Stability of Uncertain Piecewise Affine Systems with Time-Delay

Kaveh Moezzi, Luis Rodrigues, and Amir G. Aghdam

Abstract—This paper addresses the problem of robust stability of piecewise affine (PWA) uncertain systems with unknown time-varying delay in the state. It is assumed that the uncertainty is norm-bounded and that upper bounds on the state delay and its rate of change are available. A set of linear matrix inequalities (LMI) is derived providing sufficient conditions for the stability of the system. These conditions depend on the upper bound of the delay. The main contributions of the paper are as follows. First, new delay-dependent LMI conditions are derived for the stability of PWA time-delay systems. Second, the stability conditions are extended to the case of uncertain PWA time-delay systems. Numerical examples are presented to show the effectiveness of the approach.

I. INTRODUCTION

Continuous-time piecewise affine (PWA) systems have attracted considerable interest in the control literature in recent years [1], [2], [3], [4], [5], [6], [7], [8]. The theory of PWA systems has found important applications in CPU processing control [9], boost DC-DC converters [10] and aerospace [11], to name only a few. In brief, a PWA system consists of a set of affine subsystems (representing different operating conditions of a system, or an approximation of a complex nonlinear system) and a switching law that enables switching between different subsystems. It is to be noted that switching is also used in control to stabilize and regulate highly uncertain systems [12], [13], [14], [15].

Many practical systems, on the other hand, are subject to input and/or state delay. Examples of time-delay systems include power systems [16] and communication networks [17]. It is known that time-delay can cause poor performance or even instability if its effect is neglected in control design. The existing results for robust stability of time-delay systems can be categorized as delay independent and delay dependent results. Different delay independent robust stability criteria have been developed in [18] and [19]. Delay independent stability results are conservative in general because they do not take into account any available information on the delay. Delay dependent approaches for the systems subject to parameter uncertainty, on the other hand, are investigated in [20], [21], [22], [23], [24]. Stability analysis for switched systems with time-delay is provided in [25], [26], [27]. In [25], a common Lyapunov functional is used for robust stability analysis of switched uncertain time-delay systems with arbitrary switching. However, stability analysis using a common quadratic Lyapunov function is typically known to be conservative. In [27], sufficient conditions for exponential stability of linear time-delay systems with a class of switching signals is developed. To the best of the knowledge of the authors, however, the stability problem for PWA time-delay systems has only been addressed in [2], where a piecewise quadratic Lyapunov function is used to derive LMIs for stability analysis following the approach of [1]. Nevertheless, the important and practically relevant case of robust stability of PWA time-delay systems in presence of parametric uncertainty has not been considered in [2]. Furthermore, the affine term of the dynamics did not have a delay in that paper.

Based on the considerations of the previous paragraph, PWA uncertain systems with unknown time-delay are investigated in this paper, and LMI-based conditions for asymptotic stability are derived following the approach of [8]. It is assumed that the parameter uncertainties are norm bounded and that upper bounds on the time-varying delay and its rate of change are given. In order to reduce the conservatism of the results, piecewise quadratic Lyapunov functions are employed for stability analysis. The main contributions of this work are as follows. First, new delay dependent LMI conditions are derived for the stability of PWA time-delay systems. Second, the stability conditions are extended to the case of uncertain PWA time-delay systems.

This paper is organized as follows. The problem statement and formulation are given in Section II. The main result of the paper is provided in Section III, followed by robustness analysis in Section IV. Simulation results are presented in Section V. Finally, some concluding remarks are drawn in Section VI.

II. PROBLEM FORMULATION

Consider an uncertain piecewise affine system with time-delay described as

\[ x(t) = (A_i + \Delta A_i) x(t) + (A_{di} + \Delta A_{di}) x(t - \tau(t)) + (a_i + \Delta a_i) + (b_i + \Delta b_i) 1(t - \tau(t)), \quad x(t) \in X_i \]

