Reduced Order Observer Design for Nonlinear Systems with Control Applications

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Abstract—This paper presents a reduced-order observer design for nonlinear dynamic systems. The proposed reduced-order observer can be applied to a larger class of nonlinear dynamic systems than the systems with linear observer dynamics. The proposed observer design has also been used for control design of a class of nonlinear dynamic systems with the nonlinearity in the unmeasured system state variables. An example is included to demonstrate that the proposed observer and control design method can be used to stabilize a nonlinear dynamic system with unstable nonlinear zero dynamics.

I. INTRODUCTION

Nonlinear observer design has attracted attention of control researchers from 1970s [1], [2]. One significant breakthrough was the nonlinear observers with linear observer dynamics [3]. Many other results have been presented in literature (for example, see [4], [5], [6] etc). More recently, results have been reported on reduced-order design for nonlinear dynamic systems [7], [8]. In this paper, we present a basic design and analysis of reduced-order observers for general nonlinear dynamic systems, and then extend this basic design for some specific nonlinear systems, including nonlinear dynamic systems with the nonlinearity of unmeasured state variables.

One purpose of nonlinear observer design is to provide state estimate for control design using output feedback. Related results on output feedback control can be found in a number of references [9], [10], [11], [12], [13], [14], [15], [16], [17]. However, there are very few results on output feedback control of nonlinear systems with the nonlinearity of the unmeasured system state variables or nonlinear unstable zero dynamics. In this paper, the proposed reduced order observer design will be applied to tackle some of these output feedback control problems. An example of output feedback control of a nonlinear system with nonlinear unstable zero dynamics is included to demonstrate the proposed observer design for output feedback control.

II. A REDUCED-ORDER NONLINEAR OBSERVER DESIGN

Consider a nonlinear system
\[
\begin{align*}
\dot{x} &= f(x) \\
y &= h(x)
\end{align*}
\]
where \(x \in \mathbb{R}^n\) is the state vector, \(y \in \mathbb{R}^m\) is the output, \(f: \mathbb{R}^n \rightarrow \mathbb{R}^n\), is a nonlinear smooth vector field.

We have the following assumption of the system.

Assumption 1. There exists a nonlinear map \(\beta: \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that
\[
T = \begin{bmatrix} h(x) \\ \beta(x) \end{bmatrix}
\]
is a diffeomorphism. Furthermore, with the state transformation defined by \(y = h(x), z = \beta(x)\), the dynamics of \(z\) can be written in the form
\[
\dot{z} = p(z) + q(y),
\]
with \(p(0) = 0\). Furthermore, for the dynamics \(\dot{z} = p(z)\), there exists a \(V: \mathbb{R}^{n-m} \rightarrow \mathbb{R}\) such that for \(z, \tilde{z} \in \mathbb{R}^{n-m}\)
\[
\frac{\partial V(z-\tilde{z})}{\partial z} (p(z) - p(\tilde{z})) \leq -c_3 \|z - \tilde{z}\|^2
\]
\[
\|\frac{\partial V(z)}{\partial z}\| \leq c_4 \|z\|^2
\]
for some positive real constants \(c_i\) for \(i = 1\) to \(4\).

With the conditions specified in Assumption 1 being satisfied, we can propose a reduced order observer design as
\[
\begin{align*}
\dot{\hat{z}} &= p(\hat{z}) + q(y) \\
\hat{x} &= T^{-1} \begin{bmatrix} y \\ \hat{z} \end{bmatrix}
\end{align*}
\]

Theorem 1: The state of the original system \(z\) and of the observer \(\hat{z}\) are bounded if the system output is bounded. The error of the reduced order observer \(\hat{z}\) defined by \(e = z - \hat{z}\) exponentially converges to zero and the estimate given in \((4)\) asymptotically converges to the state variable of \((1)\).

