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An Improved Nonlinear H_∞ Synthesis for Parameter-Dependent Polynomial Nonlinear Systems Using SOS Programming

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Abstract— **State feedback control synthesis problems for a class of polynomial nonlinear systems are investigated in this paper. Less conservative sufficient conditions to achieve the closed-loop stability with or without H_∞ performance are formulated in terms of state dependent matrix inequalities. By introducing additional matrix variables, we succeed in eliminating the involved coupling between system matrices and the Lyapunov matrix. Hence the proposed methodology can be extended to the synthesis for the parameter-dependent polynomial systems.**

I. INTRODUCTION

Despite many important advances for nonlinear stability analysis and feedback synthesis presented [1], [2], the construction of stabilizing nonlinear control laws remains a challenging task.

Recently, a new computational relaxation based on the sum of squares decomposition for multivariable polynomials and semidefinite programming [3] provides potentially effective ways for the analysis and synthesis of nonlinear systems. This crucial property of the new nonlinear analysis method motivates great interest in sum of squares polynomials and sum of squares optimization in nonlinear control problems [4], [5], [7], [9], [10].

In this paper we revisit the Lyapunov-based state feedback synthesis for a class of nonlinear systems whose dynamics are described by polynomials. In particular, we represent the nonlinear systems in a state dependent linear-like form. Sufficient conditions to guarantee the closed-loop stability with or without bounded H_∞ performance are derived in terms of nonlinear matrix inequalities, where polynomial state feedback controllers (instead of rational controllers) are provided.

By introducing additional matrix variables, we derive a more general sufficient condition for the stabilizing controller synthesis with bounded H_∞ performance, which provides extra freedom for the controller design and the conservatism involved can be reduced. Because of the decoupling between system matrices and the Lyapunov matrix, the proposed methodology is extended to the synthesis for the parameter-dependent polynomial systems, and the feasibility of state dependent matrix inequality (SDMI) based conditions derived for parameter-dependent systems is tested by different Lyapunov matrices. Finally, these SDMI based conditions are formulated as sum-of-squares based constraints, which can be solved via the semidefinite programming relaxations based on the sum of squares decomposition.

In stead of assuming system's controller and disturbance matrices have some zero rows and Lyapunov matrix $P(\tilde{x})$ only depends on states $\tilde{x}$ whose corresponding rows in controller and disturbance matrices are zeroes [8], in this paper we define the Lyapunov matrix $P(x)$ that depends on the full order states x. Following the technique in [6], a more general relaxation is provided to deal with the nonconvex terms in the matrix inequalities. In the synthesis for the parameter-dependent systems, parameter and state dependent Lyapunov functions (instead of fix ones) are used to reduce the conservatism involved in the controller design.

The remainder of the paper is organized as follows: Section II provides preliminary material on multivariate polynomial. The basic idea and the proposed state feedback design methodology for nonlinear stability control problem and H_∞ performance design problem are described in Section III and IV. The extension of the results to the parameter-dependent polynomial systems is given in Section V. Concepts on the sum of squares decomposition and its application to solving SDMIs are also presented. Examples are provided in Section VI to demonstrate the proposed methodology. Finally, some concluding remarks are given in Section VII.

II. PRELIMINARIES

Let $\mathbb{R}$ denote the set of the real numbers, $\mathbb{R}^n$ is the n-dimensional real space. I_n stands for the identity matrix of $n\times n$ dimension. $P > 0$ ($P \geqslant 0$) means the matrix is positive (semi)definite.

We define $\mathcal{R}_n$ to be the set of all polynomials in n variables with real coefficients. A polynomial vector field $f : \mathbb{R}^n \to \mathbb{R}^n, f(x) = [f_1(x), \ldots, f_n(x)]^T$ is a vector field with $f_i \in \mathcal{R}_n$, i.e., the entries of the vector field are polynomial functions in $x \in \mathbb{R}^n$, and is denoted as $f(x) \in \mathcal{R}_n^n$.

*Definition 1 (Sum of Squares Polynomial):*A polynomial $p(x)$ in n variables is a sum of squares (SOS) polynomial if there exist $f_i(x) \in \mathcal{R}_n$, $i = 1, \ldots, m$ such that

$$p(x) = \sum_{i=1}^{m} f_i^2(x) \tag{1}$$

Define Σ_{sos} to be the set of sum of squares polynomials. Given a polynomial function $p(x)$, checking the global nonnegativity of $p(x)$ is in fact NP-hard. In [3] SOS decomposition (1) has been shown equivalent to the existence of a positive semidefinite matrix Q such that

$$p(x) = Z^T(x)QZ(x) \tag{2}$$

where $Z(x)$ is the vector of all monomials of degree less than or equal to the half degree of $p(x)$ given by different products of x. The equivalence between (1) and (2) makes the SOS decomposition computable via semidefinite programming.

