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Global Output Feedback Stabilization of a Class of Upper-Triangular Nonlinear Systems

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Abstract— **This paper considers the global output feedback stabilization of a class of upper-triangular nonlinear systems with a single input and multiple output. It is shown that under a structural condition, the output feedback stabilization problem can be solved for this class of systems by coupling with a finite-time convergent observer and a saturated homogeneous stabilizer. More general results are also achieved for some complex nonlinear systems without the homogeneous growth condition.**

I. INTRODUCTION

This paper considers a class of nonlinear systems with a single input and multiple outputs (SIMO) described by the following equations:

$$\begin{aligned}
\dot{x}_{i,1} &= x_{i,2} + \phi_{i,1}(\mathbf{x}_{i+1}, u),\\
\dot{x}_{i,2} &= x_{i,3} + \phi_{i,2}(\mathbf{x}_{i+1}, u),\\
&\vdots\\
\dot{x}_{i,n_i} &= x_{i+1,1} + \phi_{i,n_i}(\mathbf{x}_{i+1}, u), \qquad i = 1, \cdots, m\\
y &= [x_{1,1}, x_{2,1}, \cdots, x_{m,1}]^T\\
u &= x_{m+1,1}
\end{aligned} \tag{1}$$

where

$$\mathbf{x}_i := \left[x_{i,1}, \cdots, x_{i,n_i}, \cdots\cdots, x_{m,1}, \cdots, x_{m,n_m}\right], \quad i = 1, \cdots, m$$

and $\mathbf{x}_{m+1} := Null$ are system states, $u \in \mathbb{R}$ is the control input, and $y \in \mathbb{R}^{\mathrm{m}}$ is the measurable output. The functions $\phi_{i,j}(\cdot),\ j = 1, \cdots, n_i,\ i = 1, \cdots, m$, are C^1 nonlinear functions with $\phi_{i,j}(0) = 0$ and

$$\lim_{u \to 0} \phi_{m,n_m}(u)/u = 0. \tag{2}$$

The objective of this paper is to design an output feedback controller of the form

$$\dot{\eta} = F(\eta, y), \qquad u = u(\eta, y) \tag{3}$$

such that the closed-loop system (1)-(3) is globally asymptotically stable.

A notable feature of the feedforward system (1) is that the system has only one control input but multiple outputs. It is actually not an unrealistic assumption since most of higher-dimensional mechanical systems in the real world have multiple displacement variables (angular and/or linear) which are usually easily measurable.

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The stabilization of feedforward systems is an interesting problem and has attracted a great deal of attention. In the literature, a number of *state feedback* stabilizers have been developed using nested-saturation [16], [8] or forwarding [10], [15]. However, when only partial states are measurable, the global output feedback stabilization of feedforward systems is more challenging and has received very little attention compared to lower-triangular systems. In fact, as illustrated by a counter-example in [14], the structure of upper-triangular systems leads to an intrinsic obstacle which makes it impossible to achieve even *semi-global output feedback stabilization* of the general feedforward systems.

In the area of nonlinear control there are a small number of existing results in dealing with the problem of output feedback stabilization for feedforward systems. One existing approach employed bounded control to deal with feedforward systems with marginally stable free dynamics [7], [11]. For marginally unstable system with upper-triangular nonlinearities, global output feedback stabilizers were constructed under the linear growth condition [3] including the Lipschitz condition [12]. From both practical or theoretical viewpoints, it is too restrictive to require the nonlinearities to satisfy the linear growth condition. By developing a homogeneous domination approach, the linear growth condition was lifted in [14] where global output feedback stabilization was achieved for more general nonlinearities under a homogeneous growth condition. Later the paper [4] extended the result [14] to a class of upper-triangular nonlinear systems with uncontrollable linearization.

Note that most of these existing results required the nonlinearities to satisfy the homogeneous growth condition (linear growth is a special case of homogeneous growth). This is not surprising since even for low-triangular systems whose output feedback stabilization problem has been well-studied, the homogeneous growth condition is also widely assumed even for the most general results. Under the homogeneous growth condition, the homogeneous domination approach [13] has enabled us to deal with nonlinear systems with higher-order nonlinearities either in lower or upper triangular forms. But on the other hand, the homogeneous condition is still a quite restrictive condition which limits us to solve the output feedback stabilization problem of some of practical systems with non-homogeneous nonlinearities.

In this paper, we show that if we have more than one measurable outputs, we can significantly enlarge the class of nonlinear feedforward systems whose output feedback stabilization problem is solvable. From practical point of view, most of higher-dimensional mechanical systems (num-

ber of states ≥ 4) usually have more than one displacement variables which can be easily measured. With the help of the recently developed finite-time convergent observer [2], [5], [6], we will develop a cascaded observer consisting of several sub-observers, which provides us the real states of the system in a finite time. This new observer, together with a new saturated homogeneous state feedback stabilizer, will globally stabilize the feedforward system (1).