(1)

where \( A_i, A_{di} \in R^{n \times n}, \quad a_i, b_i \in R^n, \) and \( \{X_i\} \subseteq R^n \) form a partition of the state space into a number of open (possibly unbounded) polyhedral cells with pairwise empty intersection. The index set of the cells is denoted by \( I = \{1, \ldots, M\} \). The set of cells that include the origin is denoted by \( I_0 \subseteq I \), and its complement is represented by \( I = I - I_0 \). It is assumed that \( a_i = 0, \Delta a_i = 0, b_i = 0, \Delta b_i = 0 \) for \( i \in I_0 \). In addition, \( \Delta A_i, \Delta A_{di}, \Delta a_i, \Delta b_i \) are norm-bounded uncertainties which will be defined later. Furthermore, \( 1(t) \) is the step function. In (1), \( \tau(t) \) is a positive time-varying delay such that

\[ 0 \leq \tau(t) \leq h, \quad \tau(t) \leq d < 1 \]

(2)
where $h$ and $d$ are known positive constants.

Assume the initial condition
\[ x(\theta) = \phi(\theta), \quad \theta \in [-h, 0] \]
for the system (1) such that $\phi(\theta)$ is a differentiable vector-valued initial function on $[-h, 0]$, $h > 0$. Assume also $x(t)$ is a continuous piecewise $C^1$ function of time. Following [1], [8], the state space is partitioned based on $x(t)$ such that $x(t) \in \bigcup X_i$ as follows. Let $E_i = [E_i, e_i]$, (with $e_i = 0$, \forall \in h_i) such that
\[ \bar{E}_i \begin{bmatrix} x(t) \\ 1 \end{bmatrix} \geq 0 \quad \forall x(t) \in X_i, i \in I \quad (3) \]

Let $\mathcal{N}_i$ denote the set of neighboring cells that share a common facet with the cell $X_i$. The facet boundary between the cells $X_i$ and $X_k$ is contained in the set $\{ x \in \mathbb{R}^n | c_{ik}^T x(t) - d_{ik} = 0 \}$, where $c_{ik} \in \mathbb{R}^n$, $d_{ik} \in \mathbb{R}$, for all $i \in I$, $k \in \mathcal{N}_i$. Moreover, we use a parametric description of the boundaries as follows
\[ X_i \cap X_k \subseteq \{ l_{ik} + F_{ik}s | s \in \mathbb{R}^{n-1} \} \quad (4) \]

for all $i \in I$, $k \in \mathcal{N}_i$, where $F_{ik} \in \mathbb{R}^{n \times (n-1)}$ is a full rank matrix whose columns span the null space of $c_{ik}^T$, and $l_{ik} \in \mathbb{R}^n$ is given by $l_{ik} = c_{ik} (c_{ik}^T c_{ik})^{-1} d_{ik}$.

The main objective of this paper is to determine a set of computationally tractable conditions under which (1) is asymptotically stable. In the next section, a Lyapunov functional will be introduced to determine the stability of PWA systems.

### III. Nominal Analysis

In this section, sufficient LMI conditions will be established for the stability of (1) without uncertainties. These conditions will then be extended to the systems with uncertainties in Section IV. To proceed further, we define the following matrices and sets
\[ \tilde{A}_i := \begin{bmatrix} A_i & A_{di} \\ 0 & 0 \end{bmatrix}, \quad \tilde{A}_{di} := \begin{bmatrix} A_{di} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{b}_i := \begin{bmatrix} b_i \\ 0 \end{bmatrix} \]
\[ \mathcal{A} = \left\{ \begin{bmatrix} A_j & 0 \\ 0 & 0 \end{bmatrix}, \forall j \in I \right\}, \quad \mathcal{A}_d = \left\{ A_{dj}, \forall j \in I \right\}, \quad \mathcal{E} = \left\{ a_j, \forall j \in I \right\} \]
\[ \mathcal{B} = \left\{ b_j, \forall j \in I \right\}, \quad \mathcal{B}_d = \left\{ A_{dj}, \forall j \in I \right\} \]

Note that system (1) without uncertainties can be rewritten as follows
\[ \dot{x} = \tilde{A}_i \tilde{x}(t) + \tilde{A}_{di} \tilde{x}(t - \tau(t)) + \tilde{b}_i 1(t - \tau(t)) \quad (5) \]
where $\tilde{x}(t) = [x^T(t), 1]^T$, with $x(t) \in X_i$. We use the expression
\[ \tilde{x}(t - \tau(t)) = \tilde{x}(t) - \int_{t-\tau(t)}^{t} \dot{x}(s) ds \quad (6) \]

