A Fast Joint Tracking-Registration Algorithm for Multi-Sensor Systems

Shuqing Zeng

Abstract—Sensor fusion of multiple sources plays an important role in robotic systems to achieve refined target position and velocity estimates. In this paper, we address the general registration problem, which is a key module for a fusion system to accurately correct systematic errors of sensors. A fast maximum a posterior (FMAP) algorithm for joint registration and tracking is presented. The algorithm uses a recursive two-step optimization that involves orthogonal factorization to ensure numerically stability. Statistical efficiency analysis based on Cramèr-Rao lower bound theory is presented to show asymptotical optimality of FMAP. Also Givens rotation is used to derive a fast implementation with complexity $O(n)$ ($n$ denoting number of targets). Experiments are presented to demonstrate the promise and effectiveness of FMAP.

I. INTRODUCTION

Sensor systems in a robotic system are typically calibrated manually. However, sensor orientation and signal output may drift during the life of the sensor, such that the orientation of the sensor relative to the robot frame is changed. When the sensor orientation drifts, measurements become skewed relative to the robot. When there are multiple sensors, this concern is further complicated. It is thus desirable to have sensor systems that automatically align sensor output to the robot frame.

Two categories of approaches have been attempted the registration problem. The first category decouples tracking and registration into separate problems. In [3], [6], filters are designed to estimate the sensor biases by minimizing the discrepancy between measurements and associated fused target estimates, from a separated tracking module. However, these methods are not optimal in term of Cramer-Rao bounds [4]. In the second category, the approaches jointly solve for target tracking and sensor registration. For example, [1], [2], [5] have applied the extended Kalman filtering (EKF) to an augmented state vector combining the target variables and sensor system errors. However, there are a few difficulties with these approaches. Since the registration parameters are constant, the error model of state-space is degenerated. This not only makes the estimation problem larger-leading to higher computational cost (complexity $O(n^3)$ with $n$ denoting number of targets), but also results degenerated covariance matrices for the process noise vectors.

In this paper, we propose a fast maximum a posterior (FMAP) registration algorithm to tackle the problems. We combine all the measurement equations and process equations to form a linearized state-space model. The registration and target track estimates are obtained by maximizing a posterior function in the state space. The performance of FMAP estimates is examined using Cramèr-Rao lower bound theory. By exploiting the sparseness of the model, a fast implementation whose complexity scales linearly with the numbers of targets and measurements is derived.

The rest of this paper is organized as follows. Section II is devoted to the algorithm derivation. An illustrative example is given in Section III. The results of experiments are presented in Section IV. Finally we give conclusions in Section V.

Throughout this paper italic upper and lower case letters are used to denote matrices and vectors, respectively. A Gaussian distribution is denoted by information array. For example, a multivariate $x$ with density function $N(x, Q)$ is denoted as $p(x) \propto e^{-\frac{1}{2}(x-z)^TQ^{-1}(x-z)}$ or the information array $[R, z]$ in short, where $z = Rx$ and $Q = R^{-T}R^{-1}$.

II. ALGORITHM DERIVATION

In this section, we address the computational issues in performing EKF to solve joint tracking-registration problem. Fig. 1 shows the results of a joint problem with ten tracks and six registration parameters. The normalized covariance of the joint state from EKF is visualized in Fig. 1(a). Dark entries indicate strong correlations. It is clear that not only the tracks $x$ and the registration $a$ are correlated but also each track in $x$ is mutually correlated. The checkerboard appearance of the joint covariance matrix illustrates this fact. Therefore, the approximation that ignores the off-diagonal correlated entries [3], [6] is not asymptotical optimal.

