Further Results on Lyapunov-Krasovskii Functionals via Nonlinear Small-Gain Conditions for Interconnected Retarded iISS Systems

Hiroshi Ito, Pierdomenico Pepe and Zhong-Ping Jiang

Abstract—This paper presents further results on the problem of establishing stability of retarded nonlinear interconnected systems comprising integral input-to-state stable subsystems. It is shown that the stability of the interconnected systems with respect to external signals can be verified by constructing Lyapunov-Krasovskii functionals explicitly whenever small-gain type conditions are satisfied. The primary result [12] is generalized in two aspects. One is to introduce a new flexibility in constructing Lyapunov-Krasovskii functionals to deal with distributed delays more effectively. The other is to cover systems involving time-varying delays in interconnecting channels.

I. INTRODUCTION

Time-delay usually has a great influence on systems stability and performance. Recently, the popularity of research on teleoperation, networked control systems and consensus in cooperative control among multiple agents has regained attention to the importance of coping with time-delays in interconnected systems. There are various sources of delays in such networks and spatially distributed systems. One of typical delays is the transmission delay in communication between subsystems or agents, which is often time-varying.

There is a large body of literature on stability criteria for retarded systems, and some of them deal with time-varying delays (e.g., [7], [17], [5], [24] to name a few). Stability of interconnected systems and networks has been also investigated for many decades. Small-gain theorem states that an interconnected system is stable if the loop-gain is less than one. Although the small gain theorem has found numerous applications in control theory[3], the practical limitation of the $L^p$-gain framework had been also pointed out. The emergence of the small-gain theorem for input-to-state (ISS) systems has provided a way to remove the limitation[13], [23], and the ISS small-gain theorem is now widely used in many areas of nonlinear systems and control design. An ISS small-gain condition has been also developed for general ISS dynamical systems with weak semigroup properties in [14]. The philosophy of the ISS small-gain theorem has been generalized to a larger class of systems covering integral input-to-state stable (iISS) systems[10].

Recently, in [12], the iISS small-gain theorem has been generalized to systems involving discrete delays and distributed delays which can appear everywhere in the systems. Interconnections of iISS retarded nonlinear systems are targeted, and the iISS property of the overall system is established by constructing iISS Lyapunov-Krasovskii functionals explicitly. The purpose of this paper is to improve the result of [12] further in the following two points:

(G1) to equalize the treatment of discrete delays and distributed delays, and construct more flexible iISS Lyapunov-Krasovskii functionals explicitly;

(G2) to allow time-varying delays in the communication between subsystems.

The results in this paper include [12] as a special case.

Notations: The symbol $|\cdot|$ stands for the Euclidean norm. The interval $[0, \infty)$ is denoted by $\mathbb{R}_+$. For a measurable and essentially bounded function $u : \mathbb{R}_+ \to \mathbb{R}_m$, $|u|_{\infty} = \text{ess sup}_{t \geq 0} |u(t)|$. We indicate with $u(t_1,t_2) : \mathbb{R}_+ \to \mathbb{R}_m$ the function given by $u(t_1,t_2)(t) = u(t)$ for all $t \in [t_1,t_2)$ and $= 0$ elsewhere. A function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is denoted by $\omega \in \mathcal{P}$ if it is continuous and satisfies $\omega(0) = 0$. A function $\omega \in \mathcal{P}$ is said to be positive definite if $\omega(s) > 0$ holds for all $s > 0$, and written as $\omega \in \mathcal{P}$. A function is of class $\mathcal{K}$ if it belongs to $\mathcal{P}$ and is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed $t$ the function $s \mapsto \beta(s,t)$ is of class $\mathcal{K}$ and for each fixed $s$ the function $t \mapsto \beta(s,t)$ is non-increasing and goes to zero as $t \to +\infty$. The symbols $\forall$ and $\land$ denote logical sum and logical product, respectively.

