Real-Time Dynamic Pricing for Multiproduct Models with Time-Dependent Customer Arrival Rates

Jr-Shin Li and Shuo Chen

Abstract—In this article, we study the revenue management problem of multiproduct dynamic pricing in the retail industry. Given a fixed initial inventory and assortment, the retailer monitors the inventory and sets the price to affect the behavior of customer choices over a selling season. We consider the Multinomial Logit (MNL) model of customer choice over substitutes and formulate the problem of optimal dynamic pricing as an optimal stochastic intensity control problem. We derive the optimal dynamic pricing policy for the MNL model with a time-dependent customer arrival rate. Furthermore, we propose a real-time dynamic pricing (RTDP) procedure that provides on-line optimal dynamic pricing policies based on the estimation of total customer arrivals over the entire selling season. This method is realized by a simple integral transformation by which a time-inhomogeneous model is transformed to time-homogeneous with a constant customer arrival rate. A dynamic programming based numerical algorithm is presented to compute the optimal solutions and to demonstrate the robustness of the RTDP procedure.

I. INTRODUCTION

In the past, revenue management is mainly concerned in industries such as airlines and hotels. Most of the research in revenue management deals with the airline industry, covering the developments in forecasting, overbooking, seat inventory control, and pricing [14]. Prices were assumed to be fixed in the early stage of the research in revenue management. Capacity management was the major concern and managers were in charge of opening and closing different pricing classes as demand evolved. Today, dynamic pricing practices in revenue management are very active, especially in the field of nonrenewable inventory with short selling time and a both preference-based and price-dependent customer demand [5]. Advances in Information Technology (IT) make managers be able to collect more information from past sales data and implement more effective tools to analyze data for dynamic pricing policies. These improved policies can then be incorporated with new technologies to increase profits [8].

In this paper, we study the revenue management problem for a retailer who confronts a long supply leadtime and short selling season with substitute products when replenishment products are unavailable. Therefore, the inventory and assortment of the products are fixed at the beginning of the selling season. As a result, the retailer controls the price of the product with time to affect the behavior of customer choices so as to maximize the revenue. In normal circumstances in a selling season, customers arrive and choose substitutable products following their preferences. While a utility of a product depends on its features, such as brand, style, quality, and price, an individual preference can be represented as a fluctuation around a “nominal” utility function. There is a broad class of customer choice models. Multiple Competitive Interactions and Multinomial Logit (MNL) model are two very commonly used [1], [4]. In the substitute product market, a popular approach is to incorporate the MNL model for a stochastic demand process with the utility defined as a fixed mean plus a Gumbel random error [3]. The complex form of the customer choice process, however, adds lots of difficulties to the theoretical analysis of the dynamic pricing problem, because the objective function with full substitution has a nonconcavity property [15]. In addition, typically customer arrival rates vary during the selling season, usually assumed to be a nonhomogeneous Poisson process. The distribution of the arrival rate could be forecasted by previous information using a piecewise polynomial function [11]. The cumulative intensity function for a nonhomogeneous Poisson process could also be estimated by a nonparametric method [2]. An Markov Decision Process (MDP) with a nonhomogeneous arrival rate has been investigated for the optimal dynamic pricing problem [16].

In this study, we model customer choice behavior among substitutable products by the MNL model [3], [9], characterized by the transition probabilities between inventory states described in Section II-A. Moreover, we consider a time-dependent customer arrival rate assumed as a nonhomogeneous Poisson process. A time-homogeneous model with a constant arrival rate has been proposed for the MNL model and the optimal dynamic pricing policy has been derived [7]. In addition, we assume that the products are equally dissimilar (the addition and removal of one product affects the choice of all other products in the same portion) [7]. The nested MNL model with an asymmetric dissimilarity is beyond the scope of this paper.