Hence, considering (5), the equation (6) can be rewritten as
\[ \dot{\tilde{x}}(t) = (\tilde{A}_i + \tilde{A}_{di}) \tilde{x}(t) + \tilde{b}_i 1(t - \tau(t)) - \tilde{A}_{di} \int_{t-\tau(t)}^{t} \tilde{A}_{dj}(s) \tilde{x}(s) ds - \tilde{A}_{di} \int_{t-\tau(t)}^{t} \tilde{A}_{dj}(s) \tilde{x}(s - \tau(s)) ds - \tilde{A}_{di} \int_{t-\tau(t)}^{t} \tilde{b}_j(s) 1(s - \tau(s)) ds \quad (7) \]

Note that $j(s)$ in (7) is a piecewise constant function which represents the index of the matrices $\tilde{A}_{j}(s) \in \mathcal{A}_d$, $\tilde{b}_j(s) \in \mathcal{B}_d$. $\tilde{A}_{dj}(s) \in \mathcal{A}_d$ at time $s$. In order to proceed further, the following well-known lemma is borrowed from [28].

**Lemma 1:** For any vectors or matrices $z$ and $y$ with appropriate dimensions and any symmetric matrix $P > 0$, the following inequalities are satisfied:
\[ -z^T y - y^T z \leq z^T P z + y^T P^{-1} y \]
\[ z^T y + y^T z \leq z^T P z + y^T P^{-1} y \quad (8) \]

**Proof:** See [28]. The following theorem presents sufficient conditions for the stability of the PWA system (5).

**Theorem 1:** Consider the symmetric matrices $U_i, U_i$ and $W_i, \bar{W}_i$, which are composed of non-negative entries, and
\[ \begin{bmatrix} H_i' & h P A_{di} \\ * & -h M_{ii} \end{bmatrix} \begin{bmatrix} S_{31} + [0_{n \times n} & R_3 + S_3] \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_2 - (1-d) \bar{R} + [0_{n \times n} & 0_{n \times 1}] \\ 0_{1 \times n} & S_2 + R_2 + \bar{R} \end{bmatrix} \begin{bmatrix} \Pi \end{bmatrix} < 0 \quad (9) \]
\[ \Pi = \begin{bmatrix} 0_{n \times (n+1)} \\ S_{32} \end{bmatrix} \begin{bmatrix} 0_{(n+1) \times n} \\ S_{32}^T \end{bmatrix} \begin{bmatrix} h Q_i \begin{bmatrix} h P A_{di} A_j & h P A_{di} a_j \end{bmatrix} & \begin{bmatrix} h P A_{di} A_d \end{bmatrix} \end{bmatrix} \begin{bmatrix} \bar{S}_1 - \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ S_{32}^T \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & S_2 \end{bmatrix} \begin{bmatrix} \bar{S}_1 \\ \bar{S}_3 \end{bmatrix} \begin{bmatrix} \bar{S}_2 \\ \bar{S}_3 \end{bmatrix} \begin{bmatrix} \bar{S}_1 \end{bmatrix} = \begin{bmatrix} 0_{n \times n} \end{bmatrix} \begin{bmatrix} 0_{1 \times n} & S_2 + R_2 + \bar{R} \end{bmatrix} \begin{bmatrix} 0_{n \times (n+1)} \\ S_{32} \end{bmatrix} \begin{bmatrix} 0_{(n+1) \times n} \\ S_{32}^T \end{bmatrix} \begin{bmatrix} \bar{S}_1 - \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ S_{32}^T \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & S_2 \end{bmatrix} \begin{bmatrix} \bar{S}_1 \\ \bar{S}_3 \end{bmatrix} \begin{bmatrix} \bar{S}_2 \\ \bar{S}_3 \end{bmatrix} \begin{bmatrix} \bar{S}_1 \end{bmatrix} = \begin{bmatrix} 0_{n \times n} \end{bmatrix} \begin{bmatrix} 0_{1 \times n} & S_2 + R_2 + \bar{R} \end{bmatrix} \begin{bmatrix} 0_{n \times 1} \end{bmatrix} \begin{bmatrix} \Pi \end{bmatrix} < 0 \]
\[ P_i - E_i^T U_i E_i > 0, \quad M_{ii} > 0 \quad (10) \]
\[ H_i' = P_i (A_i + A_{di}) + (A_{di} + A_i)^T P_i + S_3 + R_1 + h Q_i + E_i^T U_i E_i \quad (12) \]