II. SYSTEM DESCRIPTION

Consider an interconnected system $\Sigma$ described by the following functional differential equations

$$\Sigma \left\{ \begin{array}{ll}
\Sigma_1 : & x_1(t) = f_1(t,x_1,t,x_2,t,r_1(t)), \\
\Sigma_2 : & x_2(t) = f_2(t,x_2,t,x_1,t,r_2(t)), \\
\end{array} \right. \quad (1)$$

where, for $i = 1, 2$, $x_i(t) \in \mathbb{R}^{n_i}; r_i(t) \in \mathbb{R}^{n_i}$ is an external input (measurable, locally essentially bounded); they are functions of time $t \in \mathbb{R}_+; n_i$ and $m_i$ are positive integers. For $t \in \mathbb{R}_+, x_{i,t} : [-\Delta,0] \to \mathbb{R}^{n_i}$ denotes the function $x_{i,t}(\tau) = x_i(t+\tau)$, where $\Delta > 0$ is the maximum involved delay. Let $\mathcal{C}_i$ denote the space of continuous functions mapping the interval $[-\Delta,0]$ into $\mathbb{R}^{n_i}$. For $\phi_i \in \mathcal{C}_i$, we use $|\phi_i|_{\infty} = \sup_{-\Delta \leq \theta \leq 0} |\phi_i(\theta)|$. Suppose that $\xi_{i,0} \in \mathcal{C}_i$ and that $f_i : \mathbb{R}_+ \times \mathcal{C}_i \times \mathcal{C}_i \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is a functional which is Lipschitz on any bounded set in $\mathcal{C}_i \times \mathcal{C}_i \times \mathbb{R}^{m_i}$ uniformly in $t \in \mathbb{R}_+$. We combine vectors as $x(t) = [x_1(t)^T, x_2(t)^T]^T \in \mathbb{R}^n, n = n_1 + n_2, r(t) = [r_1(t)^T, r_2(t)^T]^T \in \mathbb{R}^m, m = m_1 + \ldots$
m_2, \xi_0 = [T_{0,0}, T_{0,2}, T_{2,0}]^T \in C := C_1 \times C_2, f(\cdot) = [f_1(\cdot)^T, f_2(\cdot)^T]^T, 
\phi = [\phi_1^T, \phi_2^T]^T \in C$. We define \( x_t \) and \( \| \phi \|_{\infty} \) as done for their \( i \)-th components. If \( f_1 \) and \( f_2 \) are independent of \( t \), let \( t_0 = 0 \) without loss of generality. It is assumed that \( f_i(t, 0, 0, 0) = 0, \ i = 1, 2, \forall t \in \mathbb{R}_+ \), thus ensuring that \( x(t) = 0 \) is the solution corresponding to zero input and zero initial conditions (i.e., the trivial solution). Note that the formulation (1) accepts non-commensurate discrete as well as distributed time-delays not only in the interconnecting channels, but also in the individual subsystems \( \Sigma_i \).

In this paper, we let \( M_{a,i} : C_1 \rightarrow \mathbb{R}_+ \) and \( M_a : C \rightarrow \mathbb{R}_+ \), \( i = 1, 2 \), be continuous functions such that there exist \( \gamma_{a,i}, \gamma_a, \tau_a \in \mathbb{K}_+ \) such that
\[
\gamma_{a,i}([\phi_i(0)]) \leq M_{a,i}(\phi_i) \leq \gamma_{a,i}([\phi_i]_{\infty}), \ \forall \phi_i \in C_i \tag{2}
\]
\[
\gamma_a([\phi(0)]) \leq M_a(\phi) \leq \gamma_a([\phi]_{\infty}), \ \forall \phi \in C. \tag{3}
\]

We borrow the definitions of ISS and iISS properties of the system (1) from the references[21], [22], [1], [20].

**Definition 1:** If the solution \( x(t) \) of the interconnected system (1) exists for all \( t \geq t_0 \) and satisfies
\[
\chi([x(t)]) \leq \beta(\|\xi_0\|_{\infty}, t - t_0) + \int_{t_0}^t \gamma_r(\|r(\tau)\|)d\tau \tag{4}
\]
for all \( t \geq t_0 \), with \( \beta \in \mathcal{K}_\mathcal{L}, \chi \in \mathbb{K}_\infty, \gamma_r \in \mathbb{K}, \) the system (1) is said to be iISS with respect to input \( r \) and state \( x \).