This paper is organized as follows. In the following section, we summarize the basics of the MNL model, from which we motivate an intensity control problem. We then derive the analytical optimal dynamic pricing policy for the time-inhomogeneous model by using HJB sufficient conditions. The derivations basically follow the time-homogeneous model proposed in [7]. In Section III, we present a numerical algorithm to compute the optimal solution derived in Section II. Finally, we develop the real-time dynamic pricing (RTDP) procedure that provides a systematic scheme to compute the on-line optimal dynamic pricing for a time-inhomogeneous model. We show that RTDP is independent of the estimation of total customer arrivals over the entire selling season. This method is realized by a simple integral transformation by which a time-inhomogeneous model is transformed to time-homogeneous with a constant customer arrival rate.
of the customer arrival rate.

II. CONTROL MODEL OF DYNAMIC PRICING POLICY

In this section, we summarize the basics of the MNL model and formulate a stochastic intensity control problem of revenue management. The optimal dynamic pricing policy for a time-dependent customer arrival rate model is obtained by application of HJB sufficient conditions. A simple integral transformation is proposed to remove the time-dependence of the customer arrival rate, and as a result the optimal dynamic pricing scheme of a time-inhomogeneous model can be obtained from the corresponding time-homogeneous model with a constant arrival rate.

A. A Summary of the MNL Model

We consider a multiproduct dynamic pricing problem and assume the products are nominal, i.e., they can not be ordered in a meaningful way. The MNL model is used to describe dynamic consumer choice preferences over substitute products as prices are varied. The customer makes choices from various products to maximize his utility. The utility of the $i$th product or variant, $i = 1, \ldots, n$, is defined by a logit demand function $v^i(r^i) = \exp((q^i - r^i)/\mu)$, where $v^i(r^i)$ is a positive function of the price $r^i$, $q^i$ is the quality, and $\mu$ is a constant representing the stochastic preference of the choice process [1], [7]. The customer expected demand probability of product $i$ is defined as

$$P^i(r^i) = \frac{v^i(r^i)}{v^0 + \sum_{j=1}^n v^j(r^j)}, \quad i = 1, \ldots, n,$$

where $v^0$ denotes the utility of no-purchase choice.

In the model, the selling season moves backward in time indexed by $t \in [T, 0], T > 0$ with an initial inventory $C_T$, where $t = T$ and $t = 0$ represent the beginning and the end of the selling season, respectively. We assume that customer arrivals follow a nonhomogeneous Poisson process with a time-dependent rate $\lambda(t)$ and that in a small time interval $\delta t$, the probability of one arrival is $\lambda(t)\delta t$. The price of the products at time $t$, denoted as $r_i = (r^1_i, \ldots, r^n_i) \in \mathbb{R}^n$, is set according to the current inventory level $c_t$. If any product $i$ is sold out before the end of the selling season, it is out of stock, i.e., $i \notin \mathbb{R}(c_t) = \{i | c_t^i > 0\}$, for the remaining season because of the no-replenishment assumption, and then the price is set as $r^i_t = \infty$. If all products are sold out at time $t \in (T, 0)$, then the selling season ends at $t$. Otherwise, all unsold products are salvaged at $t = 0$.

Let $r = \{r_i, t \in [T, 0]\} = (r^1, \ldots, r^n)$ denote a pricing policy for the entire selling season. The probability for a customer arriving at time $t$ to choose product $i \in \mathbb{N} = \{1, \ldots, n\}$ is denoted by $P^i_t(r_t)$, and $P^0_t(r_t)$ denotes no-purchase probability. In addition, $\sum_{i \in \mathbb{N}} P^i_t(r_t) + P^0_t(r_t) = 1$, and $P^i_t(r_t) \geq 0$ for all $i \in \mathbb{N}$ and the equality holds when the product $i$ is out of stock. Then as defined in (1) with the no-purchase utility $v_0 = \exp(u_0/\mu)$, we have the demand probability of product $i$ and of no-purchase [7]

$$P^i_t(r_t) = \frac{\exp((q^i - r^i_t)/\mu)}{\sum_{i \in \mathbb{N}} \exp((q^i - r^i_t)/\mu) + \exp(u_0/\mu)},$$

$$P^0_t(r_t) = \frac{\exp(u_0/\mu)}{\sum_{i \in \mathbb{N}} \exp((q^i - r^i_t)/\mu) + \exp(u_0/\mu)}.$$ Note that $P^i_t(r_t) + \sum_{i \in \mathbb{N}} P^0_t(r_t) = 1$. The revenue rate at time $t$ is