II. Output feedback stabilizer design

In this section, a finite-time convergent observer will be first constructed to obtain the real states of the system in a finite time. Then we show that the separation principle holds for feedforward system (1) with various state feedback stabilizers with bounded magnitude.

A. Observer Design

For $i = 1, \cdots, m$, design the observer

$$\begin{aligned}
\dot{\hat{x}}_{i,1} &= \hat{x}_{i,2} + \phi_{i,1}(\hat{\mathbf{x}}_{i+1}, u) + a_{i,1}[x_{i,1} - \hat{x}_{i,1}]^{r_2}, \\
\dot{\hat{x}}_{i,2} &= \hat{x}_{i,3} + \phi_{i,2}(\hat{\mathbf{x}}_{i+1}, u) + a_{i,2}[x_{i,1} - \hat{x}_{i,1}]^{r_3}, \\
&\vdots \\
\dot{\hat{x}}_{i,n_i} &= \hat{x}_{i+1,1} + \phi_{i,n_i}(\hat{\mathbf{x}}_{i+1}, u) + a_{i,n_i}[x_{i,1} - \hat{x}_{i,1}]^{r_{n_i+1}}
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\hat{x}_{m+1,1} &= u, \\
\hat{\mathbf{x}}_i &= \left[\hat{x}_{i,1}, \cdots, \hat{x}_{i,n_i}, \cdots\cdots, \hat{x}_{m,1}, \cdots, \hat{x}_{m,n_m}\right], \\
\hat{\mathbf{x}}_{m+1} &= Null, \\
r_1 &= 1, \ r_{j+1} = r_j + \hat{\tau}, \ j = 1, \cdots, n_i
\end{aligned}$$

with $\hat{\tau} = p/q \in (-1/n_i, 0)$ for an odd integer q and an even integer p.

First we consider the last subsystem of (4) where $i = m$

$$\begin{aligned}
\dot{\hat{x}}_{m,1} &= \hat{x}_{m,2} + \phi_{m,1}(u) + a_{m,1}[x_{m,1} - \hat{x}_{m,1}]^{r_2}, \\
\dot{\hat{x}}_{m,2} &= \hat{x}_{m,3} + \phi_{m,2}(u) + a_{m,2}[x_{m,1} - \hat{x}_{m,1}]^{r_3}, \\
&\vdots \\
\dot{\hat{x}}_{m,n_m} &= u + \phi_{m,n_m}(u) + a_{m,n_m}[x_{m,1} - \hat{x}_{m,1}]^{r_{n_m+1}}.
\end{aligned} \tag{5}$$

Defining the errors as

$$e_{m,j} = x_{m,j} - \hat{x}_{m,j}, \quad j = 1, \cdots, n_m,$$

we have the following error dynamics based on (1)-(5)

$$\begin{aligned}
\dot{e}_{m,1} &= e_{m,2} - a_{m,1}[e_{m,1}]^{r_2}, \\
\dot{e}_{m,2} &= e_{m,3} - a_{m,2}[e_{m,1}]^{r_3}, \\
&\vdots \\
\dot{e}_{m,n_m} &= -a_{m,n_m}[e_{m,n_m}]^{r_{n_m+1}}
\end{aligned} \tag{6}$$

By Lemma A.4, there exist constants $a_{m,j}, \ j = 1, \cdots, n_m$ such that the above system is globally finite-time stable[1]. Hence, after a finite time T_1,

$$\hat{x}_{m,j} = x_{m,j}, \quad j = 1, \cdots, n_m.$$

As a consequence,

$$\phi_{m-1,j}(\hat{\mathbf{x}}_m, u) = \phi_{m-1,j}(\mathbf{x}_m, u), \quad j = 1, \cdots, n_{m-1}.$$

Hence, the dynamics of error $e_{m-1,j} = x_{m-1,j} - \hat{x}_{m-1,j}, \ j = 1, \cdots, n_{m-1}$ for $(m-1)-$th sub-system become

$$\begin{aligned}
\dot{e}_{m-1,1} &= e_{m-1,2} - a_{m-1,1}[e_{m-1,1}]^{r_2}, \\
\dot{e}_{m-1,2} &= e_{m-1,3} - a_{m-1,2}[e_{m-1,1}]^{r_3}, \\
&\vdots \\
\dot{e}_{m-1,n_{m-1}} &= -a_{m-1,n_{m-1}}[e_{m-1,n_{m-1}}]^{r_{n_{m-1}+1}}, \quad t \geq T_1.
\end{aligned} \tag{7}$$

Similarly, we know that for appropriate constants $a_{m-1,j}$'s there is a finite time T_2 such that

$$\hat{x}_{m-1,j} = x_{m-1,j}, \quad j = 1, \cdots, n_{m-1}, \quad t \geq T_1 + T_2.$$

Following the same line, we can conclude that after a finite time $T_1 + T_2 + \cdots + T_m := T^*$,

$$\hat{x}_{i,j} = x_{i,j}, \quad \forall i, \ j,$$

which means the estimates obtained from observer (4) have become the real states of the system (1).