Proof. First, let us establish the boundedness of \(z\). We take \(V(z)\) as a Lyapunov function candidate with \(\dot{V} = 0\). From \((2)\) and \((3)\), we have
\[
\dot{V} \leq -c_3 \|z\|^2 + \frac{\partial V}{\partial z} q(y)
\]
\[
\leq -c_3 \|z\|^2 + c_4 \|z\| \|q(y)\|
\]
\[
\leq -c_3 \|z\|^2 + c_4 c_5 \|z\|^2
\]
\[
+ \frac{4c_4}{c_5} \|q(y)\|^2
\]
with \(c_5\) being a positive real constant. Set \(c_5 = \frac{c_3}{2c_4}\) and we have
\[
\dot{V} \leq -\frac{c_3}{2c_2} \|z\|^2 + \frac{8c_4^2}{c_3} \|q(y)\|^2
\]
\[
\leq -\frac{c_3}{2c_2} V + \frac{8c_4^2}{c_3} \|q(y)\|^2
\]
Therefore, from the comparison lemma [18], we can conclude that \( V \) is bounded if the output \( y \) of the original system is bounded, which implies the boundedness of \( \hat{z} \). The boundedness of \( \hat{z} \) can be established in the same way.

The dynamics of \( e \) is given by

\[
\dot{e} = p(z(t)) - p(z(t) - e)
\]

Taking \( V(e) \) as the Lyapunov function candidate, we have

\[
\dot{V} = \frac{\partial V}{\partial e}(p(z) - p(z - e)) \\
\leq -c_3\|e\|^2 \leq -\frac{c_3}{c_2} V(e)
\]

Therefore we can conclude that the estimation error exponentially converge to zero. With \( \hat{z} \) as an exponentially convergent estimate of \( z \), we can further conclude that \( \hat{z} \) is an asymptotic estimate of \( x \). This concludes the proof of Theorem 1.

To demonstrate the proposed reduced order observer design, let us consider an example.

**Example 1.** Consider a second order system

\[
\begin{align*}
\dot{x}_1 &= x_1^2 - 3x_1^2x_2 - x_1^3 \\
\dot{x}_2 &= x_2 - 6x_2x_1^2 + 3x_2^2x_1 - x_2^3 \\
y &= x_1.
\end{align*}
\]

Let us check if Assumption 1 is satisfied. For this, we need to find \( \beta(x) \). Take

\[
z = \beta(x) = x_2 - x_1.
\]

We have

\[
\dot{z} = -x_2 - (x_2 - x_1)^3 \\
= -(1 + z^2)z + y
\]

Let \( V = \frac{1}{2}z^2 \). It is easy to see the first and the third conditions in (3) are satisfied. For the second condition, we have

\[
\frac{\partial V(z - \hat{z})}{\partial z}(p(z) - p(\hat{z})) \\
= -(z - \hat{z})(z - \hat{z} + z^3 - \hat{z}^3) \\
= -(z - \hat{z})^2(1 + z^2 - z\hat{z} + \hat{z}^2) \\
= -(z - \hat{z})^2(1 + \frac{1}{2}(z^2 + \hat{z}^2 + (z - \hat{z})^2)) \\
\leq -(z - \hat{z})^2
\]

Therefore, the system satisfy the conditions specified (3). We design the reduced order observer as

\[
\begin{align*}
\hat{z} &= -(1 + \hat{z}^2)\hat{z} + y \\
\hat{x}_2 &= \hat{z} + y
\end{align*}
\]

Simulation study has been carried out, and the simulation results are shown in Figures 1 and 2.

**III. Reduced-Order Observer Design for Partially Linear Systems**

Consider a single-input-single-output (SISO) nonlinear system

\[
\begin{align*}
\dot{x} &= Ax + \phi(y, u) + f\varphi(d^T x), \\
y &= c^T x
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state vector, \( y \in \mathbb{R} \) is the output, \( u \in \mathbb{R} \) is the control, \( \phi \) is a known nonlinear smooth vector field, \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is a smooth nonlinear function, \( c, d, f \in \mathbb{R}^n \) are constant vectors, and \( A \in \mathbb{R}^{n \times n} \) is a constant matrix.

When \( \varphi = 0 \), the system (8) degenerates to the well-known form of the nonlinear systems with the linear observer error dynamics, and nonlinear observer can be easily designed by using nonlinear output injection. With this additional term \( \varphi \), the nonlinear output injection term can no longer be used to generate a linear observer error dynamics. In this section, we will convert the system to the form considered in the previous section, and they apply the reduced-order observer
design for state observation. We specify certain conditions in the next assumption for the class of systems (1).

**Assumption 2.**

2.1 \( \{c^T, A\} \) is observable.

2.2 A SISO linear system characterized by \( \{A, f, c^T\} \) has relative degree 1, and is minimum phase.

We will show that with Assumption 2, we can find a \( g(x) \) such that Assumption 1 is satisfied.