$$\Phi_t(r_t) = \lambda(t) \sum_{i=1}^n r^i_t P^i_t(r_t).$$

The retailer sets the price $r_t$ to control the probability of customer choice process so that the $\Phi_t(r_t)$ is optimized. Since $\Phi_t(r_t)$ is not quasi-concave in $r_t$ [10], the optimal $r_t$ can not be obtained by concave optimization. However, we have from (2) and (3) that $P^i_t(r_t)/P^0_t(r_t) = \exp((q^i - r^i_t - u_0)/\mu)$, and this yields

$$r^i_t = q^i - u_0 - \mu \ln P^i_t + \mu \ln P^0_t, \quad i = 1, \ldots, n.$$ Observe that $P^i_t : \mathbb{R}_{\geq 0}^n \rightarrow [0, 1]^n$, $k = 1, \ldots, n$, is a mapping from the price space into the sales probability space, and (5) defines the inverse, $r_t : [0, 1]^n \rightarrow \mathbb{R}^n_{\geq 0}$. The revenue rate as in (4) can now be written as

$$\Phi_t(P_t) = \lambda(t) \sum_{i=1}^n [q^i - u_0 - \mu \ln P^i_t + \mu \ln P^0_t] P^i_t.$$ It can be shown now that $\Phi_t(P_t)$ is strictly joint concave in $P = (P^0_t, P^1_t, \ldots, P^n_t)$ [7].

B. Optimal Control of the MNL Model

Let $N^i_t$ be the Poisson counting process of the number of product $i$ sold up to time $t$, i.e., if a customer arrives and chooses the product $i$ at time $t$, then $dN^i_t = 1$, otherwise, $dN^i_t = 0$. Thus, $E[dN^i_t] = \lambda(t)P^i_t$. Then, the vector $N = (N^1_t, \ldots, N^n_t)$ is a multivariate stochastic process controlled by the price vector $r_t$ via the probability $P_t$. Given an initial inventory $c_T$, the expected revenue of a price scheme $r$ for the entire selling season is

$$w_T^r(c_T) = E_r \left[ \int_0^T r^i_t dN^i_t \right],$$

where $t$ denotes the transpose. Then, the expected revenue of the corresponding probability scheme $P = \{P_t, t \in [T, 0]\}$ is

$$w_T^P(c_T) = \left[ \int_0^T \lambda(t) \sum_{i=1}^n P^i_t[q^i - u_0 - \mu \ln P^i_t + \mu \ln P^0_t] dt \right] = \int_0^T \Phi_t(P_t) dt.$$ The retailer aims to find a pricing policy $\hat{r}$ that maximizes the expected revenue over the selling season, equivalently, to compute

$$w_T^P(c_T) = \sup_{r_t \in [0, 1]^n, t \in [T, 0]} w_T^r(c_T).$$
This leads to the following optimal intensity control problem
\[ \max_{\lambda_t \in \mathbb{R}^n, \lambda_T} \int_0^T \lambda(t) \sum_{i=1}^n \lambda_i(t) P_i(t) \left( q^i - u_0 - \mu \ln P_i^0 + \mu \ln P_i^0 \right) dt, \]
subject to \[ d\lambda_i^t = -dN_i^t, \quad c(T) = c_T, \]
where \( d\lambda_i^t \) controlled by \( P_i^t \) through \( E[dN_i^t] = \lambda(t) P_i^t, \quad i = 1, \ldots, n. \)

Now let \[ w^P_i(c_i) = \int_0^T \lambda(t) \sum_{i=1}^n \lambda_i(t) P_i(t) \left( q^i - u_0 - \mu \ln P_i^0 + \mu \ln P_i^0 \right) ds, \]
and \[ \hat{w}^P_i(c_i) = \max_{\lambda_t \in \mathbb{R}^n, \lambda_T} w^P_i(c_i). \]