**Definition 2:** If the solution \( x(t) \) of the interconnected system (1) exists for all \( t \geq t_0 \) and satisfies
\[
x(t) \leq \beta(\|\xi_0\|_{\infty}, t - t_0) + \gamma_r(\|r(t_0, t)\|_{\infty}) \tag{5}
\]
for all \( t \geq t_0 \), with \( \beta \in \mathcal{K}_\mathcal{L}, \gamma_r \in \mathbb{K}, \) the system (1) is said to be ISS with respect to input \( r \) and state \( x \).

The property (4) and (5) implies the global asymptotic stability of \( x = 0 \) when \( r(t) \equiv 0 \). For short, it is called 0-GAS in this paper.

For a locally Lipschitz functional \( V_{cl} : \mathbb{R}_+ \times C \rightarrow \mathbb{R}_+ \), \( D^+V_{cl}(\phi, r, t) \), which plays the central role in the Lyapunov-Krasovskii methodology[2], [18], [19], is defined as follows:
\[
D^+V_{cl}(\phi, r, t) = \limsup_{h \rightarrow 0^+} \frac{V_{cl}(t+h, \phi(h)) - V_{cl}(t, \phi)}{h},
\]
\[
\phi^h(s) = \begin{cases} 
\phi(s+h), & s \in [-\Delta, -h), \\
\phi(0) + (s+h)f(t, \phi, r), & s \in [-h, 0] 
\end{cases}
\tag{6}
\]
where \( \phi \in \mathcal{C}, r \in \mathbb{R}_+, t \in \mathbb{R}_+ \).

This paper addresses the problem of constructing Lyapunov-Krasovskii functionals \( V_{cl} \) to establish 0-GAS, iISS and ISS of the interconnected system (1) with respect to input \( r \) and state \( x \) under the following assumption imposed on each subsystem \( \Sigma_i \).

**Assumption 1:** For each \( i = 1, 2 \), there exists a locally Lipschitz functional \( V_i : C_i \rightarrow \mathbb{R}_+ \) such that
\[
\alpha_i(M_{a,i}(\phi_i)) \leq V_i(\phi_i) \leq \alpha_i(M_{a,i}(\phi_i)), \tag{7}
\]
\[
D^+V_i(\phi_i, \phi_{3-i}, r_i, t) \leq \rho_i(t, \phi_i, \phi_{3-i}, r_i), \tag{8}
\]
\[
\forall \phi_j \in C_j, j = 1, 2, \forall r_i \in \mathbb{R}_+ 
\]
hold, where \( \alpha_i, \alpha_i \) are \( \mathbb{K}_\infty \) functions, \( \rho_i : \mathbb{R}_+ \times C_i \times C_{3-i} \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is a continuous functional satisfying \( \rho_i(t, 0, 0, 0) = 0 \) for all \( t \in \mathbb{R}_+ \), and
\[
D^+V_i(\phi_i, \phi_{3-i}, r_i, t) = \limsup_{h \rightarrow 0^+} \frac{V_i(\phi_i^h) - V_i(\phi_i)}{h} \tag{9}
\]
\[
\phi^h_i(s) = \begin{cases} 
\phi_i(s+h), & s \in [-\Delta, -h), \\
\phi_i(0) + (s+h)f_i(t, \phi_i, \phi_{3-i}, r_i), & s \in [-h, 0] 
\end{cases}
\tag{10}
\]
for some \( \alpha_i, \alpha_i \in \mathbb{K}_\infty, \alpha_i \in \mathbb{P}_0 \) is said to be an ISS Lyapunov-Krasovskii functional for the system (1).