From Assumption 2.1, there exists a nonsingular state transformation

\[
\begin{align*}
  w &= AOx := T_2x \\
  A &= \begin{bmatrix}
    1 & 0 & \cdots & 0 \\
    a_1 & 1 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    a_{n-1} & a_{n-2} & \cdots & 1
  \end{bmatrix}, \\
  O &= \begin{bmatrix}
    c^T \\
    c^T A \\
    \vdots \\
    c^T A^{n-1}
  \end{bmatrix},
\end{align*}
\]

with \( a_i \), for \( i = 1, \ldots, n \), being the coefficients of the characteristic polynomial of \( A \). It can be obtained that

\[
\begin{align*}
  \dot{w} &= A_ow + \bar{\phi}(w_1, u) + \bar{f}\varphi (dT_2^{-1}w) \\
  y &= w_1
\end{align*}
\]  

(9)

where \( A_o \) is the left companion matrix of the characteristic polynomial of \( A \), and \( \bar{\phi} = T_2\phi, \bar{f} = T_2f \) and \( \bar{b} = T_2b \). Note that Assumption 2.2 ensures \( \bar{f}_1 \neq 0 \).

Let us introduce another state transformation \( \eta = T_3w \) where

\[
T_3 = \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  -\bar{f}_n & 1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots \\
  -\bar{f}_1 & 0 & \cdots & 1
\end{bmatrix}.
\]

Based on this state transformation, we let \( z = \eta_{2:n} \), where the subscript \( (2 : n) \) denotes the vector form by the second to the \( n \)th elements of the original vector.

A direct evaluation gives

\[
\dot{z} = Fz + \phi_w(y, u)
\]  

(10)

where \( F \in \mathbb{R}^{(n-1) \times (n-1)} \) is the left companion matrix of the characteristic polynomial \( s^{n-1} + \bar{f}_2 s^{n-2} + \ldots + \bar{f}_n = 0 \),

\[
\bar{f}_w = \begin{bmatrix}
  \bar{f}_2 \\
  \bar{f}_1 \\
  \vdots \\
  \bar{f}_1
\end{bmatrix}^T,
\]

and

\[
\phi_w(y, u) = \bar{\phi}_{2:n}(y, u) - \bar{\phi}_1(y, u) f_w - (a_{2:n} - a_1 f_w)y
\]

with \( a = [a_1, a_2, \ldots, a_n]^T \). Comparing (10) with (2), we have \( p(z) = Fz \) and \( q(y) = \phi_w(y, u) \). Assumption 2.2 ensures that \( F \) is Hurwitz, and therefore there exist two positive definite matrices \( P \) and \( Q \) such that

\[
F^T P + PF = -Q.
\]

It can be easily shown that \( V = z^TPz \) satisfy the conditions shown in (3). Hence, we design the reduced order observer as

\[
\begin{align*}
  \dot{\hat{z}} &= F\hat{z} + \phi_w(y, u) \\
  \hat{x} &= T_2^{-1}T_3^{-1} \begin{bmatrix}
    y \\
    \hat{z}
  \end{bmatrix}.
\end{align*}
\]  

(12)

Since Assumption 1 is satisfied for (8) with \( g(x) = \eta_{2:n} \), we can conclude from Theorem 1 that the observer error \( \epsilon \) converges to zero exponentially. In fact it is easy to see \( \dot{\epsilon} = Fe \) for this case. Since \( T = T_1T_2 \) is a linear transformation, the state estimate \( \hat{x} \) converges to \( x \) exponentially.

The above result is summarized in the following lemma.

**Lemma 2:** For the system (8), the conditions specified in Assumption 2 ensure that Assumption 1 is satisfied. Furthermore, the reduced-order observer (11) ensures that the observer error exponentially converges to zero, and the estimate \( \hat{x} \) in (12) exponentially converge to the state variable \( x \).

**IV. APPLICATION TO CONTROL DESIGN OUTPUT FEEDBACK SYSTEMS**

For the control design, we need to further assume the structure information of the system. In particular, we replace \( \phi(y, u) \) in (8) by \( \phi(y) + b\sigma(y)u \), i.e., the system is now described by

\[
\begin{align*}
  \dot{x} &= Ax + \phi(y) + f\varphi (dT_x) + b\sigma(y)u, \\
  y &= c^T x
\end{align*}
\]  

(13)

where \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous nonlinear function and \( \sigma(y) \neq 0, \forall y \in \mathbb{R} \), \( b \in \mathbb{R}^n \) is constant vector. For the control design, we introduce two more conditions in the following assumptions.