This optimal control problem can then be solved by applying HJB sufficient conditions [6], namely, the optimal revenue rate
\[ \frac{\partial}{\partial t} \hat{w}^P_i(c_i) = \max_{\lambda_t \in \mathbb{R}^n, \lambda_T} \sum_{i=1}^n \lambda(t) P_i^t \left( q^i - u_0 - \mu \ln P_i^0 + \mu \ln P_i^0 - \Delta' \hat{w}^P_i(c_i) \right), \]
where \( \Delta' \hat{w}^P_i(c_i) \) is defined by (8) and \( \hat{w}^P_i(c_i) \) is an n-vector with the i-th entry equals to 1, and \( \hat{w}^P_i(c_i) \) satisfies the boundary and initial conditions
\[ \hat{w}^P_i(0) = 0, \quad \forall \ t, \]
\[ \hat{w}^P_i(c_0) = 0, \quad \forall \ c_0. \]

The optimal pricing policy corresponds to the maximizer \( P_i^* \) of the HJB equation (8). Let \[ \Psi^P_i(c_i) = \sum_{i=1}^n \lambda(t) P_i^t \left[ q^i - u_0 - \mu \ln P_i^0 + \mu \ln P_i^0 - \Delta' \hat{w}^P_i(c_i) \right]. \]

Observe that \( \Psi^P_i(c_i) \) is concave in \( P_i \) since the Hessian \( H(P_i) < 0 \) for all \( P_i \). Therefore,
\[ \frac{\partial \Psi^P_i(c_i)}{\partial P_i} = 0, \]
for all \( i = 1, \ldots, n \), defines a global optimum. Simple calculation then gives
\[ \frac{\partial \Psi^P_i(c_i)}{\partial P_i} = \lambda(t) \left[ q^i - u_0 - \mu \ln P_i^0 + \mu \ln P_i^0 - \Delta' \hat{w}^P_i(c_i) - \mu \right], \]
and
\[ \frac{\partial \Psi^P_i(c_i)}{\partial P_i^0} = \lambda(t) \mu \left( \sum_{i=1}^n P_i^t \right) / P_i^0, \]
where we employed \( P_i^0 = 1 - \sum_{i=1}^n P_i^t \). Thus, the maximizer
\[ (P_i^*)^c = \frac{\mu}{m_i(c_i)} \exp \left( \frac{q^i - \Delta' \hat{w}^P_i(c_i) - u_0 - m_i(c_i)}{\mu} \right), \]
where \( i \in \mathbb{R}(c_i) \) and \( m_i \) is defined through the algebraic equation
\[ m_i(c_i) = \frac{\mu}{1 - \sum_{i=1}^n (P_i^*)^c} = \frac{\mu}{P_i^0}. \]

Because
\[ \sum_{i=1}^n (P_i^*)^c = 1 - (P_i^0)^c = 1 - \frac{\mu}{m_i(c_i)}, \]
combining this with (12) and (13) yields
\[ \left( \frac{m_i(c_i) - 1}{\mu} \right) \exp \left( \frac{m_i(c_i) + u_0}{\mu} \right) = \sum_{i=1}^n \exp \left( \frac{q^i - \Delta' \hat{w}^P_i(c_i)}{\mu} \right), \]

\( m_i(c_i) \) is the unique solution. Substituting (13) and (12) into (5), we obtain the optimal price scheme
\[ (r_i^*)^c = \Delta' \hat{w}^P_i(c_i) + m_i(c_i). \]

Plugging (14) into (8), we have
\[ \frac{\partial}{\partial t} \hat{w}^P_i(c_i) = \lambda(t) [m_i(c_i) - \mu]. \]

Applying conditions (9) and (10), we get the maximum expected revenue at the beginning of the selling season,
\[ \hat{w}^P_i(c_i) = \int_0^T \lambda(t) [m_i(c_i) - \mu] dt. \]

III. DYNAMIC PROGRAMMING FOR OPTIMAL DYNAMIC PRICING

As shown in Section II-C, there is no close form solution for \( m_i(c_i) \) and neither is for \( (r_i^*)^c \) as in (14). Here, we present numerical solutions via the study of the discrete time model. Let \( T = t_n > t_{n-1} > \ldots > t_0 = 0 \) be a discretization in the time domain. Given an inventory level \( c_k \) with the price \( r_k \) at time \( t_k \), the probability of a unit sale of product \( i \) in a small time interval \( \delta t \) is \( \lambda(t_k) \delta t \) and the probability of no sale is \( \lambda(t_k) \delta t \). After a unit sale of product \( i \), the inventory level will be changed to \( c_i - e^i \).