A key insight that motivates the proposed approach is shown in Fig. 1(b). Shown there is the Cholesky factor of the inverse covariance matrix (also known as information matrix)

![Fig. 1. Typical snapshot of correlation matrices applied to the joint registration problem with 10 tracks $x$ and 6 registration parameters $a$. (a) a correlation matrix $P$ of EKF (normalized). (b) Normalized Cholesky factor $R$ of inverse covariance.](image-url)

S. Zeng is with Electrical & Controls Integration Laboratory, General Motors R&D Center, 30500 Mound Road, Warren, MI 48090 shuqing.zeng@gm.com
The system dynamics equation for the state is expressed as:
\[ s(t + 1) = f(s(t), w(t)) \]
where the function relates the state at time \( t \) to the state at time \( t+1 \); and where terms \( w \) are vectors of zero-mean noise random variables that are assumed to have nonsingular covariance matrices.

The measurement process can be modeled symbolically as a function of target track \( x \), and registration parameter \( a \), such as
\[ o(t) = h(x(t), a(t)) + v(t) \]
where \( o(t) \) and \( v(t) \) denotes the measurements and the additive noise vectors at time instant \( t \).

### B. Measurement update

The FMAP registration algorithm works with the posterior density function \( p(s(t) \mid o_{o(0)\cdots t}) \), where \( o_{o(0)\cdots t} \) denotes a series of measurements \( \{o(0), \ldots, o(t)\} \).

Using the Bayes rule and Markovian assumption, we obtain the posterior function as
\[
p(s(t) \mid o_{o(0:t)}) = p(s(t) \mid o_{o(0:t-1)}, o(t)) \propto p(o(t) \mid o_{o(0:t-1)}, s(t)) p(s(t) \mid o_{o(0:t-1)})
\]
\[
\propto p(o(t) \mid s(t)) p(s(t) \mid o_{o(0:t-1)})
\]
Assuming the density functions are normally distributed, the prior function\(^1\) \( p(s \mid o_{o(0:t-1)}) \) can be expressed by the information array \( [\bar{R}, \bar{z}] \), i.e.,
\[
p(s \mid o_{o(0:t-1)}) \propto e^{-\frac{1}{2} s^T \bar{z}^T \bar{z} s}
\]
Linearizing (2) using Taylor expansion in the neighborhood \([x^*, a^*] \), produces:
\[ o = C_x x + C_a a + u_1 + v \]
with \( u_1 = h^* - C_x x^* - C_a a^* \), where \( h^* = h(x^*, a^*) \), and Jacobian matrices \( C_x \) and \( C_a \). Without loss of generality, the covariance matrix of \( v \) is assumed to be an identity matrix\(^2\).

Thus, the measurement function can be written as
\[
p(o \mid s) \propto e^{-\frac{1}{2} (s - \hat{s})^T \hat{C}_s^{-1} (s - \hat{s})}
\]

Plugging in Eqs. (4) and (6), the negative log likelihood function of (3), ignoring the constant term, is given by,
\[
J_t = \left\| \begin{bmatrix} \bar{R}_a \\ \bar{R}_{x_a} \\ C_x \\ C_a \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} - \begin{bmatrix} \bar{z}_x \\ \bar{z}_a \\ 0 \\ -u_1 \end{bmatrix} \right\|^2
\]

The principle of the maximum a posterior (MAP) estimation is to maximize the posterior function with respect to the unknown variables \( (x, a) \). Clearly, the maximization process

\(^1\)Unless it is necessary, we will not include time such as \( (t) \) in all the following equations.

\(^2\)If not, the noise term \( v \) in (5) can be transformed to a random vector with identity covariance matrix. Let \( \text{cov}(v) = L_v \) denote the covariance matrix of the measurement model. Multiplying both sides of (5) by \( L_v \), the square root information matrix of \( R_v \), results in a measurement equation with an identity covariance matrix.
is equivalent to minimization of the least squares in (7). The right side of (7) can be written as a matrix $X$ expressed as:

$$X = \begin{bmatrix} \hat{R}_x & \hat{R}_{xa} & \hat{z}_x \\ 0 & \hat{R}_a & \hat{z}_a \\ \tilde{C}_x & \tilde{C}_a & o - u_1 \end{bmatrix} \tag{8}$$