**Assumption 3.**

3.1 \( \{d^T, A\} \) is observable.

3.2 A SISO linear system characterized by \( \{A, b, d^T\} \) has relative degree \( n \).

Since \( \{d^T, A\} \) is observable, we define a state transformation

\[
\xi = T_1x
\]

where \( T_1 = [d, A^Td, \ldots, (A^{n-1})^Td]^T \), and it is nonsingular. Under the coordinate \( \xi \), we have \( y = c^T T_1^{-1}\xi := T_1^T \xi \), and we define \( \psi(y) := T_1^T \phi(y) \).

**Assumption 4.** The nonlinear function \( \psi \) satisfies the following condition

\[
\frac{\partial \psi_i(T^T \xi)}{\partial \xi_j} = 0
\]

for \( i = 1, \ldots, n - 1 \) and \( j > i \).

**Remark 1:** The condition specified in Assumption 4 is similar to the triangular condition imposed on the strict feedback form [12], and it is required for control design.

For the convenience of the control design, we transform the system (13) to the state variable \( \xi \). It can be obtained as

\[
\begin{align*}
  \dot{\xi} &= A_c \xi + \psi(y) + g\varphi(\xi_1) + h\sigma(y)u, \\
  y &= T^T \xi
\end{align*}
\]  

(14)
where $A_c$ is the lower companion matrix of the characteristic polynomial of $A$, $h = T_1 b$ and in this section, we denote $g = T_1 f$. Based on Assumption 3.2, we have $h_i = 0$ for $i = 1, \ldots, n - 1$ and $h_n \neq 0$. Hence from the structure of $A_c$ and $h$, we can write the dynamics for the individual states as

$$
\begin{align*}
\dot{\xi}_1 &= \xi_2 + \psi_1(y) + g_1 \varphi(\xi_1) \\
\vdots \\
\dot{\xi}_{n-1} &= \xi_n + \psi_{n-1}(y) + g_{n-1} \varphi(\xi_1) \\
\dot{\xi}_n &= h_n \sigma(y)u + \psi_n(y) + g_n \varphi(\xi_1) - \sum_{i=1}^{n} a_i \xi_{n-i+1}
\end{align*}
$$

(15)

Based on the state estimate for $z$, we have the estimate for $\dot{\xi}$ given by

$$
\dot{\xi} = T_1 T_2^{-1} T_3^{-1} \begin{bmatrix} y \\ \hat{z} \end{bmatrix}
$$

Let $\hat{\xi} = \dot{\xi} - \dot{\xi}$ and it can be obtained that

$$
\hat{\xi} = T_1 T_2^{-1} T_3^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = T_1 T_2^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

and

$$
\hat{\xi} = T_1 T_2^{-1} \begin{bmatrix} 0 \\ 0 \\ F \end{bmatrix} T_2 T_1^{-1} \dot{\xi} := G \hat{\xi}
$$

Since $\xi$ is not available, control design will be carried out with $\hat{\xi}$. For the control design, we introduce the following notations:

$$
\zeta_i := \hat{\xi}_i, \quad \xi_i := \hat{\xi}_i - \alpha_{i-1}(\hat{\xi}_1, \ldots, \hat{\xi}_{i-1}, y) \quad \text{for} \quad i = 2, \ldots, n
$$

(16)

where $\alpha_i$ are the stabilizing functions to be designed. In order to express the control designed for Step $n$ in the same form as the other steps, we define $\xi_{n+1} := h_n \sigma(y)u - \sum_{i=1}^{n} a_i \xi_{n-i+1}$ and $\hat{\xi}_{n+1} := h_n \sigma(y)u - \sum_{i=1}^{n} a_i \hat{\xi}_{n-i+1}$. Thus, we can denote $\xi_{n+1} = \xi_{n+1} - \hat{\xi}_{n+1} = -\sum_{i=1}^{n} a_i \xi_{n-i+1}$. For the convenience of notations, we introduce

$$
\begin{align*}
\bar{g}_1 &= g_1 \\
\bar{g}_i &= g_i - \sum_{j=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial \xi_j} + l_j \frac{\partial \alpha_{i-1}}{\partial y} \right) g_j
\end{align*}
$$

(17)

for $i = 2, \ldots, n$.