When there are no sales, the inventory will remain to be \( c_k \). Now, given \( c_k \) and \( n \), the expected revenue at time \( t_k \) is
\[ \hat{w}^P_i(c_k) = \max_{P_k \in [0,1]^n} \left( \sum_{i=1}^n \lambda(t_k) P_k (r_k) \delta t \left[ q^i - \Delta' \hat{w}^P_i(c_k) + m_i(c_k) \right] \right), \]
where \( \Delta' \hat{w}^P_i(c_k) \) and \( \hat{w}^P_i(c_k) \) satisfy the initial and boundary conditions defined in (9) and (10). Following the same derivation in the continuous time model, the optimal price solution is given by
\[ (r_i^*)^c = \Delta' \hat{w}^P_i(c_k) + m_i(c_k), \forall \ i \in \mathbb{R}(c_k). \]

The maximum expected revenue is
\[ \hat{w}^P_i(c_T) = \sum_{k=1}^N \lambda(t_k) [m_i(c_k) - \mu]. \]

A numerical algorithm based on backward dynamic programming is developed to compute the optimal dynamic pricing policy.
Let $C_{tk} = \{c \in \mathbb{R}_+^n \mid c \leq c_{tk}N \text{ and } \sum_{i \in \mathbb{R}(c_{tk})} c_i^j - \sum_{j \in \mathbb{R}(c)} c_j^i \leq N - k\}$ and let $m(k,c)$ be the unique solution of
\[
\frac{m(k,c) - 1}{\mu} \exp \frac{m(k,c) + u_0}{\mu} = \sum_{j \in \mathbb{R}(c_{tk})} \exp \frac{q_i^j - \Delta t \hat{w}_{tk-1}(c)}{\mu}
\]

Step 1. Initialization
1.1 $(r_i^j)^* = \infty \quad \forall i \notin \mathbb{R}(c_{tk})$.
1.2 $\forall c \in C_{tk}$, $\hat{w}_0(c) = 0$ and $\Delta t \hat{w}_0(c) = 0 \quad \forall i \in \mathbb{R}(c)$.

Step 2. Let $k = 1$.

Step 3. Iteration
While $k \leq N$, $\forall c \in C_{tk}$,
3.1 $(r_i^j)^* = \infty \quad \forall i \notin \mathbb{R}(c)$.
3.2 $\hat{w}_0(c) = \hat{w}_{tk-1}(c) + \lambda(t_k)(m(k,c) - \mu)$.
3.3 $\forall i \in \mathbb{R}(c)$,
   (1) $(r_i^j)^*(c) = \Delta t \hat{w}_{tk-1}(c) + m(k,c)$.
   (2) $\hat{w}_{tk}(c - e_i) = \hat{w}_{tk-1}(c - e_i) + \lambda(t_k)(m(k,c - e_i) - \mu)$.
   (3) $\Delta t \hat{w}_0(c) = \hat{w}_k(c) - \hat{w}_0(c - e_i)$.
3.4 $k = k + 1$ In the algorithm, $\mu$ is normalized to 1, $u_0$ is 0, and $\lambda(t_k)$ denotes the customer arrival rate at time $t_k$.

A. Examples and Simulation Results

Consider a three-variate model with an initial inventory level $c_T = (c_1^T, c_2^T, c_3^T) = (5,5,5)$ and the quality index $q = (q_1, q_2, q_3) = (11,10,9)$. We compute the optimal revenue for various types of customer arrival rate models, as shown in Figure 1, using the dynamic programming algorithm presented in Section III. The simulation results are shown in Table I. Note that in the simulations, we choose $\max \lambda(t_k) \delta t \leq 0.1$ so that the probability of more than one arrival in a unit time interval is negligible. The optimal pricing policies for each model are shown in Figure 2, where we see that the higher quality of the product, the higher the optimal price. The optimal price for each product converges at the end of the selling season. The corresponding intensity control is shown in Figure 3, where the demand probability vector $p^*$ represents the choice behavior of the customer.