The following notation is adopted: $*$ indicates symmetric entries in a symmetric matrix.

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III. STABILITY SYNTHESIS

A. Problem Formulation

Consider the following input-affine nonlinear time-invariant (NLTI) system which is in the state dependent linear-like representation:

$$\dot{x} = A(x)Z(x) + B(x)u \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input. $A(x)$ and $B(x)$ are polynomial matrices in x with appropriate dimensions, and $Z(x)$ is an $N \times 1$ vector of monomials in x satisfying the following assumption.

Assumption 1 $Z(x) = 0$ iff $x = 0$.

Remark 3.1: It should be noted that, given $f(x) \in \mathcal{R}_n^n$, the representation $f(x) = A(x)Z(x)$ is highly non-unique. Notice that for any $L(x)$ with $L(x)Z(x) = 0$, $A(x) + L(x)$ can also be used as a representation for $f(x)$. A special case of the representation corresponds to $Z(x) = x$, while at the other extreme, $Z(x)$ can be selected to contain all the monomials in $f(x)$ (in which case, $A(x)$ becomes a constant matrix).

Let $M(x)$ be a $N \times n$ polynomial matrix whose $(i,j)^{\text{th}}$ entry is given by

$$M_{ij}(x) = \frac{\partial Z_i}{\partial x_j}(x), \quad i = 1, \ldots, N, j = 1, \ldots, n \tag{4}$$

Our objective is to design a state feedback control law (5) which stabilizes the closed-loop system. Here we have not included any performance objective in the synthesis.

$$u = K(x)Z(x) \tag{5}$$

We have the following closed-loop system

$$\dot{x} = [A(x) + B(x)K(x)]\, Z(x) = \tilde{A}(x)Z(x) \tag{6}$$

B. Stabilizing State Feedback Design

Consider the following Lyapunov function

$$V(x) = Z^T(x)P(x)Z(x) \tag{7}$$

with $P(x)$ is nonsingular and $P(x) = P^T(x) > 0$.

Theorem 1: Consider system (3). If there exist $Q(x) = Q^T(x) > 0$ and $Y(x)$ such that the following nonlinear matrix inequality is satisfied

$$\begin{aligned} &M(x)A(x)Q(x) + Q(x)A^T(x)M^T(x) + M(x)B(x)Y(x) \\ &\quad + Y^T(x)B^T(x)M^T(x) + Q(x)\dot{P}(x)Q(x) < 0 \end{aligned} \tag{8}$$

then the control law (5) stabilizes the system with

$$K(x) = Y(x)Q^{-1}(x) \tag{9}$$

Proof: With the closed-loop system matrix $\tilde{A}(x)$ in (6) and controller matrix in (9), it is very easy to obtain $\dot{V}(x) < 0$. Then we know that the system is stable with state feedback control law (5). □

Remark 3.2: When $Y(x)$ and $Q(x)$ are polynomial matrices, $K(x)$ in (5) is rational. Although a possible solution to avoid rational controller gain can be obtained with a constant matrix Q, it is obviously conservative. Moreover, when parameters are involved in system matrices, the stability inequality is not convex due to the product of $Q(x)$ by $\tilde{A}(x)$. A more suitable structure is proposed to deal with the non-convexity problem by the following sufficient condition.

Theorem 2: Consider system (3). If there exist $Q(x) = Q^T(x) > 0$, $Y(x)$, and constant nonsingular matrix G such that, for a positive tuning scalar β, the nonlinear matrix inequality (10) is satisfied

$$\begin{bmatrix} \begin{array}{c} [M(x)A(x)G + G^TA^T(x)M^T(x) \\ +M(x)B(x)Y(x) + Y^T(x)B^T(x)M^T(x) \\ +Q(x)\dot{P}(x)Q(x)] \end{array} & * \\ \begin{array}{c} [-Q(x) + G - \beta G^TA^T(x)M^T(x) \\ -\beta Y^T(x)B^T(x)M^T(x)] \end{array} & -\beta(G + G^T) \end{bmatrix} < 0 \tag{10}$$

then the control law (5) stabilizes the system with

$$K(x) = Y(x)G^{-1} \tag{11}$$

Proof: Due to the limit of the space, it is omitted here. □

Remark 3.3: (a) The advantage of (10) is that it separates the system matrix and Lyapunov matrix, and it only involves affine matrix variable terms in $Q(x)$, G and $Y(x)$. (b) By choosing G as a constant matrix, it is easy to see that $K(x) = Y(x)G^{-1}$ is a polynomial matrix in x.