The follow property of smooth functions is useful for the control design and the stability analysis.

**Lemma 3:** For a smooth function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, the following expression holds for any $\rho_1, \rho_2 \in \mathbb{R}$,

$$
|\varphi(\rho_1 + \rho_2) - \varphi(\rho_1)| \leq \gamma_0(\rho_2) + |\rho_1| \gamma_1(\rho_1) \gamma_2(\rho_2)
$$

(18)

where $\gamma_i : \mathbb{R} \rightarrow \mathbb{R}$ for $i = 0, 1, 2$ are functions with $\gamma_i(\rho) \geq 0$ for any $\rho \in \mathbb{R},$ and $\gamma_1(\cdot)$ is smooth. Furthermore, if for a function $\rho(t) : \mathbb{R}^+ \rightarrow \mathbb{R}$ with $|\rho(t)| < \rho_3 e^{-\rho_4 t}$ for positive reals constants $\rho_3$ and $\rho_4$, then there exist some positive real constants $\rho_5$ and $\rho_6$ such that $\gamma_0(\rho(t)) \leq \rho_5 e^{-\rho_6 t}$ and $\gamma_2(\rho(t)) \leq \rho_5 e^{-\rho_6 t}$.

**Proof.** Proof is omitted due to the page limit.

The control design will be carried out in $n$ steps.

**Step 1.** We start the control design from $\zeta_1$. Its dynamics are given by

$$
\begin{align*}
\dot{\zeta}_1 &= \dot{\xi}_1 - \dot{\xi} \\
&= \xi_2 + \psi_1(y) + g_1 \varphi(\xi_1) - G_1 \dot{\xi} \\
&= \xi_2 + \alpha_1 + \psi_1(y) + g_1 \varphi(\xi_1) - G_1 \dot{\xi} + \xi_2
\end{align*}
$$

where $G_1$ denotes the first row of matrix $G$. From Assumption 3, we have

$$
\begin{align*}
\gamma_{11} g_1(\varphi(\xi_1) - \varphi(\xi_1)) &\leq |\gamma_{11}(\xi_1) + |\xi_1| \gamma_{12}(\xi_1) \gamma_{22}(\xi_1)) \\
&\leq k \xi_1^2 g_1^2 + \frac{1}{4k} \gamma_{22}(\xi_1) + k \xi_1^2 g_1^2 \xi_1^2 \gamma_{22}(\xi_1) + \frac{1}{4k} \gamma_{22}(\xi_1)
\end{align*}
$$

where we have used the fact that $|ab| \leq ka^2 + \frac{1}{4k} b^2$ for any positive real $a,b,k$. Based on the above, we design the first stabilizing function as

$$
\alpha_1 = -(r_1 + k) \zeta_1 - \psi_1(y) - g_1 \varphi(\xi_1)
$$

$$
- k \xi_1 g_1^2 (1 + \gamma_1^2(\xi_1))
$$

(19)

where $r_1$ is among the set of positive real design parameters $\{r_i\}$ for $i = 1, \ldots, n$.

**Step 2.** The dynamics of $\zeta_2$ is described by

$$
\begin{align*}
\dot{\zeta}_2 &= \dot{\xi}_2 - \alpha_1(\xi_1, y) \\
&= \xi_2 - G_2 \dot{\xi} - \frac{\partial \alpha_1}{\partial \xi_1} \xi_1 - \frac{\partial \alpha_1}{\partial y} \xi_1 \\
&= \dot{\xi}_2 - \left[ \frac{\partial \alpha_1}{\partial \xi_1} + l_1 \frac{\partial \alpha_1}{\partial y} \right] \xi_1 - G_2 \dot{\xi} + \frac{\partial \alpha_1}{\partial \xi_1} G_2 \dot{\xi}
\end{align*}
$$

**Remark 2:** To obtain the above expression, we have used $\dot{y} = \frac{\partial y}{\partial y} l_1 \xi_1$ not $\dot{y} = \sum_{j=1}^{n} \frac{\partial y}{\partial y} l_1 \xi_j$. This is due to the triangular condition specified in Assumption 2. From Assumption 2, we have $\frac{\partial y}{\partial y}(\xi_1) = 0$ if $l_2 \neq 0$. Similarly we will use $\dot{y} = \sum_{j=1}^{n} \frac{\partial y}{\partial y} l_1 \xi_j$ at Step 1. Because if $l_1 \neq 0$, then from Assumption 2 we must have $\frac{\partial y}{\partial y}(\xi_1) = 0$ for $j = 1, \ldots, i - 1$. The differences of $\dot{y}$ shown in the expressions do not mean that $\dot{y}$ is different at different steps. In fact, for a given $y$, the expression is unique, and many terms are actually zero. We use different expressions of $\dot{y}$ in different step for the convenience of notations for a generic expression of $y$. 