### TABLE I

<table>
<thead>
<tr>
<th>Model</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenue</td>
<td>80.885</td>
<td>80.848</td>
<td>80.855</td>
</tr>
</tbody>
</table>

IV. REAL-TIME DYNAMIC PRICING

We have presented the dynamic pricing of the MNL model with time-dependent customer arrival rates. In practice, however, estimating a customer arrival distribution is in general an intractable problem. Moreover, the optimal dynamic pricing scheme derived from the numerical algorithm is sensitive to the choice of discretizations in the time domain. For example, extreme cases of arrival rates, as shown in Figure 4, may result in significant computational errors of optimal solutions. We propose a simple integral transformation that removes the inhomogeneity of customer arrival rate. In the sequel, a real-time dynamic pricing procedure can be constructed, which is based on the estimation of total customer arrivals and is independent of the shape of arrival rates.

A. Integral Transformation

Consider a time-dependent customer arrival rate $\lambda(t)$, $\lambda \in [T,0]$. Without loss of generality, we assume $\lambda(t)$ is positive and piecewise continuous. Let $\Lambda(t)$ be an integral transformation with a positive constant kernel $a$,

$$
\Lambda(t) = \int_0^t a\lambda(s)ds.
$$

It is clear that $\Lambda(t)$ is a strictly monotonic increasing function in $t$, so $t \mapsto \Lambda(t)$ is a bijection. Note that $\Lambda$ can be viewed as a new time space in which the selling season is indexed.
as \( \sigma \equiv \Lambda(t) \in [\Lambda(T),0] \), where \( t \in [T,0] \). Now consider, in the original time coordinate, \( s_1, s_2 \in [T,0] \) and \( s_1 < s_2 \). Let \( \sigma_1 = \Lambda(s_1) \) and \( \sigma_2 = \Lambda(s_2) \), which are the corresponding time points in the new coordinate. We then have \( \sigma_1 < \sigma_2 \). The expected customer arrival within an arbitrary time interval \([s_1, s_2]\) is

\[
\int_{s_1}^{s_2} \lambda(s)ds = \int_{0}^{s_2} \lambda(s)ds - \int_{0}^{s_1} \lambda(s)ds = \int_{\sigma_1}^{\sigma_2} \frac{1}{a}dv. \quad (17)
\]

The last term represents the expected customer arrival for a time-homogeneous rate \( \eta(\sigma) = \frac{1}{a} \) over \( \sigma \in [\sigma_1, \sigma_2] \). Thus, given a fixed total expected number of customer arrivals, we can associate a time-inhomogeneous model with a time-homogeneous one by employing (16). Consequently, the optimal dynamic pricing policy for the time-dependent model can be obtained by solving the corresponding time-homogeneous customer arrival rate model. The idea of the integral transformation is illustrated in Figure 5, where \( \lambda(t) \) is transformed to a constant arrival rate 0.1 and in this case \( t \in [23,0] \), \( \Lambda(t) \in [135,0] \).

B. Real Time Intensity Control

The optimal dynamic pricing scheme derived in (15) depends on the prediction of \( \lambda(t) \) for the entire selling season. Several approximation methods are applicable to give a piecewise continuous customer arrival rate function from the previous sales information [13], [12]. Our proposal is to avoid such estimations.