B. Output Feedback Stabilizers

The finite-time convergent observer presented in the previous subsection can be easily integrated with several state feedback controllers.

First, we employ the small state feedback controllers designed for feedforward systems with higher-order nonlinearities.

Theorem 2.1: Assume that for $j = 1, \cdots, n_i, \ i = 1, \cdots, m$, the following holds

$$\phi_{i,j}(\mathbf{x}_{i+1}, u) \leq (\|\mathbf{x}_{i+1}\|^2 + u^2)\rho_{i,j}(\mathbf{x}_{i+1}, u) \tag{8}$$

for a smooth function $\rho_{i,j}(\mathbf{x}_{i+1}, u)$. Then, there is an output feedback controller under which the states of system (1) are bounded and tend to the origin ultimately.

Proof. First, under the condition (8), we can design a small controller $u(\mathbf{x}), \ \mathbf{x} = \mathbf{x}_1$ using either the saturation design [16] or the forwarding design [9]. Since the states $\mathbf{x}$ are not available for feedback in the very beginning, we will substitute the states $\mathbf{x}$ by their estimates $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1$.

Note that one common property of both saturation and forwarding methods is that the controller is bounded by an arbitrarily small positive number. Hence, even before the convergence time T^*, all the states of the system will stay bounded. After the convergence time T^*, the controller

$$u(\hat{\mathbf{x}}) = u(\mathbf{x})$$

which will ultimately render the system globally asymptotically stable. ∎

Note that Theorem 2.1 still requires the higher-order condition (8). However, this requirement can be lifted and more general results can be achieved using a saturated

[1]Roughly speaking, the solutions of a system are said to be finite-time stable if the solutions are stable and will reach the origin in a finite time and stay there afterwards. A more precise definition can be found in the references [2], [5].

homogeneous controller. In what follows, we show that global output feedback stabilization can be achieved without the higher-order condition (8).

In order to design the state feedback stabilizer, we consider the following system for neatness of notation.

$$\begin{aligned}\dot{z}_1 &= z_2 + f_1(z_2, z_3, \cdots, z_n, u),\\ \dot{z}_2 &= z_3 + f_2(z_3, \cdots, z_n, u),\\ &\vdots\\ \dot{z}_n &= u + f_n(u)\end{aligned} \tag{9}$$

Assumption 2.1: There is a constant $\tau \geq 0$ such that

$$\lim_{\varepsilon \to 0} f_i(\varepsilon^{h_{i+1}} z_{i+1}, \cdots, \varepsilon^{h_n} z_n, \varepsilon^{h_{n+1}} u)/\varepsilon^{h_{i+1}} = 0 \tag{10}$$

where

$$h_1 = 1, \quad h_{i+1} = h_i + \tau, \quad i = 2, \cdots, n.$$

Lemma 2.1: Under Assumption 2.1, there are constants b_i and saturation functions defined as

$$\sigma_i(s) = \begin{cases} sign(s) \cdot M_i, & |s| > M_i \\ s, & |s| \leq M_i \end{cases} \quad \text{for a constant } M_i$$

such that the following controller

$$\begin{aligned} u = -b_n \sigma_n (z_n^{\frac{h_n+\tau}{h_n}} &+ b_{n-1}\sigma_{n-1}(z_{n-1}^{\frac{h_n+\tau}{h_{n-1}}} + \cdots \\ &+ b_2 \sigma_2 (z_2^{\frac{h_n+\tau}{h_2}} + b_1 \sigma_1 (z_1^{\frac{h_n+\tau}{h_1}}))\cdots)) \end{aligned} \tag{11}$$

renders system (9) globally asymptotically stable.

Proof of Lemma 2.1 We first show that for system

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \cdots, \quad \dot{z}_{n-1} = z_n, \quad \dot{z}_n = u \tag{12}$$

there is a homogeneous controller

$$u = -c_n (z_n^{\frac{h_n+\tau}{h_n}} + c_{n-1}(z_{n-1}^{\frac{h_n+\tau}{h_{n-1}}} + \cdots + c_2 (z_2^{\frac{h_n+\tau}{h_2}} + c_1 z_1^{\frac{h_n+\tau}{h_1}})\cdots))$$

where $h_1 = 1, \quad h_{i+1} = h_i + \tau, \quad i = 2, \cdots, n$.

For the simplicity, we assume $\tau = \frac{q}{p}$ with q an even integer and p an odd integer. We will employ an inductive argument to construct the controller.