**Definition 4:** A locally Lipschitz functional \( V_{cl} : C \rightarrow \mathbb{R}_+ \), satisfying
\[
\alpha_{cl}(M_a(\phi)) \leq V_{cl}(\phi) \leq \alpha_{cl}(M_a(\phi)), \forall \phi \in \mathcal{C}, \tag{12}
\]
\[
D^+V_{cl}(\phi, r, t) \leq -\alpha_{cl}(M_a(\phi)) + \sigma_r(|r|), \forall \phi \in \mathcal{C}, r \in \mathbb{R}_+ 
\]
for some \( \alpha_{cl}, \alpha_{cl} \in \mathbb{K}_\infty, \alpha_{cl} \in \mathbb{P}_0, \sigma_r \in \mathbb{P}_0 \) is said to be an iISS Lyapunov-Krasovskii functional for the system (1).

The existence of iISS (ISS) Lyapunov-Krasovskii functionals is a sufficient condition for the ISS (ISS, respectively) of the system \( \Sigma \) (See Appendix). When we require \( \alpha_{cl}([\phi(0)]) \) instead of \( \alpha_{cl}([M_a(\phi)]) \) in (11) for \( r(t) \equiv 0 \), the functional \( V_{cl} \) reduces to the well-known Lyapunov-Krasovskii functional for 0-GAS [8], [16], [2].

**B. Motivation**

In [12], the supply rate \( \rho_i \) in Assumption 1 is chosen as
\[
\rho_i = -\alpha_i(M_{a,i}(\phi_i)) + S_{i,0}\sigma_{i,0}(M_{a,3-i}(\phi_{3-i})) \tag{13}
\]
\[
+ \sum_{j=1}^{m_i} S_{i,j}\sigma_{i,j}(\gamma_{a,3-i}(\phi_{3-i}(-\Delta_j))) + \sigma_{r,i}(|r_i|) \tag{14}
\]
where \( \Delta_j \)'s are time-invariant time-delays, and \( S_{i,j} \in \{0,1\}, \alpha_i, \sigma_{i,j} \in \mathbb{K} \) and \( \sigma_{r,i} \in \mathbb{P}_0 \). This supply rate explicitly includes discrete delays in the interconnection channels, i.e., the third term in (14), while distributed delays in the
interconnection channels do not appear explicitly in (14). The distributed delays are required to be incorporated in $M_{a,3-i}$ in $\rho_i$. This requirement together with (7) implies that we need to choose $V_{3-i}$ for $\Sigma_{3-i}$ taking into account the distributed delays at the input of $\Sigma_i$. In [12], this idea is employed only for the distributed delays, and discrete delays in the interconnecting channels are handled by the composite functional $V_{id}$ instead. In fact, the iISS Lyapunov-Krasovskii functional constructed for the overall system $\Sigma$ is

$$V_{cl}(\phi) = \int_0^{V_1(\phi_1)} \lambda_1(s)ds + \int_0^{V_2(\phi_2)} \lambda_2(s)ds + \sum_{j=1}^{h} S_{1,j} \int_{-\Delta_j}^{0} F_{1,j}(\tau) \tilde{\sigma}_{1,j} \left( \gamma_{a,2}(|\phi_2(\tau)|) \right) d\tau$$

$$+ \sum_{j=1}^{h} S_{2,j} \int_{-\Delta_j}^{0} F_{2,j}(\tau) \tilde{\sigma}_{2,j} \left( \gamma_{a,1}(|\phi_1(\tau)|) \right) d\tau \quad (15)$$

The last two terms in (15) cope with the discrete delays. The definition of $\lambda_i$, $\tilde{\sigma}_{i,j}$ and $F_{i,j}$ will be shown later. It can be expected here that we can use the same idea for distributed delays. To deal with delays in the interconnecting channels, we should be able to take either of the following ways:

- preprocessing the input delays in choosing the pair of dissipative inequalities of the individual subsystems;
- leaving the input delays explicitly in the supply rates of individual subsystems, and dealing with the input delays in the process of constructing a composite Lyapunov-Krasovskii functional for the overall system.