C. Sum of Square Based Optimization

The stability condition for the state feedback design is based on a nonlinear matrix inequality. Solving this inequality means solving an infinite set of LMIs. When only symmetric polynomial matrices are involved, the SOS decomposition can provide a computational relaxation for the sufficient conditions in Theorem 2 [8].

The nonlinear matrix inequality (10) is not convex because of the existence of the nonlinear term $Q(x)\dot{P}(x)Q(x)$. Following the technique in [6], the transformation of this nonconvex term into a convex one is discussed as follows.

Notice the nonlinear term in (10), we have

$$\begin{aligned} & v_1^T \left[\sum_{j=1}^{n} \frac{\partial Q(x)}{\partial x_j} B_j(x)K(x)Z(x) \right] v_1 \\ = & \sum_{j=1}^{n} v_1^T \frac{\partial Q(x)}{\partial x_j} v_1 B_j(x)K(x)Z(x) \\ = & \left[\; v_1^T \tfrac{\partial Q(x)}{\partial x_1} v_1, \ldots, v_1^T \tfrac{\partial Q(x)}{\partial x_n} v_1 \; \right] B(x)K(x)Z(x) \\ \triangleq & \; \phi_1(x, v_1)B(x)u(x) \end{aligned} \tag{12}$$

We can bound the effect of the nonlinear term as in (13). Then we know that if γ_1 in (13) has zero minimum, the nonlinear term $Q(x)\dot{P}(x)Q(x)$ in Theorem 2 can be replaced by the linear part $-\sum_{j=1}^{n} \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x)$.

$$\begin{bmatrix} \gamma_1 & \phi_1(x, v_1)B(x) \\ * & I \end{bmatrix} \geqslant 0 \tag{13}$$

Proposition 3.1: Consider the nonlinear system (3). If there exist $Q(x) = Q^T(x)$, $Y(x)$, and nonsingular G such

that, for a positive tuning scalars β, a constant $s_1 > 0$ and a sum of squares polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following optimization problem has zero optimum,

$$\text{Minimize} \quad \gamma_1$$

subject to

$$v_1^T \Big[Q(x) - s_1 I \Big] v_1 \in \Sigma_{sos} \tag{14}$$

$$-\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Upsilon(x) + s_2(x) I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \Sigma_{sos} \tag{15}$$

$$v_3^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_1) B(x) \\ * & I \end{bmatrix} v_3 \in \Sigma_{sos} \tag{16}$$

where $\phi_1(x, v_1)$ is as in (12) and

$$\Upsilon(x) = \begin{bmatrix} \begin{array}{c} M(x)A(x)G + G^T A^T(x) M^T(x) \\ +M(x)B(x)Y(x) + Y^T(x)B^T(x)M^T(x) \\ -\sum_{j=1}^{n} \frac{\partial Q(x)}{\partial x_j} A_j(x) Z(x) \end{array} & * \\ \begin{array}{c} -Q(x) + G - \beta G^T A^T(x) M^T(x) \\ -\beta Y^T(x) B^T(x) M^T(x) \end{array} & -\beta(G + G^T) \end{bmatrix} \tag{17}$$

then the state feedback stabilization problem is solvable, and the control law (5) stabilizes the system with (11).

IV. H_∞ PERFORMANCE SYNTHESIS

A. Problem Formulation

Consider the following nonlinear system which is in the state dependent linear-like representation:

$$\begin{aligned} \dot{x} &= A(x)Z(x) + B_u(x)u + B_w(x)w \\ z &= C_z(x)Z(x) + D_z(x)u \end{aligned} \tag{18}$$

where $w \in \mathbb{R}^p$ is the disturbance, $z \in \mathbb{R}^{n_z}$ is the objective signal to be regulated. Define $M(x)$ and $Z(x)$ as in Section III.