Initial Step. Choose

$$V_1 = \frac{h_1}{2h_n + \tau} z_1^{(2h_n+\tau)/h_1}.$$

Clearly, the virtual controller z_2^* defined by

$$z_2^* = -z_1^{h_2/h_1} n := -(z_1^{h_{n+1}/h_1})^{h_2/h_{n+1}} \beta_1,$$

yields

$$\dot{V}_1(z_1) \leq -n z_1^{2h_{n+1}/h_1} + z_1^{(2h_n+\tau)/h_1 - 1}[z_2 - z_2^*]. \tag{13}$$

Inductive Step. Suppose at step $k-1$, there are a C^1 Lyapunov function $V_{k-1} : \mathbb{R}^{k-1} \to \mathbb{R}$, which is positive definite and proper, and a set of virtual controllers $z_1^*, \cdots, z_k^*$, defined by

$$\begin{aligned} z_1^* &= 0 & \xi_1 &= z_1^{h_{n+1}/h_1} - z_1^{*h_{n+1}/h_1},\\ z_2^* &= -\xi_1^{h_2/h_{n+1}} \beta_1 & \xi_2 &= z_2^{h_{n+1}/h_2} - z_2^{*h_{n+1}/h_2},\\ &\vdots & &\vdots\\ z_k^* &= -\xi_{k-1}^{h_k/h_{n+1}} \beta_{k-1} & \xi_k &= z_k^{h_{n+1}/h_k} - z_k^{*h_{n+1}/h_k}, \end{aligned} \tag{14}$$

with constants $\beta_1 > 0, \cdots, \beta_{k-1} > 0$, such that

$$\dot{V}_{k-1} \leq -(n-k+2)\sum_{l=1}^{k-1} \xi_l^2 + \xi_{k-1}^{1+\frac{h_n-h_{k-1}}{h_{n+1}}}(z_k - z_k^*). \tag{15}$$

Obviously, (15) reduces to the inequality (13) when $k = 2$.

We claim that (15) also holds at step k. To prove the claim, we consider the Lyapunov function $V_k : \mathbb{R}^k \to \mathbb{R}$, defined by

$$V_k(z_1, \cdots, z_k) = V_{k-1} + \int_{z_k^*}^{z_k} \left[s^{h_{n+1}/h_k} - z_k^{*h_{n+1}/h_k} \right]^{1+\frac{h_n-h_k}{h_{n+1}}} ds.$$

The derivative of the Lyapunov function V_k along the system (9) is

$$\begin{aligned} \dot{V}_k &\leq \dot{V}_{k-1} + \xi_k^{1+\frac{h_n-h_k}{h_{n+1}}} z_{k+1} + c|\xi_k|^{\frac{h_n}{h_{n+1}}} |\dot{\xi}_{k-1}|\\ &\leq -(n-k+2)\sum_{l=1}^{k-1} \xi_l^2 + \xi_{k-1}^{1+\frac{h_n-h_{k-1}}{h_{n+1}}}(z_k - z_k^*).\\ &\quad + \xi_k^{1+\frac{h_n-h_k}{h_{n+1}}} z_{k+1} + c|\xi_k|^{\frac{h_n}{h_{n+1}}} |\dot{\xi}_{k-1}|. \end{aligned} \tag{16}$$

Next we estimate each term in the right hand side of (16). First, it follows from Lemmas A.1 and A.2 that

$$\begin{aligned} |\xi_{k-1}^{1+\frac{h_n-h_{k-1}}{h_{n+1}}}(z_k - z_k^*)| &\leq c|\xi_{k-1}|^{1+\frac{h_n-h_{k-1}}{h_{n+1}}} |\xi_k|^{h_k/h_{n+1}}\\ &\leq \frac{1}{2}\xi_{k-1}^2 + c_k \xi_k^2, \end{aligned} \tag{17}$$

for a constant $c_k > 0$.

The last term in (16), namely $c|\xi_k|^{\frac{h_n}{h_{n+1}}}|\dot{\xi}_{k-1}|$ can be estimated as the follows. First, by the definition of the ξ_l, we have

$$\begin{aligned} c|\xi_k|^{\frac{h_n}{h_{n+1}}}|\dot{\xi}_{k-1}| &\leq c|\xi_k|^{\frac{h_n}{h_{n+1}}} \left(|z_{k-1}^{\frac{h_{n+1}}{h_{k-1}}-1} z_k| + \cdots + |z_1^{\frac{h_{n+1}}{h_1}-1} z_2| \right)\\ &\leq c|\xi_k|^{\frac{h_n}{h_{n+1}}} \left(|z_1|^{\frac{h_{n+1}+\tau}{h_1}} + \cdots + |z_k^{\frac{h_{n+1}+\tau}{h_k}}| \right)\\ &\leq c|\xi_k|^{\frac{h_n}{h_{n+1}}} \left(|\xi_1|^{\frac{h_{n+1}+\tau}{h_{n+1}}} + \cdots + |\xi_k|^{\frac{h_{n+1}+\tau}{h_{n+1}}} \right)\\ &\leq \frac{1}{2}\left(\xi_1^2 + \cdots + \xi_{k-1}^2\right) + \xi_k^2 \hat{c}_k \end{aligned} \tag{18}$$

where $\hat{c}_k$ is a constant. Note that the last inequality follows from Lemma A.2.