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Using the dynamics of \( \xi_1 \) and \( \xi_2 \), and \( \xi = \hat{\xi} + \tilde{\xi} \), we have
\[
\dot{\xi}_2 = \dot{\zeta}_2 = \zeta_3 + \alpha_2 + \psi_2(y) + \bar{g}_2 \phi(\xi_1) - \left[ \frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} \right] \xi_2 - G_2 \dot{\xi}_2 + \frac{\partial \alpha_1}{\partial \hat{\xi}_2} G_1 \dot{\xi}_2 
\]  

(20)

The stabilizing function \( \alpha_2 \) is designed as
\[
\alpha_2 = -\zeta_1 - (r_2 + k) \xi_2 - \psi_2(y) - \bar{g}_2 \phi(\hat{\xi}_1) 
+ \left[ \frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} \right] (\hat{\xi}_2 + \psi_1(y)) 
- k_2 \xi_2 \bar{g}_1^2 (1 + \xi_1^2) \xi_1 
- k_2 \xi_2 \frac{\partial \alpha_1}{\partial \hat{\xi}_2} + \frac{\partial \alpha_1}{\partial y} \left[ \hat{\xi}_2 \right] \xi_2 
\]  

(21)

Step 1. Similar to the procedures shown in Step 2, in the subsequent steps, for \( i = 3, \ldots, n \), we have
\[
\dot{\xi}_i = \zeta_{i+1} + \alpha_i + \psi_i(y) + \bar{g}_i \phi(\xi_1) 
- \sum_{j=1}^{i-1} \left[ \frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} \right] \xi_{j+1} + \psi_j(y) 
+ \hat{\xi}_{i+1} - \hat{\xi}_i \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} \xi_{j+1} 
- G_i \hat{\xi}_i + \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} G_j \hat{\xi}_j 
\]  

(22)

The stabilizing function \( \alpha_i \) is obtained as
\[
\alpha_i = -\zeta_{i-1} - (r_i + k) \xi_i - \psi_i(y) - \bar{g}_i \phi(\hat{\xi}_1) 
- \sum_{j=1}^{i-1} \left[ \frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} \right] \xi_{j+1} + \psi_j(y) 
- k_i \xi_i \bar{g}_1^2 (1 + \xi_1^2) \xi_1 
- k_i \xi_i \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} \xi_{j+1} \right] \xi_2 + \psi_1(y) 
\]  

(23)

When \( i = n \), we have \( \alpha_n \) as defined in (24). The control input is then designed by setting \( \zeta_{n+1} = 0 \), which results in
\[
u = \alpha_n + \sum_{i=1}^{n} a_i \hat{\xi}_{n-i+1} \sigma(y) 
\]  

(25)

Remark 3: In the control design, we use \( k \) to be denote a generic positive real design constant, instead of using \( k_{i,j} \) for different \( i \) and \( j \) in \( \alpha_i \), for \( i = 1, \ldots, n \).

We have the stability result for the proposed control design.

Theorem 4: Under Assumptions 2 to 4, the output feedback control (25) based on the output \( y \) and observer state \( \dot{\xi} \) obtained in (11) globally and asymptotically stabilize the dynamic system (13).

Proof. Proof is omitted due to the page limit.

We introduce the control design from an estimated state variable to demonstrate the application of the reduced-order observer design proposed in this paper. One of the advantages of backstepping design from an estimated state variable is its capability to deal with nonlinear systems with unstable zero dynamics. In the remaining part of this section, we include an example to illustrate the proposed control design for the stabilization of such a system.