These two ways should be applicable to discrete delays and distributed delays equally. This section formulates a stability criterion providing the above two options explicitly.

C. A Generalized Small-Gain Theorem

The following is the main result in this section.

**Theorem 1**: Suppose that supply rate functionals $\rho_i$, $i = 1, 2$, are as follows:

$$\rho_i(\phi_i, \phi_{3-i}, r_i) = -\alpha_i(M_{a,i}(\phi_i)) + S_{0,0} \sigma_i(0)(M_{a,3-i}(\phi_{3-i}))$$

$$+ \sum_{j=1}^{h+h_d} S_{i,j} \int_{-\Delta_j}^{0} \sigma_{i,j} \left( \gamma_{a,3-i}(|\phi_{3-i}(\Delta_j)|) \right) d\tau$$

$$+ \sigma_{i,r}(r_i), \quad (16)$$

where $h$ and $h_d$ are non-negative integers and, for $i = 1, 2$, $\alpha_i$ are functions of class $K_\infty$, $\sigma_{i,r} \in P_0$, and for $j = 0, 1, \ldots, h + h_d$, $S_{i,j}$ belong to $\{0, 1\}$, $\sigma_{i,j}$ are functions of class $K$, and for $j = 1, 2, \ldots, h + h_d$, $\Delta_j \in (0, \Delta]$. Assume that one of the following three conditions holds:

- **(H1)** $\lim_{s \to \infty} \alpha_2(s) = \infty$ \quad $\land$ \quad $\lim_{s \to \infty} \alpha_1(s) = \infty$,
- **(H2)** $\lim_{s \to \infty} \alpha_2(s) = \infty$ \quad $\land$ \quad $\lim_{s \to \infty} \sigma_2(s) < \infty$,
- **(H3)** $\lim_{s \to \infty} \sigma_1(s) < \infty$ \quad $\land$ \quad $\lim_{s \to \infty} \sigma_2(s) < \infty$,

where $\sigma_i$, $i = 1, 2$, are defined as

$$\sigma_i(s) = \left( \sum_{k=0}^{h+h_d} S_{i,k} \right) \max_{j=0,1,2} \max_{k=0,1,2} S_{i,j} \sigma_{i,j}(s)$$

$$\max_{j=0,1,2} S_{i,j} \Delta_j \sigma_{i,j}(s) \quad (17)$$

Suppose that there exist $\epsilon_i > 1$, $i = 1, 2$, such that

$$\epsilon_i \alpha_1 \circ \alpha_2^{-1} \circ \bar{\pi}_2 \circ \alpha_2^{-1} \circ \bar{\pi}_2 \alpha_2(s) \leq \alpha_1 \circ \alpha_1^{-1} \circ \bar{\pi}_1 \alpha_1(s), \forall s \in \mathbb{R}_+ \quad (18)$$

Then, the interconnected system (1) is iISS with respect to input $r$ and state $x$. In addition, it is ISS with respect to input $r$ and state $x$ in the case of (H1). Furthermore, an iISS (ISS in the (H1) case) Lyapunov-Krasovskii functional for (1) is

$$V_{cl}(\phi) = \int_0^{V_1(\phi_1)} \lambda_1(s)ds + \int_0^{V_2(\phi_2)} \lambda_2(s)ds$$

$$+ \sum_{j=1}^{h} S_{1,j} \int_{-\Delta_j}^{0} F_{1,j}(\tau) \tilde{\sigma}_{1,j} \left( \gamma_{a,2}(|\phi_2(\tau)|) \right) d\tau$$

$$+ \sum_{j=1}^{h} S_{2,j} \int_{-\Delta_j}^{0} F_{2,j}(\tau) \tilde{\sigma}_{2,j} \left( \gamma_{a,1}(|\phi_1(\tau)|) \right) d\tau$$

$$+ \sum_{j=1}^{h} S_{2,j} \int_{-\Delta_j}^{0} F_{2,j}(\tau) \tilde{\sigma}_{2,j} \left( \gamma_{a,1}(|\phi_1(\tau)|) \right) d\tau \quad (19)$$