Substituting the estimates (17) and (18) into (16), we have

$$\begin{aligned} \dot{V}_k &\leq -(n-k+1)\sum_{l=1}^{k-1} \xi_l^2 + \xi_k^{1+\frac{h_n-h_k}{h_{n+1}}}(z_{k+1} - z_{k+1}^*)\\ &\quad + \xi_k^{1+\frac{h_n-h_k}{h_{n+1}}} \left(z_{k+1}^* + (c_k + \hat{c}_k)\xi_k^{h_{k+1}/h_{n+1}} \right). \end{aligned}$$

Observe that a virtual controller of the form

$$z_{k+1}^* = -\xi_k^{h_{k+1}/h_{n+1}} \beta_k = -\xi_k^{h_{k+1}/h_{n+1}} [n-k+1+c_k+\hat{c}_k], \tag{19}$$

renders

$$\dot{V}_k \leq -(n-k+1)\sum_{l=1}^{k} \xi_l^2 + \xi_k^{1+\frac{h_n-h_k}{h_{n+1}}}(z_{k+1} - z_{k+1}^*).$$

This completes the inductive proof.

The inductive argument shows that (15) holds for $k=n+1$ with a set of virtual controllers (14). Hence, at the last step, choosing

$$\begin{aligned} u &= -\xi_n^{h_{n+1}/h_{n+1}}\beta_n \\ &= -c_n(z_n^{\frac{h_n+\tau}{h_n}} + c_{n-1}(z_{n-1}^{\frac{h_n+\tau}{h_{n-1}}} + \cdots + c_2(z_2^{\frac{h_n+\tau}{h_2}} + c_1 z_1^{\frac{h_n+\tau}{h_1}})\cdots)) \end{aligned} \tag{20}$$

yields

$$\dot{V}_n \le -\left(\xi_1^2 + \cdots + \xi_n^2\right) < 0, \qquad \forall\, z \neq 0 \tag{21}$$

where $V_n(z_1,\cdots,z_n)$ is a positive definite and proper Lyapunov function. As a result, (12)–(20) is *globally asymptotically stable.*

Based on the homogeneous controller (20), following the same line in the work [17] we are able to choose appropriate saturation constants for functions $\sigma_i(s)$'s and adjust controller coefficients b_i's such that

$$\begin{aligned} u = -b_n\sigma_n(z_n^{\frac{h_n+\tau}{h_n}} &+ b_{n-1}\sigma_{n-1}(z_{n-1}^{\frac{h_n+\tau}{h_{n-1}}} + \cdots \\ &+ b_2\sigma_2(z_2^{\frac{h_n+\tau}{h_2}} + b_1\sigma_1(z_1^{\frac{h_n+\tau}{h_1}}))\cdots)) \end{aligned}$$

globally stabilizes the system (9) under Assumption 2.1. The procedure to choose the constants is very similar to the subtle procedure [17] that was introduced for a class of upper-triangular systems with uncontrollable linearization. For the sake of space, here we will not repeat the lengthy detailed procedure which can be recovered from [17]. ∎

Remark 2.1: It can be verified that system (1) satisfies Assumption 2.1 automatically. To show this, we first denote

$$x_{i,j} = z_{k_i+j}, \quad k_i = \sum_{l=1}^{i-1} n_l, \quad \phi_{i,j} = f_{k_i+j}(z_{k_i+n_i+1},\cdots,z_n,u).$$

Under the new notations, together with the fact that $\sum_{l=1}^{m} n_l = n$, it is clear that system (1) is in the form of (9). Moreover, due to the fact that $\phi_{i,j}(\mathbf{x}_{i+1},u)$ is C^1 and $\phi_{i,j}(0)=0$, we know

$$f_{k_i+j}(z_{k_i+n_i+1},\cdots,z_n,u) = (z_{k_i+n_i+1},\cdots,z_n,u)\tilde{f}_{k_i+j}(\cdot)$$

for a continuous function $\tilde{f}_{k_i+j}(\cdot)$. As a result, one has

$$\begin{aligned} &\left|\frac{f_{k_i+j}(\varepsilon^{h_{k_i+n_i+1}}z_{k_i+n_i+1},\cdots,\varepsilon^{h_n}z_n,\varepsilon^{h_{n+1}}u)}{\varepsilon^{h_{k_i+j}}}\right| \\ &\le \varepsilon^{h_{k_i+n_i+1}-h_{k_i+j}}(|z_{k_i+n_i+1}| + |\varepsilon^{\tau}z_{k_i+n_i+2}| + \cdots \\ &\quad + |\varepsilon^{h_n-h_{k_i+j}}z_n| + |\varepsilon^{h_{n+1}-h_{k_i+j}}u|)\|\tilde{f}_{k_i+j}(\cdot)\|. \end{aligned}$$

Note that $1 \le j \le n_i$. Hence $h_{k_i+n_i+1} - h_{k_i+j} = (1+n_i-j)\tau > 0$, which implies

$$\lim_{\varepsilon\to 0}\frac{f_{k_i+j}(\varepsilon^{h_{k_i+n_i+1}}z_{k_i+n_i+1},\cdots,\varepsilon^{h_n}z_n,\varepsilon^{h_{n+1}}u)}{\varepsilon^{h_{k_i+j}}} = 0.$$

This, together with (2), implies that the assumption (10) holds for system (1). ∎

Theorem 2.2: There is an output feedback controller under which the states of system (1) are bounded and tend to the origin ultimately.