Our objective is to design a state feedback control law (5) such that the closed-loop system is stable and the induced L_2 gain from w to z is minimized. With Controller (5) we have the closed-loop system as follows:

$$\begin{aligned} \dot{x} &= [A(x) + B_u(x)K(x)]\, Z(x) + B_w(x)w \\ &= \tilde{A}(x)Z(x) + B_w(x)w \\ z &= [C_z(x) + D_z(x)K(x)]\, Z(x) = \tilde{C}(x)Z(x) \end{aligned} \tag{19}$$

B. Stabilizing State Feedback Design

We define the Lyapunov function as in Section III.

Theorem 3: Consider system (18). If there exist $Q(x) = Q^T(x) > 0$ and $Y(x)$ such that the following nonlinear matrix inequality is satisfied

$$\begin{bmatrix} \begin{array}{c} [M(x)A(x)Q(x) + Q(x)A^T(x)M^T(x) \\ +M(x)B_u(x)Y(x) + Y^T(x)B_u^T(x)M^T(x) \\ +Q(x)\dot{P}(x)Q(x)] \\ B_w^T(x)M^T(x) \\ C_z(x)Q(x) + D_z(x)Y(x) \end{array} & \begin{array}{cc} * & * \\ -\gamma^2 I & * \\ 0 & -I \end{array} \end{bmatrix} < 0 \tag{20}$$

then the control law (5) stabilizes the system and achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$ with

$$K(x) = Y(x)Q^{-1}(x) \tag{21}$$

Proof: With the closed-loop system matrix $\tilde{A}(x)$ and $\tilde{C}(x)$ in (19) and controller matrix in (21), the nonlinear matrix inequality (20) is equivalent to

$$\begin{bmatrix} \begin{array}{c} [Q(x)\tilde{A}^T(x)M^T(x) + M(x)\tilde{A}(x)Q(x) \\ +Q(x)\dot{P}(x)Q(x)] \\ B_w^T(x)M^T(x) \\ \tilde{C}(x)Q(x) \end{array} & \begin{array}{cc} * & * \\ -\gamma^2 I & * \\ 0 & -I \end{array} \end{bmatrix} < 0 \tag{22}$$

Multiplying (22) from both sides by diag $\{P(x), I, I\}$, from the Lyapunov function, and by the Schur complement we have $\dot{V} + z^T z - \gamma^2 w^T w < 0$. Then with the zero initial condition, the system is stable and the H_∞ performance is achieved as $\|z\|_2 < \gamma \|w\|_2$ with state feedback control law (5). □

Just like what has been discussed in Remark 3.2, a sufficient condition is proposed to deal with the non-convexity problem, which is more suitable for the case when parameters are involved in system matrices.

Theorem 4: Consider system (18). If there exist $Q(x) = Q^T(x) > 0$, nonsingular constant matrix G_{11}, $G_{21}(x)$, $G_{22}(x)$, $H_{21}(x)$, $H_{22}(x)$ and $Y(x)$ such that, for some positive tuning scalar β, the nonlinear matrix inequality (23) is satisfied,

$$\Gamma_h(x) = \begin{bmatrix} \Gamma_{h_{11}}(x) & \hat{B}(x) & \Gamma_{h_{13}}(x) \\ * & -\gamma^2 I & 0 \\ * & * & -H(x) - H^T(x) \end{bmatrix} < 0 \tag{23}$$

where $\Gamma_{h_{11}}(x)$ and $\Gamma_{h_{13}}(x)$ are as in (25) on the top of the next page and

$$\begin{aligned} \hat{B}(x) &= \begin{bmatrix} M(x)B_w(x) \\ 0 \end{bmatrix} \\ G(x) &= \begin{bmatrix} G_{11} & 0 \\ G_{21}(x) & G_{22}(x) \end{bmatrix} \\ H(x) &= \begin{bmatrix} \beta G_{11} & 0 \\ H_{21}(x) & H_{22}(x) \end{bmatrix} \end{aligned} \tag{24}$$

then the control law (5) stabilizes the system and achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$ with

$$K(x) = Y(x)G_{11}^{-1} \tag{26}$$

Proof: Due to the limit of the space, it is omitted here. □

C. Sum of Square Based Optimization

Based on the result in Theorem 4, using the similar technique in Section III to deal with the nonlinear term $Q(x)\dot{P}(x)Q(x)$. Define $\phi_1(x, v_{21})$ as

$$\phi_1(x, v_{21}) = \begin{bmatrix} v_{21}^T \frac{\partial Q(x)}{\partial x_1} v_{21}, \ldots, v_{21}^T \frac{\partial Q(x)}{\partial x_n} v_{21} \end{bmatrix} \tag{27}$$

We have the following sum of squares relaxation problem.