Consider an optimal dynamic pricing problem for a time-inhomogeneous model with a time-dependent customer arrival rate \( \lambda(t) \). To solve this problem, we first apply the integral transformation as in (16) and then obtain a constant arrival rate \( \eta \). According to (17), \( \eta = \frac{1}{a} \) and the total expected number of customer arrivals is preserved under this transformation. Therefore, given the total arrivals \( N \) of the entire selling season, it is straightforward to compute the optimal dynamic pricing scheme for the time-homogeneous model with the arrival rate \( \eta \) over \( \sigma \in [aN,0] \) by application of the numerical algorithm described in Section III. Suppose that \( r^*_\lambda(t), t \in [T,0] \), and \( r^*_\eta(\sigma), \sigma \in [aN,0] \), are corresponding optimal pricing schemes in the time-inhomogeneous and time-homogeneous models, respectively. Let

\[
n_1 = \int_{T-t_1}^{T} \lambda(s)ds,
\]

and then we have

\[
r^*_\lambda(T-t_1) = r^*_\eta(aN-an_1), \quad (18)
\]

with \( r^*_\lambda(T) = r^*_\eta(aN) \), where \( r^*_\lambda(T) \) denotes the price at the beginning of the selling season.

The idea presented above provides a simple but powerful real-time pricing scheme. Starting with \( r^*_\lambda(T) \), the retailer monitors the inventory level and customer arrivals. After a certain period of time, say after the first day of sale, the optimal pricing for the second day can be easily derived based on the current inventory level \( c_t \) and the total customer arrivals \( n_1 \) to date. The procedure can be readily implemented in the numerical algorithm. For example, we assume \( N = 100 \) and the kernel \( a = 10 \). Then, \( \eta = 0.1 \). Suppose that there are 2 customers in the first day, then the optimal price for the second day can be computed in the time-homogeneous model as \( r^*_\lambda(1000-2 \cdot 10) = r^*_\eta(980) \) and the initial price \( r^*_\lambda(T) = r^*_\eta(1000) \). At any time \( t \), the retailer can control the price by observing the customer arrival prior to \( t \) and the current inventory level.

Now we consider an example of three-variate model with the initial inventory \( c_T = (5,5,5) \) and quality index \( q = (11,10,9) \) and assume a precise estimate of total customer
arrivals \( N = 10 \) from previous data. We take \( a = 10 \) and hence the corresponding constant arrival rate \( \eta = 0.1 \) and \( T = 100 \). Suppose \( T \) represents a 4-day sale. The observations of the inventory level and the customer arrivals are shown in Table II. Starting with the price \( r^{\ast}_T(100) \) in the time-homogeneous model, the optimal price under inventory level \((3,4,5)\) at time 60 in the time-homogeneous model is the optimal price for day 2. The rest can be derived similarly. The optimal dynamic pricing policy \( r^* \) is shown in Figure 6. Note that we obtain \( r^* \) without knowing the distribution of the customer arrival rate \( \lambda(t) \).

**Table II**

<table>
<thead>
<tr>
<th>Time</th>
<th>Day1</th>
<th>Day2</th>
<th>Day3</th>
<th>Day4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Customer Arrivals</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Inventory Level</td>
<td>(3,4,5)</td>
<td>(2.3,5)</td>
<td>(1,3,4)</td>
<td>(1,3,4)</td>
</tr>
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</table>

V. CONCLUSIONS AND FUTURE WORKS

Dynamic pricing is an indispensable course in revenue management of the retail industry. We formulated the dynamic pricing problem as an optimal stochastic intensity control of MNL model. We derived an analytical optimal dynamic pricing policy for a selling season with time-dependent customer arrival rates and provided a numerical algorithm to compute optimal solutions. Moreover, we developed a real-time dynamic pricing procedure that is able to output an on-line optimal dynamic pricing scheme based only on the observations of cumulative customer arrivals to date and the current inventory level. This procedure is powerful in the sense that only the estimation of total customer arrivals over the whole selling season is required, which can be statistically easily analyzed, and that one only needs to work on the time-homogeneous model, by which the computational complexity is significantly reduced. We plan to consider, in the future, the model with time-dependent quality, \( q(t) \), of the product. For example, the quality of a perishable product will decrease as time goes by. Another example is the airline ticket problem, where customers are generally willing to pay more for the ticket at a later time meaning that the utility of the ticket increases with time. The fidelity of the real-time dynamic pricing policy obtained by the proposed method depends on the estimation of the expected total customer arrivals. The development of a better estimation of total arrivals and the ability of adapting observations to improve the estimation and model performance are of fundamental and practical importance. This is the idea of adaptive feedback control.

**References**