Proof: By Remark 2.1, we know system (1) satisfies Assumption 2.1 automatically. Consequently, by Lemma 2.1 we can construct a state feedback controller $u(\mathbf{x})$ bounded by a small positive constant. Similar to the proof of Theorem 2.1, we also construct a finite-time convergent observer (4) and use the observer states $\hat{\mathbf{x}}$, instead of the unmeasurable states $\mathbf{x}$, in the small controller. With the bounded controller $u(\hat{\mathbf{x}})$, all the states of the closed-loop system (1) are bounded for any finite time. Due to the finite-time convergence of the observer, after a finite time T^*,

$$u(\hat{\mathbf{x}}) = u(\mathbf{x})$$

which will eventually globally stabilize system (1) according to Lemma 2.1. ∎

C. An Example

In this subsection, computer simulation is conducted for the following feedforward system

$$\begin{aligned} \dot{x}_{1,1} &= x_{1,2} + x_{2,1} + \sqrt{|x_{2,2}|} + x_{2,2}^2 \\ \dot{x}_{1,2} &= x_{2,1} + \sin(x_{2,2}) \\ \dot{x}_{2,1} &= x_{2,2} \\ \dot{x}_{2,2} &= u \end{aligned} \tag{22}$$

where $y = [x_{1,1}, x_{2,1}]^T$ is the measurable output. The nonlinearities of system (22) apparently are in the feedforward form. However, the nonlinearities are not higher-order as required in the most existing results. Hence even the state feedback stabilization problem is unsolved. On the other hand, by choosing $\tau = 2$, it can be verified that Assumption 2.1 holds for the system. As a result, we can design an output feedback stabilizer for system (22) as follows:

$$\begin{aligned} \dot{\hat{x}}_{1,1} &= \hat{x}_{1,2} + x_{2,1} + \sqrt{|\hat{x}_{2,2}|} + \hat{x}_{2,2}^2 + [x_{1,1} - \hat{x}_{1,1}]^{r_1} \\ \dot{\hat{x}}_{1,2} &= \hat{x}_{2,1} + \sin(\hat{x}_{2,2}) + [x_{1,1} - \hat{x}_{1,1}]^{r_2} \\ \dot{\hat{x}}_{2,1} &= \hat{x}_{2,2} + [x_{2,1} - \hat{x}_{2,1}]^{r_1} \\ \dot{\hat{x}}_{2,2} &= u + [x_{2,1} - \hat{x}_{2,1}]^{r_2} \\ u &= -b_4\sigma_4(\hat{x}_{2,2}^{9/7} + b_3\sigma_3(x_{2,1}^{9/5} + b_2\sigma_2(\hat{x}_{1,2}^{9/3} + b_1\sigma_1(x_{1,1}^{9/1})))) \end{aligned}$$

Fig. 1 shows that the trajectories of the states $x_{1,1}$ and $x_{2,1}$ converge to the zero. Fig 2 plots the trajectories of the unmeasurable states $x_{1,2}$ and $x_{2,2}$ compared to their corresponding estimates $\hat{x}_{1,2}$ and $\hat{x}_{2,2}$. Note that the estimated states converge to the real states in a finite time and the real stats are asymptotically stable.

III. Extensions

In this section, we show that the techniques presented in the previous section can be extended to a class of more

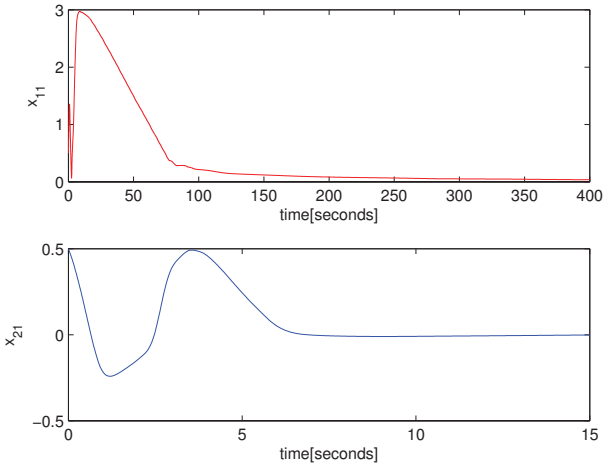

Fig. 1. Trajectories of $x_{1,1}$ and $x_{2,1}$

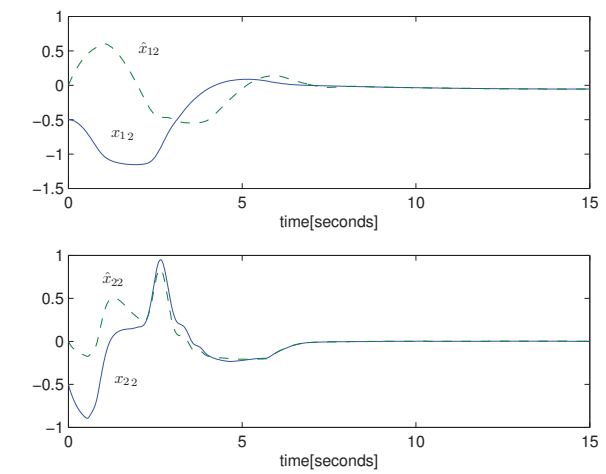

Fig. 2. Trajectories of unmeasurable states and their estimates

general nonlinear systems.