Example 2. Consider a nonlinear system
\[
x_1 = x_2 + (x_1 + x_2 + x_3)^2 
\]
\[
x_2 = x_3 + 2(x_1 + x_2 + x_3)^2 - y^3 + u 
\]
\[
x_3 = 3(x_1 + x_2 + x_3)^2 + y^3 - u 
\]
\[
y = x_1 
\]

Comparing with the structure of (13), we have
\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, 
\]
\[
d = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, f = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \phi(y) = \begin{bmatrix} 0 \\ -y^3 \\ y^3 \end{bmatrix}, 
\]
\[
\varphi(d^T x) = (x_1 + x_2 + x_3)^2, \sigma = 1. 
\]

It can be verified that Assumptions 2, 3 and 4 are satisfied. Furthermore, the zero dynamics is obtained as
\[
\dot{z}_0 = x_0 + 6x_0^2 
\]
where \( x_0 = x_2 + x_3 \). This zero dynamics is nonlinear and unstable. For the state transformation, we have \( T_2 = I \) and
\[
T_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}. 
\]

We have
\[
z_1 = x_2 - 2y 
z_2 = x_3 - 3y 
\]
and it is easy to obtain that
\[
\dot{z} = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} z + \begin{bmatrix} -y - y^3 + u \\ -6y + y^3 - u \end{bmatrix}. 
\]

We therefore design the reduced order observer as
\[
\dot{\hat{z}} = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} -y - y^3 + u \\ -6y + y^3 - u \end{bmatrix} 
\]
and the estimates of the unmeasured states are given by
\[
\dot{x}_2 = \dot{\hat{z}} + 2y 
\]
\[
\dot{x}_3 = \dot{\hat{z}} + 3y 
\]
For the state transformation \( T_1 \), we have
\[
\xi = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x 
\]
and the system is transformed to
\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 + 6\xi_2^2 \\
\dot{\xi}_2 &= \xi_3 + 5\xi_2^2 \\
\dot{\xi}_3 &= 3\xi_2 + y^3 - u \\
y &= \xi_1 - \xi_2
\end{align*}
\]
For the nonlinear function \( \varphi(\xi_1) = \xi_1^2 \), we take \( \gamma_1 = 2 \).
The control design starts from \( \tilde{\xi}_1 \), for which we have \( \dot{\tilde{\xi}}_1 = y + \tilde{x}_2 + \tilde{x}_3 \). From the control design proposed earlier, we have
\[
\begin{align*}
\alpha_1 &= - (r_1 + k)\xi_1 - g_1\xi_1^2 - kg_1\xi_1(1 + 4\xi_1^2) \\
\alpha_2 &= - \xi_1 - (r_2 + k)\xi_2 - \tilde{g}_2\xi_2^2 + \frac{\partial\alpha_1}{\partial\xi_2}\xi_2^2 \\
&\quad - k\xi_2\tilde{g}_2^2(1 + 4\xi_1^2) - k\xi_2\left[\frac{\partial\alpha_1}{\partial\xi_1}\xi_1^2\right]^2 \\
\alpha_3 &= - \xi_2 - (r_3 + k)\xi_3 - y^3 - \tilde{g}_3\xi_3^2 \\
&\quad + \frac{\partial\alpha_1}{\partial\xi_2}\xi_2 + \frac{\partial\alpha_2}{\partial\xi_3}\xi_3 - k\xi_3\tilde{g}_3^2(1 + 4\xi_1^2) \\
&\quad - k\xi_3\left[\frac{\partial\alpha_1}{\partial\xi_1}\xi_1^2 + \frac{\partial\alpha_2}{\partial\xi_2}\xi_2^2\right]^2 \\
u &= - \alpha_3
\end{align*}
\]
where \( g_1 = 6, \tilde{g}_2 = 5 - 6\frac{\partial\alpha_1}{\partial\xi_2} \) and \( \tilde{g}_3 = 3 - 6\frac{\partial\alpha_1}{\partial\xi_1} - 5\frac{\partial\alpha_2}{\partial\xi_2} \). In the simulation study, we set \( r_1 = r_2 = r_3 = 1 \) and \( k = 0.01 \).

The simulation results are shown in Figures 3 and 4.

\[ \text{Fig. 3. The state variables and the input} \]

\[ \text{Fig. 4. Unmeasured state and its estimate} \]

V. CONCLUSIONS

In this paper, we have presented a basic design of reduced-order observers for nonlinear dynamic systems. The conditions have been identified for the convergence of the observer errors. More detailed design procedures have been proposed for a specific class of nonlinear systems, which is larger than the nonlinear systems with linear observer errors for the full order observers. The proposed reduced order observer design has been applied to output feedback stabilization of nonlinear systems with nonlinearity of the unmeasured state and nonlinear unstable zero dynamics.