for any fixed $i \in I_0$ and for all $A_j \in \mathcal{A}$, $b_j \in \mathcal{B}$, $a_j \in \mathcal{E}$, $A_d \in \mathcal{A}_d$, such that
\[ \bar{S}_1 = \begin{bmatrix} S_1 \\ S_{31} \\ S_2 \end{bmatrix}, \quad \bar{S}_3 = \begin{bmatrix} S_{32} \\ S_3 \end{bmatrix}, \quad S_1 \in R^{n \times n}, \quad S_2 \in R \]
\[ S_3 \in R^{n \times 1}, \quad S_{31} \in R^{n \times (n+1)}, \quad S_{32} \in R^{1 \times (n+1)} \]
\[ \bar{R} = \begin{bmatrix} R_1 \\ R_{31} \\ R_2 \end{bmatrix}, \quad R_1 \in R^{n \times n}, \quad R_2 \in R, \quad R_3 \in R^{n \times 1} \]
satisfying
\[ \begin{bmatrix} \bar{S}_1 & \bar{S}_3 \end{bmatrix} > 0, \quad \bar{R} > 0 \quad (13) \]
for $S_1$, $S_2$ and $S_3$. Furthermore, let the following inequalities hold

$$
\begin{bmatrix}
  H' \quad \bar{S}_1 \\
  * \quad -M_{1i} \\
  * \quad * \quad -hM_{2i} \\
  * \quad * \quad * \quad S_2 - (1 - d)\bar{R}
\end{bmatrix} < 0
$$

(14)

$$
\begin{bmatrix}
  hQ_i - h\bar{P}_{i}\bar{A}_{di}\bar{A}_{dij} & h\bar{P}_{i}\bar{A}_{dij} - S_3 \\
  * \quad S_1 - 0 & 0 \\
  * \quad * \quad * \quad S_2
\end{bmatrix} \geq 0
$$

(15)

$$
P_i - E_i^T W_i E_i > 0, \quad M_{ki} > 0, \quad k = 1, 2
$$

(16)

for any fixed $i \in I_1$ and for all $\bar{A}_{ji} \in \mathcal{A}$, $b_j \in \mathcal{B}$, $\bar{A}_{dji} \in \mathcal{A}_d$, where

$$
\bar{H}_i := \bar{P}(\bar{A}_{i} + \bar{A}_{di}) + (\bar{A}_{di} + \bar{A}_{i})^T \bar{P} + \bar{S}_1 + \bar{R} + 
\begin{bmatrix}
  0 & 0 & 0 \\
  b_i^T M_{1i} \bar{b}_j & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix}
$$

(17)

Assume also that for all $i \in I$ and $k \in \mathcal{K}_i$,

$$
F_{ik}^T (P_i - P_k) F_{ik} = 0
$$

(18a)

$$
F_{ik}^T (P_i - P_k) I_{jk} + F_{ik}^T (q_i - q_k) = 0
$$

(18b)

$$
l_{ik}^T (P_i - P_k) I_{jk} + 2(q_i - q_k)^T l_{jk} + (r_i - r_k) = 0
$$

(18c)

where $\bar{P}_i := \begin{bmatrix} P_i & \bar{q}_i \\ \bar{q}_i^T & r_i \end{bmatrix}$, for all $i \in I$. Under conditions (2), (3) and (9)-(18), every piecewise $C^1$ trajectory $x(t)$, governed by (5) for $t \geq 0$, tends to zero asymptotically in the absence of sliding modes.