Proposition 4.1: Consider the nonlinear system (18). If there exist $Q(x) = Q^T(x)$, G_{11}, $G_{21}(x)$, $G_{22}(x)$, $H_{21}(x)$,

$$\Gamma_{h_{11}}(x) = \begin{bmatrix} M(x)A(x)G_{11} + [M(x)A(x)G_{11}]^T + M(x)B_u(x)Y(x) & * \\ +[M(x)B_u(x)Y(x)]^T + Q(x)\dot{P}(x)Q(x) & \\ C_z(x)G_{11} + D_z(x)Y(x) - \frac{1}{2}G_{21}(x) & -\frac{1}{2}G_{22}(x) - \frac{1}{2}G_{22}^T(x) \end{bmatrix}$$

$$\Gamma_{h_{13}}(x) = -\begin{bmatrix} Q(x) & 0 \\ 0 & I \end{bmatrix} + G^T(x) - \begin{bmatrix} \beta M(x)A(x)G_{11} + \beta M(x)B_u(x)Y(x) & 0 \\ \beta C_z(x)G_{11} + \beta D_z(x)Y(x) - \frac{1}{2}H_{21}(x) & -\frac{1}{2}H_{22}(x) \end{bmatrix} \quad (25)$$

$H_{22}(x)$ and $Y(x)$ such that, for a given H_∞ upper bound γ, some tuning scalar β, a constant $s_1 > 0$ and a sum of squares polynomial $s_2(x)$ with $s_2(x) > 0$ for $x \neq 0$, the following optimization problem has zero optimum,

$$\text{Minimize} \quad \gamma_1$$

subject to

$$v_1^T\Big[Q(x) - s_1 I\Big]v_1 \in \Sigma_{sos} \quad (28)$$

$$-\begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_h(x) + s_2(x)I) \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (29)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_1(x, v_{21})B_u(x) & \phi_1(x, v_{21})B_w(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_5 \in \Sigma_{sos} \quad (30)$$

where $G(x)$ and $H(x)$ possess the structure of (24), $\phi_1(x, v_{21})$ is as in (27),

$$\Upsilon_h(x) = \begin{bmatrix} \Upsilon_{h_{1,1}} & \hat{B}(x) & \Upsilon_{h_{1,3}} \\ * & -\gamma^2 I & 0 \\ * & * & -H(x) - H^T(x) \end{bmatrix} \quad (31)$$

with

$$\Upsilon_{h_{1,1}} = \begin{bmatrix} M(x)A(x)G_{11} + [M(x)A(x)G_{11}]^T & * \\ +M(x)B_u(x)Y(x) + [M(x)B_u(x)Y(x)]^T & \\ -\sum_{j=1}^n \frac{\partial Q(x)}{\partial x_j} A_j(x)Z(x) & \\ C_z(x)G_{11} + D_z(x)Y(x) - \frac{1}{2}G_{21}(x) & -\frac{1}{2}G_{22}(x) - \frac{1}{2}G_{22}^T(x) \end{bmatrix}$$

$$\Upsilon_{h_{1,3}} = \Gamma_{h_{13}}(x) \text{ in (25)} \quad (32)$$

then state feedback controller (5) with (26) achieves the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$.

Remark 4.1: The improvement of the result proposed in Proposition 4.1 lies in that it provides a more general sufficient condition involving additional matrix variables $G(x)$ and $H(x)$ for the stabilizing controller design, which potentially leads to extra freedom in the design so that the conservatism involved can be reduced. We can see that when $G(x) = [G_{11}, 0; 0, I]$ and $H(x) = [\beta G_{11}, 0; 0, I]$, it is the result of Theorem 6 in [11].

V. ROBUST STATE FEEDBACK CONTROLLER

The nonlinear systems considered in Section III and IV assume that all parameters of the systems are known. In this section, we consider systems (3) and (18) whose matrices are not exactly known.