$X$ can be turned into a triangular matrix by applying an orthogonal transformation $\hat{T}$:

$$\hat{T}X = \begin{bmatrix} \hat{R}_x & \hat{R}_{xa} & \hat{z}_x \\ 0 & \hat{R}_a & \hat{z}_a \end{bmatrix} \tag{9}$$

where $e$ is the residual corresponding to the second term of the right side of (7), reflecting the discrepancy between the model and measurement. Substituting (9) into (7), and using the orthogonal property of $\hat{T}$, produces:

$$J_t = | e | + \left| \begin{bmatrix} \hat{R}_x & \hat{R}_{xa} \\ 0 & \hat{R}_a \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} - \begin{bmatrix} \hat{z}_x \\ \hat{z}_a \end{bmatrix} \right|^2 \tag{10}$$

Therefore, the posterior density given the measurements $o(0:t)$ is written as

$$p(s | o(0:t)) \propto e^{-\frac{1}{2} \left| \begin{bmatrix} \hat{R}_x & \hat{R}_{xa} \\ 0 & \hat{R}_a \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} - \begin{bmatrix} \hat{z}_x \\ \hat{z}_a \end{bmatrix} \right|^2} \tag{11}$$

or in the information array form

$$\begin{bmatrix} \hat{R}_t, \hat{z}_t \end{bmatrix} = \begin{bmatrix} \hat{R}_x & \hat{R}_{xa} \\ 0 & \hat{R}_a \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} - \begin{bmatrix} \hat{z}_x \\ \hat{z}_a \end{bmatrix} \tag{12}$$

(12) allows us to solve the estimates of the track variables $\hat{x}$ and registration parameters $\hat{a}$ by back-substitution using $\hat{R}$ and right hand side $\hat{z}$, i.e.,

$$\begin{bmatrix} \hat{R}_x & \hat{R}_{xa} \\ 0 & \hat{R}_a \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} = \begin{bmatrix} \hat{z}_x \\ \hat{z}_a \end{bmatrix} \tag{13}$$

C. Time propagation

The linear approximation of the system dynamics in (1) in the neighborhood $[s^*, w^*]$ can be expressed as

$$\begin{bmatrix} x(t+1) \\ a(t+1) \end{bmatrix} = \begin{bmatrix} \Phi_x & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} + \begin{bmatrix} G_x \\ 0 \end{bmatrix} w + u_2 \tag{14}$$

where $\Phi_x$ and $G_x$ are Jacobian matrices; and the nonlinear term $u_2 = f(s^*, w^*) - \Phi_x x^* - G_x w^*$.

If multivariate $s$ and $w$ are denoted by the information arrays in (12) and $[\tilde{R}_w, \tilde{z}_w]$, respectively, the joint density function given the measurements $o(0:t)$ can be expressed in terms of $x(t+1)$ and $a(t+1)$ as

$$p(s(t+1), w | o(0:t)) \propto e^{-\frac{1}{2} A \begin{bmatrix} w(t+1) \\ a(t+1) \end{bmatrix} - \begin{bmatrix} \tilde{z}_w + \tilde{R}_x \Theta^{-1} \tilde{w}_2 \end{bmatrix}^2} \tag{15}$$

where

$$A = \begin{bmatrix} \tilde{R}_w & 0 \\ 0 & \tilde{R}_w \end{bmatrix}$$

The exponent term in (15) can be denoted as matrix $Y$. After apply an orthogonal transformation $\hat{T}$ to turn the matrix $Y$ into a triangular matrix such as

$$\hat{T}Y = \begin{bmatrix} \hat{R}_w(t+1) & \hat{R}_{wa}(t+1) & \hat{z}_w(t+1) \\ 0 & \hat{R}_a(t+1) & \hat{z}_a(t+1) \end{bmatrix} \tag{16}$$

with

$$Y = \begin{bmatrix} -\hat{R}_w \Theta^{-1} G_x & \hat{R}_w \Theta^{-1} \hat{R}_a & \hat{z}_w + \hat{R}_w \Theta^{-1} \tilde{w}_2 \\ 0 & 0 \end{bmatrix} \tag{17}$$