**Proof:** Define the candidate Lyapunov-Krasovsky functional

$$
\bar{V}_i = \bar{V}_{i1} + \bar{V}_{i2} + \bar{V}_{i3}
$$

(19)

where, for $x(t) \in X_i$, $i \in I$

$$
\bar{V}_{i1} = \bar{x}^T(t) \bar{P} \bar{x}(t)
$$

(20a)

$$
\bar{V}_{i2} = \int_{t - \tau(t)}^{t} \bar{x}^T(s) \bar{R} \bar{x}(s) ds
$$

(20b)

$$
\bar{V}_{i3} = h^{-1} \int_{-h}^{0} \int_{-h}^{s} \begin{bmatrix}
  \bar{x}(\theta) \\
  \bar{x}(\theta - \tau(\theta))
\end{bmatrix}^T 
\begin{bmatrix}
  S_1 & S_2 \\
  \tilde{S}_1 & \tilde{S}_2
\end{bmatrix} 
\begin{bmatrix}
  \bar{x}(\theta) \\
  \bar{x}(\theta - \tau(\theta))
\end{bmatrix} \tilde{\theta} \tilde{\theta} ds
$$

(20c)

The conditions that guarantee the continuity of the Lyapunov function at the boundaries are give in (18a-c), and can be obtained using the same approach as the one in [8]. Note that the candidate Lyapunov functional is positive definite because of (16) and (13). Applying Leibnitz integral rule and using (2), the derivative of this Lyapunov functional is
Then by adding inequality (24) to the right hand side of (23) and considering (25a-c), one can write the following for all $x(t) \in X_i, i \in I$

$$\dot{V}_i \leq x^T(t) \tilde{Z}_i \tilde{Z}_i^T(t, \tau(t))$$

$$+ h^{-1} \int_{t-\tau}^{t} \tilde{Y}_{i(s)} \tilde{Y}_{i(s)}^T(t, \tau(s)) ds$$

(26)

where $\tilde{Y}_i(t) = [\tilde{x}_i(t), \tilde{x}_i(t-\tau(t))]^T$ and $\tilde{Y}_i(t, \tau(s)) = [\tilde{x}_i^T(s), \tilde{x}_i^T(s-\tau(s))]^T$. Note that (14) and (16) imply

$$\tilde{Z}_i^T(\cdot) \tilde{Z}_i \tilde{Z}_i^T(\cdot) < 0$$

(27)

using the Schur complement, where $\tilde{U}_i = \text{diag}[\tilde{U}_i, 0], \tilde{E}_i = [\tilde{E}_i, 0]$, and $\tilde{U}_i$ has only non-negative entries. Note also that from (3), the inequality $\tilde{E}_i \tilde{x}_i(t) \geq 0$ holds for all $x(t) \in X_i$. This leads to

$$\tilde{E}_i \tilde{x}_i(t) \geq 0, \forall x(t) \in X_i, i \in I$$

and consequently it follows that

$$\tilde{Z}_i^T(\cdot) \tilde{Z}_i \tilde{Z}_i^T(\cdot) \geq 0, x(t) \in X_i, i \in I$$

(28)

Therefore, the relations (3), (16) and (14) imply $\tilde{Z}_i^T(\cdot) \tilde{Z}_i \tilde{Z}_i^T(\cdot) < 0$, for all $x(t) \in X_i, i \in I$. Furthermore, (15) implies $\tilde{Y}_{i(s)} \leq 0$ and from (26), $\dot{V}_i < 0, x(t) \in X_i, i \in I$. A similar procedure can be repeated for the case when the switching index belongs to $I_0$, leading to (9)-(11) and $\dot{V}_i < 0$, for all $x(t) \in X_i, i \in I_0$. Thus the system is asymptotically stable.

Remark 1: Theorem 1 assumes the absence of sliding modes. To avoid sliding modes at the boundaries, the following conditions can be added. Let the set $\{x \in \mathbb{R}^n : \sigma_k = c_k^T x - d_k = 0\}$ denote the sliding surface between the cells $X_i$ and $X_k$. According to [8], $\sigma_k$ must be continuous across the boundary described in (4), which yields

$$\sigma_k^T \left[ c_k^T(A_k F_k s + I_k) + A_k d_k x(t - \tau(t)) + a_k + b_k 1(t - \tau(t)) \right]$$

$$= c_k^T \left[ A_k F_k s + I_k + A_k d_k x(t - \tau(t)) + a_k + b_k 1(t - \tau(t)) \right]$$

for all $s \in \mathbb{R}^{n-1}, k \in \mathcal{M}_i, i \in I$. The above equation can be rewritten as follows

$$\sigma_k^T(A_k - A_k) F_k = 0$$

(29a)

$$\sigma_k^T(A_k d_k - A_k d_k) = 0$$

(29b)

$$\sigma_k^T \left[ (A_k - A_k) I_k + (a_k - a_k) \right] = 0$$

(29c)

$$\sigma_k^T(b_k - b_k) = 0$$

(29d)