$$\begin{aligned}
\dot{x}_{i,1} &= x_{i,2} + \phi_{i,1}(\mathbf{x}_{i+1}, y, u), \\
\dot{x}_{i,2} &= x_{i,3} + \phi_{i,2}(\mathbf{x}_{i+1}, y, u), \\
&\vdots \\
\dot{x}_{i,n_i} &= x_{i+1,1} + \phi_{i,n_i}(\mathbf{x}_{i+1}, y, u), \qquad i = 1, \cdots, m \quad (23)
\end{aligned}$$

where $\mathbf{x}_i$, $y = [x_{1,1}, \cdots, x_{m,1}]^T$, and $u = x_{m+1,1}$ are defined same as in (1).

Since the additional variable y in the nonlinear functions is measurable, a finite-time observer can be still constructed for the system (23). As a matter of fact, we can design the following observer:

$$\begin{aligned}
\dot{\hat{x}}_{i,1} &= \hat{x}_{i,2} + \phi_{i,1}(\hat{\mathbf{x}}_{i+1}, y, u) + a_{i,1}[x_{i,1} - \hat{x}_{i,1}]^{r_2}, \\
\dot{\hat{x}}_{i,2} &= \hat{x}_{i,3} + \phi_{i,2}(\hat{\mathbf{x}}_{i+1}, y, u) + a_{i,2}[x_{i,1} - \hat{x}_{i,1}]^{r_3}, \\
&\vdots \\
\dot{\hat{x}}_{i,n_i} &= \hat{x}_{i+1,1} + \phi_{i,n_i}(\hat{\mathbf{x}}_{i+1}, y, u) + a_{i,n_i}[x_{i,1} - \hat{x}_{i,1}]^{r_{n_i+1}} \qquad (24)
\end{aligned}$$

for $i = 1, \cdots, m$. Following the same argument in Section II, it is straightforward to see that there is a finite time T^* after which all the states of observer (24) will become the real states of (1) as long as the real states of (1) are still bounded in $[0, T^*]$.

In addition, we assume that there is a state feedback controller $u(\mathbf{x})$ satisfying the following conditions:

(H1) $u(\mathbf{x})$ is a global asymptotic stabilizer for (23).

(H2) For any finite time T, the state feedback controller $u(\hat{\mathbf{x}})$ using the estimate from the observer (24) keeps the states of system (23) away from infinity for $t \in [0, T]$.

Under these two conditions, we can see that after obtaining the real values of the states of system (23) after a finite time T^*, the controller $u(\hat{\mathbf{x}}) = u(\mathbf{x})$ which renders the system (1) globally asymptotically stable.

Remark 3.1: Clearly, for the general feedforward system (23), the main task to obtain an output feedback controller is to find state feedback controller satisfying conditions (H1)-(H2). In the literature, there are vast amount of results on how to design small controllers. For example, we can use the "forwarding" technique in [9] to easily design a controller $u(x)$ for the following system in the form of (23).

$$\begin{aligned}
\dot{x}_{1,1} &= x_{1,2} + x_{1,1}(x_{1,2}^2 + x_{2,2}^2) \\
\dot{x}_{1,2} &= x_{2,1} + x_{2,2}^2 \\
\dot{x}_{2,1} &= x_{2,2} + \ln(1 + x_{2,1}^2)u^2 \\
\dot{x}_{2,2} &= u \qquad (25)
\end{aligned}$$

The controller $u(x)$ constructed in [9] is arbitrarily small and hence we can verify that (H2) holds for any finite time. Hence, the global output feedback stabilization of (23) can be achieved by combining the finite-time observer and the existing state feedback stabilizer.

In the light of the above design philosophy, global output feedback stabilizer can be achieved for the following feedforward system that is not exactly in the form of (1).

$$\begin{aligned}
\dot{x}_{1,1} &= x_{1,2} \\
\dot{x}_{1,2} &= -\sin x_{2,1} + x_{1,1} \cdot x_{2,2}^2 \\
\dot{x}_{2,1} &= x_{2,2} \\
\dot{x}_{2,2} &= u \qquad (26)
\end{aligned}$$

where $y = [x_{1,1}, x_{2,1}]^T$ is the measurable output. Design

$$\begin{aligned}
\dot{\hat{x}}_{1,1} &= \hat{x}_{1,2} + [x_{1,1} - \hat{x}_{1,1}]^{r_1} \\
\dot{\hat{x}}_{1,2} &= -\sin x_{2,1} + [x_{1,1} - \hat{x}_{1,1}]^{r_2} + x_{1,1} \cdot \hat{x}_{2,2}^2 \\
\dot{\hat{x}}_{2,1} &= \hat{x}_{2,2} + [x_{2,1} - \hat{x}_{2,1}]^{r_1} \\
\dot{\hat{x}}_{2,2} &= u + [x_{2,1} - \hat{x}_{2,1}]^{r_2} \qquad (27)
\end{aligned}$$