where $\lambda_1$, $\lambda_2$ and $\tilde{\sigma}_{i,j}$ are given in (54), (55) and (46) with

$$\omega_{i,j} = 1, \quad j = 0, 1, \ldots, h + h_d, \quad (20)$$

and $F_{i,j} : [-\Delta_j, 0] \to \mathbb{R}$ is defined for $0 < \epsilon_i < \epsilon_i - 1$ as

$$F_{i,j}(\tau) = -\frac{\tau}{\Delta_j} + (1 + \epsilon_i) \tau + \frac{\tau}{\Delta_j} \quad (21)$$

Proof: Using $F_{i,j}(\tau) \leq 1 + \epsilon_i$ for $\tau \in [-\Delta_j, 0]$, we can verify that $V_{cl}$ given by (19) satisfies (12) for some $\gamma_{a,2}, \gamma_{a,1} \in K_\infty$ and some $M_a : \mathcal{C} \to \mathbb{R}_+$ fulfilling (3) with $\gamma_a$, $\gamma_{a,2} \in K_\infty$. The following inequality also holds:

$$D^+ V_{cl}(\phi, r, t) \leq \sum_{i=1}^{2} \lambda_i(V_i(\phi_i)) D^+ V_i(\phi_i, \phi_{3-i}, r_i, t)$$

$$+ \sum_{j=1}^{h} (1 + \epsilon_i) S_{1,j} \tilde{\sigma}_{1,j} \left( \gamma_{a,3-i}(|\phi_{3-i}(0)|) \right)$$

$$- S_{i,j} \tilde{\sigma}_{i,j} \left( \gamma_{a,3-i}(|\phi_{3-i}(i) - (\Delta_j)|) \right)$$

$$- S_{i,j} \tilde{\sigma}_{i,j} \left( \gamma_{a,3-i}(|\phi_{3-i}(i) - (\Delta_j)|) \right) d\tau$$

$$+ \sum_{j=1}^{h} (1 + \epsilon_i) S_{1,j} \Delta_j \tilde{\sigma}_{1,j} \left( \gamma_{a,3-i}(|\phi_{3-i}(i)) \right)$$

$$- S_{i,j} \tilde{\sigma}_{i,j} \left( \gamma_{a,3-i}(|\phi_{3-i}(i)) \right) d\tau$$

$$- S_{i,j} \Delta_j \tilde{\sigma}_{i,j} \left( \gamma_{a,3-i}(|\phi_{3-i}(i)) \right) d\tau \quad (22)$$

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Combining (22) with Lemma 1, we obtain the following for suitable functions $\tilde{\alpha}_i, \tilde{\sigma}_{i,j}, \in K$ and $\tilde{\sigma}_{r,i} \in P_0$.

$$
D^+ V_{cl} \leq \sum_{i=1}^{2} -\tilde{\alpha}_i(M_{a,i}(\phi_i)) + S_{i,0,0}(M_{a,3-i}(\phi_{3-i}))
+ \left\{ \begin{array}{l}
\sum_{j=1}^{h} (1+\epsilon_j) S_{i,j} \tilde{\sigma}_{i,j} \left( 2_{a,3-i}(\phi_{3-i}(0)) \right) \\
- S_{i,j} \int_{0}^{\Delta_j} \tilde{\sigma}_{i,j} \left( 2_{a,3-i}(\phi_{3-i}(t)) \right) dt \\
- \sum_{j=h+1}^{h+h_d} S_{i,j} \int_{\Delta_j}^{0} \tilde{\sigma}_{i,j} \left( 2_{a,3-i}(\phi_{3-i}(\tau)) \right) d\tau \\
+ \sum_{j=h+1}^{h+h_d} (1+\epsilon_j) S_{i,j} \tilde{\sigma}_{i,j} \left( 2_{a,3-i}(\phi_{3-i}(0)) \right) \\
- S_{i,j} \int_{0}^{\Delta_j} \tilde{\sigma}_{i,j} \left( 2_{a,3-i}(\phi_{3-i}(\tau)) \right) dt \\
r_{r,i}(\tau_{1}) \right\}
\right\}
$$