Remark 2: Using a procedure similar to the one presented here, one can apply the results of [29] and define the following Lyapunov-Krasovskiy functional

$$\dot{V}_i' = \tilde{x}_i^T(t) \tilde{P}_i \tilde{x}_i(t) + \int_{t-\tau}^{t} \tilde{x}_i^T(s) e^{B(s-i)} \tilde{P}_i \tilde{x}_i(s) ds$$

$$+ h^{-1} \int_{t-h}^{t} \int_{t+\tau}^{t} \tilde{x}_i^T(\theta) e^{B(\theta-i)} \tilde{P}_i \tilde{x}_i(\theta - \tau(\theta)) \tilde{x}_i^T(\theta - \tau(\theta)) d\theta ds$$

to obtain the LMIs that determine the exponential stability of the system (5). It is to be noted that exponential stability is stronger than asymptotic stability, at the cost of more conservative LMIs.

IV. ROBUSTNESS ANALYSIS

Consider now the system (1) and define the matrices $\tilde{A}_i = A_i + \Delta A_i, \tilde{A}_d = A_d + \Delta A_d$, $\bar{a}_i = a_i + \Delta a_i, \bar{b}_i = b_i + \Delta b_i (i \in I)$ and

$$\bar{A}_i = A_i + \Delta A_i, \quad \bar{A}_d = A_d + \Delta A_d$$

$$\bar{b}_i = b_i + \Delta b_i$$

$$\Delta A_i = \begin{bmatrix} \Delta A_{i1} \\ 0 \\ 0 \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} \Delta A_{d1} \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta b_i = \begin{bmatrix} \Delta b_{i1} \end{bmatrix}$$

$$\Delta A_{i} = \{ \Delta A_i, \forall j \in I \}$$

$$\Delta A_{d} = \{ \Delta A_d, \forall j \in I \}$$

$$\Delta b_i = \{ \Delta b_i, \forall j \in I \}$$

Let $\| \cdot \|$ denote the 2-norm. The following bounds are assumed to be given for the norm of respective matrices

$$\| \Delta A_i \| \leq \bar{a}_i, \quad \| \Delta A_d \| \leq \bar{b}_i$$

$$\| \bar{A}_i \| \leq \bar{a}_i, \quad \| \bar{A}_d \| \leq \bar{b}_i$$

$$\max_{X \in \Delta A_i} \| X \| \leq \bar{a}^*, \quad \max_{X \in \Delta A_d} \| X \| \leq \bar{b}^*$$

$$\max_{X \in \Delta A_i} \| X \| \leq \bar{a}^*, \quad \max_{X \in \Delta A_d} \| X \| \leq \bar{b}^*$$

The following theorem presents sufficient conditions for the stability of uncertain PWA systems described by (1).

Theorem 2: Consider the symmetric matrices $\tilde{U}_i, U_i$ and $\tilde{W}_i, W_i$ composed of non-negative entries. Then, the uncertain PWA time-delay system (1) is asymptotically stable in the absence of sliding modes if (11), (16), (13) and (18a-c) hold, and there exist positive definite matrices $L_{ki}, k = 1, \ldots, 10, L_{ki}, k = 1, \ldots, 9, M_{ki}, i \in I_0$ and $M_{pe}, p = 1, 2, i \in I_1$ such that

$$L_{ki} = \begin{bmatrix} L_{k1} & \cdots & L_{k10} \\ \vdots & \ddots & \vdots \\ L_{k9} & \cdots & L_{k10} \end{bmatrix}$$

$$M_{ki} = \begin{bmatrix} M_{k1} & \cdots & M_{k10} \\ \vdots & \ddots & \vdots \\ M_{k9} & \cdots & M_{k10} \end{bmatrix}$$

$$M_{pe} = \begin{bmatrix} M_{p1} & \cdots & M_{p10} \\ \vdots & \ddots & \vdots \\ M_{p9} & \cdots & M_{p10} \end{bmatrix}$$