Therefore, given the measurements $o(0:t)$, the updated prior function $q$ can be produced by marginalization on variable $w$, i.e.,

$$q \equiv p(s(t+1) | o(0:t)) = \int p(s(t+1), w | o(0:t))dw$$

and be expressed as, ignoring normalization factor

$$q \propto e^{-\frac{1}{2} \left| \begin{bmatrix} \hat{R}_w(t+1) & \hat{R}_{wa}(t+1) & \hat{z}_w(t+1) \\ 0 & \hat{R}_a(t+1) & \hat{z}_a(t+1) \end{bmatrix} \right|^2} \tag{18}$$

Therefore, we obtain the updated prior information array $[\tilde{R}(t+1), \tilde{z}(t+1)]$ at time $t+1$ as:

$$[\tilde{R}(t+1), \tilde{z}(t+1)] = \begin{bmatrix} \tilde{R}_w(t+1) & \tilde{R}_{wa}(t+1) & \tilde{z}_w(t+1) \\ 0 & \tilde{R}_a(t+1) & \tilde{z}_a(t+1) \end{bmatrix} \tag{19}$$

The derivation leading to (19) illuminates the purposes of using information arrays $[\tilde{R}, \tilde{z}]$, which is recursively updated to $[\tilde{R}(t+1), \tilde{z}(t+1)]$ at time instant $t+1$.

D. Statistical Efficiency Analysis

The Cramér-Rao lower bound (CRLB) is a measure of statistical efficiency of an estimator. Let $p(o(0:t), s)$ be the joint probability density of the state (parameters) $s$ at time instant $t$ and the measured data $o(0:t)$, and let $g(o(0:t))$ be a function of an estimate of $s$. The CRLB for the estimation error has the form

$$P = E \left[ (g(o(0:t)) - s) [g(o(0:t)) - s]^T \right] \geq J_t \tag{20}$$

where $J_t$ is the Fisher information matrix with the elements

$$J_t^{(ij)} = E \left\{ \frac{\partial^2 \log p(o(0:t), s)}{\partial s_i \partial s_j} \right\}$$

Proposition 2.1: Consider the system defined by (1) and (2), the Fisher information matrix of the system $J_t$ at time $t$ is identical to $[\tilde{R}(t)]'[\tilde{R}]$, with $\tilde{R}(t)$ defined in (12).

Proof: The join probability density can be derived from equality $p(o(0:t), s) = p(s | o(0:n))p(o(0:n))$. Since $p(o(0:n))$ is a function of measured data, not depending on the state $s$; therefore, we have $J_t = E(\Delta \log p(s | o(0:n))$ where $\Delta$ denotes the second-order partial derivative, i.e., $\Delta L(s) = \frac{\partial L(s)}{\partial s} \frac{\partial L(s)}{\partial s}$. \hspace{1cm} \Box$

Plugging (11) in, the logarithm of the posterior function $\log p(s | o(0:n))$ reads

$$- \log p(s | o(0:n)) = c_0 + \frac{1}{2} \left| \tilde{R}s - \tilde{z} \right|^2$$
where $c_0$ denotes a constant independent of $s$. Then the Fisher information matrix $J_t$ reads $J_t = \tilde{R}(t)^t \tilde{R}$. 

**Remark** If $s$ is estimated by $g(s) = E(s|a(0:t)) = \tilde{R}^{-1} \tilde{z}$, then (20) is satisfied with equality. Therefore the FMAP algorithm is optimal in sense of CRLB.

**E. Fast Implementation**

We have observed the FMAP algorithm comprises two factorization operations outlined in Eqs. (9) and (16), and back substitution operation (13). However, directly applying matrix factorization techniques (e.g., QR decomposition) can be computational ineffective as EKF since the complexity of QR is $O(n^3)$ ($n$ denotes number of targets).

Note that the matrix in (8) is sparse. Fig. 3(a) illustrates an example schematically with two tracks, two registration parameters, and six measurements. The non-zero elements of the matrix in (8) are denoted by crosses; and blank position represents a zero element.