A. Robust Stability Synthesis

Consider the system

$$\dot{x} = A(x;\theta)Z(x) + B(x;\theta)u \quad (33)$$

where $A(x;\theta)$ and $B(x;\theta)$ are polynomial matrices of the form

$$A(x;\theta) = \sum_{i=1}^q A_i(x)\theta_i, \quad B(x;\theta) = \sum_{i=1}^q B_i(x)\theta_i \quad (34)$$

The uncertain constant parameter vector $\theta = [\theta_1, \ldots, \theta_q]^T \in \mathbb{R}^q$ satisfies

$$\theta \in \mathbf{\Theta} \triangleq \Big\{\theta \in \mathbb{R}^q : \theta_i \geqslant 0, i = 1, \ldots, q, \sum_{i=1}^q \theta_i = 1\Big\} \quad (35)$$

With the state feedback controller (5), we have the following closed-loop system:

$$\dot{x} = [A(x;\theta) + B(x;\theta)K(x)]Z(x) = \tilde{A}(x;\theta)Z(x) \quad (36)$$

with

$$\tilde{A}(x;\theta) = A(x;\theta) + B(x;\theta)K(x) = \sum_{i=1}^q \theta_i \tilde{A}_i(x) \quad (37)$$

where $\tilde{A}_i(x) = A_i(x) + B_i(x)K(x)$ is closed-loop matrix at the vertex.

Define the following parameter-dependent Lyapunov function

$$V(x) = Z^T(x)P(x;\theta)Z(x) \quad (38)$$

with $P(x;\theta)$ is nonsingular, $P(x;\theta) = P^T(x;\theta) > 0$ and $P(x;\theta) = \sum_{i=1}^q \theta_i P_i(x)$.

With the results in Section III, the main result for robust stabilizing synthesis can be proposed directly.

Proposition 5.1: Consider the nonlinear system (33). If there exist $Q_i(x) = Q_i^T(x)$, $Y(x)$ and G such that, for a positive tuning scalar β, constants $s_i > 0$ and and sum of squares polynomials $s_{il}(x)$ with $s_{il}(x) > 0$ for $x \neq 0$ $(i, l = 1, \cdots, q)$, the following optimization problem has zero optimum,

$$\text{Minimize} \quad \gamma_1$$

subject to

$$v_1^T\Big[Q_i(x) - s_i I\Big]v_1 \in \Sigma_{sos} \quad (39)$$

$$-\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T (\Upsilon_{il}(x) + s_{il}(x)I) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \Sigma_{sos} \quad (40)$$

$$v_3^T \begin{bmatrix} \gamma_1 & \phi_{1i}(x, v_1)B_l(x) \\ * & I \end{bmatrix} v_3 \in \Sigma_{sos} \quad (41)$$

where $\phi_{1i}(x, v_1)$ is defined as

$$\phi_{1i}(x, v_1) = \left[v_1^T \frac{\partial Q_i(x)}{\partial x_1} v_1, \ldots, v_1^T \frac{\partial Q_i(x)}{\partial x_n} v_1 \right] \quad (42)$$

and

$$\Upsilon_{il}(x) = \begin{bmatrix} M(x)A_i(x)G + G^T A_i^T(x) M^T(x) \\ +M(x)B_i(x)Y(x) + Y^T(x)B_i^T(x)M^T(x) \\ -\sum_{j=1}^n \frac{\partial Q_i(x)}{\partial x_j} A_{lj}(x)Z(x) \\ \\ -Q_i(x) + G - \beta G^T A_i^T(x) M^T(x) \\ -\beta Y^T(x) B_i^T(x) M^T(x) \end{bmatrix} \begin{matrix} * \\ -\beta(G + G^T) \end{matrix} \Bigg] \quad (43)$$

then state feedback controller (5) stabilizes the system with $K(x) = Y(x)G^{-1}$

B. Robust H_∞ Synthesis

Consider the system

$$\begin{aligned} \dot{x} &= A(x;\theta)Z(x) + B_u(x;\theta)u + B_w(x;\theta)w \\ z &= C_z(x;\theta)Z(x) + D_z(x;\theta)u \end{aligned} \quad (44)$$

where constant parameter θ is defined as in (35). $A(x;\theta)$, $B_u(x;\theta)$, $B_w(x;\theta)$, $C_z(x;\theta)$ and $D_z(x;\theta)$ are polynomial matrices of the form

$$\begin{aligned} A(x;\theta) &= \sum_{i=1}^q A_i(x)\theta_i, B_u(x;\theta) = \sum_{i=1}^q B_{ui}(x)\theta_i \\ B_w(x;\theta) &= \sum_{i=1}^q B_{wi}(x)\theta_i, C_z(x;\theta) = \sum_{i=1}^q C_{zi}(x)\theta_i \\ D_z(x;\theta) &= \sum_{i=1}^q D_{zi}(x)\theta_i \end{aligned} \quad (45)$$