$$\cdots$$

$$\cdots$$

$$\cdots$$
and let the cell partition be given by
\[
E_1 = -E_3 = \begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix}
\]
\[E_2 = -E_4 = \begin{bmatrix}
-1 & 1 \\
1 & 1
\end{bmatrix}
\]
(40)

for any fixed \( i \in I_1 \) and for all \( \bar{A}_j \in \mathcal{A}, \bar{b}_j \in \mathcal{B}, \bar{A}_{dj} \in \mathcal{A}_d, \) where the following inequalities are satisfied
\[
\rho_{l_{ki}} > 0, \quad L_{ki} - \rho_{l_{ki}} I > 0, \quad k = 1, 3, 4, 6, 7, \ldots, 10 \quad \forall i \in I_0
\]
(36)
\[
\rho_{X_{ki}} > 0, \quad \rho_{X_{ki}} I - X_{ki} < 0, \quad k = 1, 2 \quad \forall i \in I_0
\]
(38)
\[
\tilde{H}_i = H_i^T + L_{7i} + L_{8i} + \begin{bmatrix} 0 & 0 \\ 0 & h_{l_{1i}} + h_{l_{2i}} \end{bmatrix}
\]
(39)

where \( H_i^T \) and \( \tilde{H}_i \) are defined in (12) and (17), respectively.

**Proof:** The proof is similar to the proof of Theorem 1, and is omitted here due to space restrictions.

**V. NUMERICAL EXAMPLES**

In this section, two examples are provided to show the effectiveness of the proposed approach.

**Example 1:** In this example, the stability of a time-delay system is investigated and it is shown that while the LMI's proposed in [2] are infeasible, the ones introduced in this paper are quite effective. Consider the piecewise linear time-delay system \( \dot{x}(t) = A x(t) + A_d x(t - \tau) \) with the system matrices given by

\[
A_1 = A_3 = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix}
0.3 & 0 \\
0 & 0.3
\end{bmatrix}
\]
\[
A_{d1} = A_{d3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{d2} = A_{d4} = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}
\]

and let the cell partition be given by

\[
E_1 = -E_3 = \begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix}, \quad E_2 = -E_4 = \begin{bmatrix}
-1 & 1 \\
1 & 1
\end{bmatrix}
\]

(40)
One can verify that using the LMIs proposed in [2], stability of the system is guaranteed only for time-delays less than 0.005, which is a very small margin. However, the LMIs derived in Theorem 1 ensure the stability for time-delays as large as \( h = 10^3 \).

**Example 2:** Consider the piecewise linear time-delay system \( \dot{x}(t) = A_1 x(t) + A_{d1} x(t-\tau), \) with the same cell partition as in (40), and the system matrices given by

\[
A_1 = A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}
\]

\[
A_{d1} = A_{d3} = \begin{bmatrix} 0.1 & 5.0 \\ -5.0 & 0.1 \end{bmatrix}, \quad A_{d2} = A_{d4} = \begin{bmatrix} 1.0 & 5.0 \\ -5.0 & -1.0 \end{bmatrix}
\]

The LMIs derived in Theorem 1 are feasible for time-delays less than or equal to \( h = 0.0264 \) in this example using simulations, the system is unstable for \( \tau_{\text{max}} = 0.031 \). This suggests that the result obtained in this example using the approach proposed for systems with no uncertainty is not too conservative.

Assume now that the matrices \( A_i \) and \( A_{d_i} \) \((i = 1, \ldots, 4)\) in the above example are subject to uncertainty. It can be verified that for \(|\Delta A_i| \leq 0.1\) and \(|\Delta A_{d_i}| \leq 0.1\) \((i = 1, \ldots, 4)\) the LMIs given in Theorem 2 are feasible for the time-delays less than or equal to \( h = 0.024 \).

**VI. Conclusions**

In this paper, robust stability of a class of piecewise affine (PWA) systems with time-varying delay is considered. It is assumed that the system is subject to bounded uncertainty. It is also assumed that the time delay is unknown and time-varying, but upper bounds on the magnitude of the delay and its rate of variation exist. Sufficient conditions in the form of conservative LMIs using different Lyapunov functionals; (ii) robust performance analysis of time-delay PWA systems; (iii) finding stability conditions for other types of uncertainties such as polytopic and polynomial, and (iv) stability analysis for neutral-type time-delay PWA systems.

**References**