Givens rotation is used to eliminate the non-zero elements of the matrix $C_x$ in (8), shown in Fig. 3(a) as crosses surrounded by cycle. Givens rotation is applied from the left to the right and for each column from the top to the bottom. Each non-zero low-triangular element in the $i$-th row of the matrix block $C_x$, is combined with the diagonal element in the same column in the matrix block $\tilde{R}_x$ to construct the rotation. If the element in $C_x$ is zero, then no rotation is needed.

**Lemma 2.1:** If $\tilde{R}_x$ in (8) has block-diagonal form, then 1) the result $\tilde{R}_x(t+1)$ in (16) has the same form as $\tilde{R}_x$; and 2) the complexity of the triangulation process in (16) is $O(nk)$, with $n$ denoting the number of targets.

**Proof:** Each individual target has its own system dynamics equation. For the $i$-th target, (14) can be expressed as:

$$x_i(t+1) = \Phi_i x(t) + G_i \xi_i(t) + u_{2i}$$

where $\Phi_i$ and $G_i$ are Jacobian matrices; the nonlinear term $u_{2i} = f(x_i^*, w_i^*) - \Phi_i x_i^* - G_i \xi_i^*$; and $\xi_i(t)$ is represented by the information array $[\tilde{R}_{w_i}, \tilde{z}_{w_i}]$. Therefore, the corresponding collective quantities $\Phi_x, \Xi_x, G_x$, and $\tilde{R}_w$ are in the same block-diagonal form. Since each off-diagonal zero entry in $\tilde{R}_x \Phi_x^{-1} G_x$ pairs up with a zero entry in the rotation operations, the off-diagonal entry remains zero. Thus 1) is established.

Also as Fig. 3(b) shows, to eliminate an element to zero, a maximum $O(2N_x + kn_a)$ rotations are needed. Providing $n$ tracked objects, $O(nN_x^2)$ elements need to be eliminated to zero. Givens rotation is applied a maximum $O(2kn^2 + kn_n N_x^2)$ times to transform the matrix $Y$ into a triangular matrix, which is equivalent to $O(nk)$ additive and multiplicative operations. 2) is established.

**Lemma 2.3:** The complexity of the back substitution operation expressed in (13) is $O(n + k^3)$.

**Proof:** Since $\tilde{R}_x$ is a block-diagonal matrix, $O(nN_x^2)$ and $O((kn_a)^2)$ operations are needed to solve $\tilde{x}$ and $\tilde{a}$, respectively. Therefore, the complexity is $O(n + k^3)$.

To summarize, we establish the following proposition:

**Proposition 2.2:** The complexity of the FMAP algorithm is $O(m + n)k^2 + k^3)$, where $m, n$ and $k$ denote the numbers of measurement equations, targets and sensors.
Proof: The FMAP algorithm comprises of three matrix operations whose complexity are specified by Lemmas 2.1, 2.2 and 2.3, respectively. Thus the total complexity is $O(mk^2 + nk + n + k^3) = O((m + n)k^2 + k^3)$. 

Remark The above proposition shows that the complexity of the FMAP algorithm scales linearly with the number of measurements and with the number of target tracks.

III. AN ILLUSTRATIVE EXAMPLE

By way of an example, we restate the flow chart of the presented algorithm. The schematic illustration of Fig. 4 includes the sensors mounted on the exemplary vehicle at positions A and B, preferably mounted at the front of the vehicle. A single target T, in front and in the same lane as the vehicle, moves away from the vehicle.

![Fig. 4. An example](image)

In the scenario illustrated in Fig. 4, the positions of the sensors A and B are denoted by $(x_{A0}, y_{A0})$ and $(x_{B0}, y_{B0})$, respectively. The orientations of the sensors A and B are denoted by $\psi_A$ and $\psi_B$, respectively. The target is located in position $(x, y)$ in the XY-coordinate system.