With controller (5), we have the closed-loop system:

$$\begin{aligned} \dot{x} &= \tilde{A}(x;\theta)Z(x) + B_w(x;\theta)w \\ z &= \tilde{C}(x;\theta)Z(x) \end{aligned} \quad (46)$$

with

$$\begin{aligned} \tilde{A}(x;\theta) &= A(x;\theta) + B_u(x;\theta)K(x) = \sum_{i=1}^q \theta_i \tilde{A}_i(x) \\ \tilde{C}(x;\theta) &= C_z(x;\theta) + D_z(x;\theta)K(x) = \sum_{i=1}^q \theta_i \tilde{C}_i(x) \end{aligned} \quad (47)$$

where $\tilde{A}_i(x) = A_i(x) + B_{ui}(x)K(x)$ and $\tilde{C}_i(x) = C_{zi}(x) + D_{zi}(x)K(x)$ are closed-loop matrices at the vertex. With the results in Section IV, the main result for robust H_∞ synthesis can be proposed directly.

Proposition 5.2: Consider the nonlinear system (44). If there exist $Q_i(x) = Q_i^T(x)$, G_{11}, $G_{21}(x)$, $G_{22}(x)$, $H_{21}(x)$, $H_{22}(x)$ and $Y(x)$ such that, for the given H_∞ upper bound γ, some tuning scalar β, constants $s_i > 0$ and sum of squares polynomials $s_{il}(x)$ with $s_{il}(x) > 0$ for $x \neq 0$ $(i, l = 1, \cdots, q)$, the following optimization problem has zero optimum,

$$\text{Minimize} \qquad \gamma_1$$

subject to

$$v_1^T \Big[Q_i(x) - s_i I \Big] v_1 \in \Sigma_{sos} \quad (48)$$

$$-\begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}^T (\Upsilon_{hil}(x) + s_{il}(x)I) \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \Sigma_{sos} \quad (49)$$

$$v_5^T \begin{bmatrix} \gamma_1 & \phi_{1i}B_{ul}(x) & \phi_{1i}B_{wl}(x) \\ * & I & 0 \\ * & * & I \end{bmatrix} v_5 \in \Sigma_{sos} \quad (50)$$

where $G(x)$ and $H(x)$ possesses the structure of (24), $\phi_{1i}(x, v_{21})$ abbreviated as ϕ_{1i} here is defined as

$$\phi_{1i}(x, v_{21}) = \left[v_{21}^T \frac{\partial Q_i(x)}{\partial x_1} v_{21}, \ldots, v_{21}^T \frac{\partial Q_i(x)}{\partial x_n} v_{21} \right] \quad (51)$$

and

$$\Upsilon_{hil}(x) = \begin{bmatrix} \Upsilon_{hil_{1,1}} & \hat{B}_i(x) & \Upsilon_{hil_{1,3}} \\ * & -\gamma^2 I & 0 \\ * & * & -H(x) - H^T(x) \end{bmatrix} \quad (52)$$

with $\Upsilon_{hil_{1,1}}$ and $\Upsilon_{hil_{1,3}}$ in (53) on the top of next page, then the control law (5) stabilizes the system and achieves the H_∞ performance $\|z\|_2 < \gamma \|w\|_2$ with $K(x) = Y(x)G_{11}^{-1}$

VI. EXAMPLES

In this section state feedback design examples are presented to demonstrate the proposed controller design approach in Section V.

A. Example 1

Consider a parameter-dependent nonlinear system of the form (33) with system matrices given by

$$\begin{aligned} A_1(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix} \\ A_2(x) &= \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 & \frac{1}{4} - x_1^2 \\ 0 & 0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \end{aligned} \quad (54)$$

and $Z(x) = [x_1 \;\; x_2]^T$.

