaptation scheme of $\hat{\theta}_i$ and $\hat{\Phi}_{ij}$. $\hat{\theta}_i$ and $\hat{\Phi}_{ij}$ are tuned by the same adaptive laws as (67), (68) with the new definition of $g_i$, (79). Then, we obtain the following theorem for the multi-agent system (1).

**Theorem 4** It is assumed that $J_L(y)$ satisfies the condition (84). Then the adaptive $H_\infty$ formation control system composed of (1), (47), (62) and the adaptation law (67), (68) with the new definition of $g_i$ (79) is uniformly bounded, and the same relation as (69), (70) holds, and the desired velocity tracking (74) is achieved approximately by the accuracy proportional to $\sum_{i=1}^{N} \sqrt{\gamma_{13} + \gamma_{14}} \parallel \Delta y_i \parallel \sim \sum_{i=1}^{N} \sqrt{\gamma_{23} + \gamma_{24}}$). Furthermore, by choosing appropriate formation constraints, such as an appropriate desirable region related to $J_G(\Delta y)$ and appropriate relative distances related to $J_L(y)$, the desired formation of the leader-follower type is achieved approximately by the accuracy proportional to $\sum_{i=1}^{N} \sqrt{\gamma_{13} + \gamma_{14}}$.

**VI. CONCLUDING REMARKS**

Design methodologies of adaptive $H_\infty$ formation control of multi-agent systems composed of Euler-Lagrange systems by utilizing neural network approximators have been proposed in the present paper. The resulting control strategies are derived as solutions of certain $H_\infty$ control problems, where estimation errors of tuning parameters, error terms in potential functions, and approximate and algorithmic errors in neural network estimation schemes are regarded as external disturbances to the process. It is shown that the resulting control systems are robust to uncertain system parameters and uncertain nonlinear properties, and that the desirable formations are achieved asymptotically via adaptation schemes.

**REFERENCES**