IV. SIMULATION COMPARISONS

As shown in Fig. 4, two sensors placed near to the front bumper are simulated. Assume that Sensor A is located at $(2, 0.6)$, oriented $10^\circ$ outwards from the vehicle’s bore-sight. Sensor B is located at $(2, -0.6)$, oriented $-10^\circ$ away from the vehicle’s bore-sight. The random walking parameters in (26) are expressed as $q_x = 0.1$ and $q_y = 0.1$. The measurement noise variance $(\sigma_r, \sigma_{rr}, \sigma_\theta)$ is set to $(0.1, 0.2, 1^\circ)$. Three algorithms are implemented: SEP, EKF, and FMAP. SEP [3] treats the registration and tracking problems separately; but the EKF and FMAP algorithms address the problems jointly. Each algorithm has been implemented in Matlab on a 2 GHz Intel Core 2 Duo processor running Windows XP. No special care has been taken to produce efficient code.

Algorithm 1 FMAP update

**Require:** Given the a priori at instant $t$ (i.e., previous results and its uncertainty measure) expressed as information array $[\hat{R}, \hat{z}]$ and measurements $o$

**Ensure:** The updated estimate of $s$, expressed by $\hat{s}$$$

1: Plugging the a priori $[\hat{R}, \hat{z}]$ and sensor measurements expressed in Eqs. (21), (22), and (23) into matrix $X$ (c.f., (8));
2: Factorizing $X$ by Givens rotation;
3: Deriving the posterior density information array as shown in (12);
4: Calculating an update of tracking and the correction of registration of the sensors as expressed in (13);
5: Plugging Eqs. (24) and (25) into $Y$;
6: Factorizing $Y$ by Givens rotation;
7: Deriving the a priori information array $[\hat{R}(t + 1), \hat{z}(t + 1)]$ for time $t + 1$ (c.f., (19)), which can be utilized when the new sensor measurements are available.

IV. SIMULATION COMPARISONS

As shown in Fig. 4, two sensors placed near to the front bumper are simulated. Assume that Sensor A is located at $(2, 0.6)$, oriented $10^\circ$ outwards from the vehicle’s bore-sight. Sensor B is located at $(2, -0.6)$, oriented $-10^\circ$ away from the vehicle’s bore-sight. The random walking parameters in (26) are expressed as $q_x = 0.1$ and $q_y = 0.1$. The measurement noise variance $(\sigma_r, \sigma_{rr}, \sigma_\theta)$ is set to $(0.1, 0.2, 1^\circ)$. Three algorithms are implemented: SEP, EKF, and FMAP. SEP [3] treats the registration and tracking problems separately; but the EKF and FMAP algorithms address the problems jointly. Each algorithm has been implemented in Matlab on a 2 GHz Intel Core 2 Duo processor running Windows XP. No special care has been taken to produce efficient code.

Proof: The FMAP algorithm comprises of three matrix operations whose complexity are specified by Lemmas 2.1, 2.2 and 2.3, respectively. Thus the total complexity is $O(mk^2 + nk + n + k^3) = O((m + n)k^2 + k^3)$.
In the first experiment, ten targets are generated in the field-of-view of the sensors. We initially set the registration parameters of the sensors randomly with small numbers. Tracks are initialized use the first measurements from one of the sensors. Fig. 5 shows the error curves of the registration parameters of Sensor B for the three algorithms. Fig. 6 shows the error curves of the estimates of a target. The error performance curves of EKF and FMAP are indistinguishable and are both superior to that of SEP.

In the next experiment, we varies the number of simulated targets from 10 to 300. Fig. 7 demonstrates that the execution time of FMAP and SEP is similar and are both an order of magnitude lower than that of EKF.

V. CONCLUSIONS

In this paper, we have addressed the recursive joint tracking-registration problem. A fast algorithm (FMAP) whose time complexity scales linearly with the numbers of measurements and targets is derived. It is proved that FMAP is asymptotically optimal and has an $O(n)$ implementation. The results from experiments on synthetic data demonstrate that, as expected, FMAP consistently performs better than methods where tracking and registration are treated separately. It has been demonstrated experimentally that the complexity of FMAP is indeed $O(n)$, and the execution time is an order of magnitude lower than that of EKF.

REFERENCES