A state feedback control law is derived using Proposition 5.1. The values of positive constants s_1, s_2 are fixed as 0.001, then for $\beta = 0.0001$, the sum of squares based problem returns constant matrices Q_1, Q_2, and a 2nd order nonlinear control law is constructed as follows so that the equilibrium point $(0, 0)$ is asymptotically stable.

$$\begin{aligned} u(x) = &-6.6065x_1 - 5.8518x_2 - 0.8940x_1^2 \\ &-0.5960x_1x_2 - 0.3351 \times 10^{-9}x_2^2 \end{aligned} \quad (55)$$

Figure 1 shows the closed-loop state trajectories of 10 interpolated systems at various values of θ between the two vertices in (54), which demonstrates that the state feedback controller derived stabilizes the parameter-dependent system and the origin point is asymptotically stable.

$$\Upsilon_{hil_{1,1}} = \begin{bmatrix} M(x)A_i(x)G_{11} + [M(x)A_i(x)G_{11}]^T + M(x)B_{ui}(x)Y(x) \\ +[M(x)B_{ui}(x)Y(x)]^T - \sum_{j=1}^{n} \frac{\partial Q_i(x)}{\partial x_j}[A_{lj}(x)Z(x)] & * \\ C_{zi}(x)G_{11} + D_{zi}(x)Y(x) - \frac{1}{2}G_{21}(x) & -\frac{1}{2}G_{22}(x) - \frac{1}{2}G_{22}^T(x) \end{bmatrix}$$

$$\Upsilon_{hil_{1,3}} = -\begin{bmatrix} Q_i(x) & 0 \\ 0 & I \end{bmatrix} + G^T(x) - \begin{bmatrix} \beta M(x)A_i(x)G_{11} + \beta M(x)B_{ui}(x)Y(x) & 0 \\ \beta C_{zi}(x)G_{11} + \beta D_{zi}(x)Y(x) - \frac{1}{2}H_{21}(x) & -\frac{1}{2}H_{22}(x) \end{bmatrix} \quad (53)$$

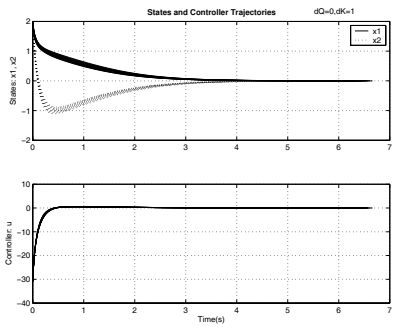

Fig. 1. States and controller trajectories of the close-loop systems

B. Example 2

Consider a parameter-dependent nonlinear system of the form (44)-(45) with system matrices given by

$$A_1(x) = \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix}$$
$$B_{u1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_{w1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_{z1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$D_{z1} = 1, D_{z2} = 1, C_y = \begin{bmatrix} 1 & -1 \end{bmatrix}$$
$$A_2(x) = \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 & \frac{1}{4} - x_1^2 \\ 0 & 0 \end{bmatrix}$$
$$B_{u2} = \begin{bmatrix} 0 \\ 1.2 \end{bmatrix}, B_{w2} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}, C_{z2} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad (56)$$

and $Z(x) = [x_1 \;\; x_2]^T$.

Using Proposition 5.2, we derive a 2nd order state feedback control law (57) with the minimization of γ. The values of positive constants s_1, s_2 are fixed as 0.00001, the sum of squares polynomials $s_{il}(x)$ for $i, l = 1, 2$ are chosen as $0.00001(x_1^2 + x_2^2)$. Then for $\beta = 0.01$, the sum of squares based optimization problem returns 1.7925 as the optimal value of γ, which implies that the L_2 gain from w to z of the closed-loop system is no greater than 1.7925. Since the Lyapunov matrices $Q_i(x)$ returned by the sum of squares optimization are constant matrices, the performance of the controller designed for the parameter-dependent nonlinear system is guaranteed over the entire state space.

$$u(x) = -2.7798x_1 - 3.7415x_2 - 0.0962x_1^2 \\ -0.0641x_1x_2 - 0.2832 \times 10^{-8}x_2^2 \quad (57)$$

VII. CONCLUSIONS

This paper discusses the state feedback synthesis problems for a class of nonlinear polynomial systems. Less conservative sufficient conditions to guarantee the closed-loop stability with or without H_∞ performance via state feedback are presented as SDMIs. We eliminate the coupling terms between system matrices and the Lyapunov matrix by introducing additional matrix variables, hence SDMI based conditions have a more suitable structure to deal with parameter uncertainty for the parameter-dependent polynomial systems. In order to reduce the conservatism involved in the controller design, parameter and state dependent Lyapunov functions (instead of fix ones) are used, and more general assumption and relaxation are provided to deal with the nonconvex terms in the matrix inequalities.

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