Now we design a state feedback controller based on the one developed in [1]. We first adopt the virtual controller $x_{2,2}^* = v(x_{1,1}, x_{1,2}, x_{2,1})$. Next, we design the controller

$$\begin{aligned}
u(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = &-\arctan([x_{2,2} - v(x_{1,1}, x_{1,2}, x_{2,1})]^{7/9}) \\
&- \dot{v}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) \qquad (28)
\end{aligned}$$

which guarantees that $x_{2,2} = v(x_{1,1}, x_{1,2}, x_{2,1})$ in a finite time. Finally, we replace the unmeasurable states $x_{1,2}$ and $x_{2,2}$ in $u(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$ with their estimate $\hat{x}_{1,2}$ and $\hat{x}_{2,2}$.

The simulation results are illustrated in Fig. 3 and Fig. 4, which show that the all the states converge to zero asymptotically and the estimates converge to the real values in a finite time.

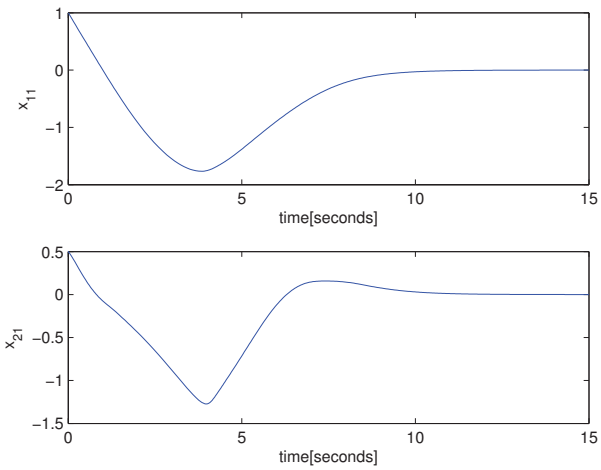

Fig. 3. Trajectories of $x_{1,1}$ and $x_{2,1}$

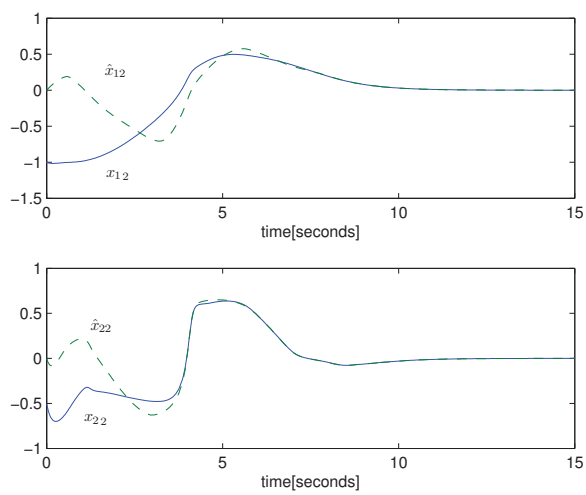

Fig. 4. Trajectories of unmeasurable states and their estimates

APPENDIX

In the appendix we include several useful lemmas.

Lemma A.1: For $x \in \mathbb{R}, y \in \mathbb{R}$, $p \geq 1$ is a constant, the following inequality holds:

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|. \tag{A.1}$$

Lemma A.2: Let c, d be positive constants. Given any positive number $\gamma > 0$, the following inequality holds:

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}. \tag{A.2}$$

Lemma A.3: Let $p \geq 1$ be a real number and x, y be real-valued functions. Then, for a constant $c > 0$

$$|x^p - y^p| \leq p|x-y|(x^{p-1} + y^{p-1}) \tag{A.3}$$

$$\leq c|x-y||(x-y)^{p-1} + y^{p-1}|. \tag{A.4}$$

Lemma A.4: Consider

$$\begin{aligned} \dot{e}_1 &= e_2 - a_1[e_1]^{r_2}, \\ \dot{e}_2 &= e_3 - a_2[e_1]^{r_3}, \\ &\vdots \\ \dot{e}_N &= -a_N[e_1]^{r_{N+1}} \end{aligned} \tag{A.5}$$

where $r_1 = 1 + \hat{\tau}$, $r_{i+1} = r_i + \hat{\tau}$, $i = 1, \cdots, N$ with $\hat{\tau} = p/q \in (-1/n, 0]$ for an odd integer q and an even integer p. There are constants a_i such that the system (A.5) is globally asymptotically stable. When $\hat{\tau} < 0$, the solution of (A.5) will tend to the origin in a finite time.

Proof: The lemma was first proved for two-dimensional version of (A.5) in [2], [5]. Later the paper [6] provided a proof for the general dimensional system (A.5).

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