(23)

By Lemma 2, taking into account (29), we obtain the following for suitable functions $\tilde{\alpha}_i, \tilde{\sigma}_{i,j}, \in K$ and $\tilde{\sigma}_{r,i} \in P_0$.

$$
\frac{d\Delta_j(t)}{dt} \leq b_j, \quad j = 1, 2, \ldots, h + h_d,
$$

(27)

Note that (27) implies $b_j \geq 0$. In the case where these time-varying delays appear in (1), we can establish 0-GAS of $\Sigma$ in the following way.

**Theorem 2:** Suppose that supply rate functions $\rho_i, i = 1, 2, $ satisfy

$$
\rho_i(t, \phi_i, \phi_{3-i}, r_{i}) = -\alpha_i(M_{a,i}(\phi_i)) + \sigma_i(\gamma_{a,3-i}(\phi_{3-i}(t)))
+ \sum_{j=1}^{h} S_{i,j} \sigma_{i,j} \left( 2_{a,3-i}(\phi_{3-i}(\tau_j)) \right) dt
$$

(28)

for $r_{i}(t) = 0, $ where $h$ and $h_d$ are non-negative integers and, for $i = 1, 2, \alpha_i \in K, \sigma_i \in P_0$ and for $j = 0, 1, \ldots, h + h_d, S_{i,j}$ belong to $\{0, 1\}, \sigma_{i,j}$ are of class $K$. Assume that

$$
\lim_{s \to \infty} \alpha_1(s) = \infty \quad \text{and} \quad \lim_{s \to \infty} \sigma_2(s) < \infty,
$$

holds, where $\sigma_i, i = 1, 2, $ are defined as

$$
\sigma_1(s) = \sum_{k=0}^{h+h_d} S_{i,k} \left( \sum_{j=1}^{h} S_{i,j} \sigma_{i,j}(s) \right)
$$

(29)

IV. TIME-VARYING DELAYS

This section considers the following time-varying delays:

$$
0 \leq \Delta_j(t) \leq \bar{\Delta}_j, \quad \frac{d\Delta_j(t)}{dt} \leq b_j < 1, \quad j = 1, 2, \ldots, h + h_d,
$$

(27)

Remark 2: The selection of the individual functionals $V_i, i = 1, 2, $ has an influence on the result of stability analysis based on the inequality (28). To reduce the associated conservativeness arising in each case, we can make use of a large literature on detailed techniques of delay systems for the effective choices of $V_i$ of the individual $\Sigma_i$.

**Remark 5:** The small-gain condition (18) can be relaxed into a non-uniform small-gain condition[11] by using a little complex $\lambda_i$'s.

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Suppose that there exist $c_i > 1$, $i = 1, 2$, such that (18) holds. Then, the interconnected system (1) is 0-GAS. Furthermore, a Lyapunov-Krasovskii functional for (1) is

$$V_c(t, \phi) = \int_0^t \lambda_1(s) ds + \int_0^t \lambda_2(s) ds$$

$$+ \sum_{j=1}^h \frac{S_{1,j}}{1 - b_j} \int_{\Delta_1(t)}^0 \sigma_{1,j} \left( \gamma_{a,j,1}(\phi_2(\tau)) \right) d\tau$$

$$+ \sum_{j=h+1}^{h+h_d} \frac{S_{1,j}}{1 - b_j} \int_{\Delta_1(t)}^0 \sigma_{1,j} \left( \gamma_{a,j,2}(\phi_2(\theta)) \right) d\theta d\tau$$

$$+ \sum_{j=1}^h \frac{S_{2,j}}{1 - b_j} \int_{\Delta_2(t)}^0 \sigma_{2,j} \left( \gamma_{a,j,1}(\phi_1(\tau)) \right) d\tau$$

$$+ \sum_{j=h+1}^{h+h_d} \frac{S_{2,j}}{1 - b_j} \int_{\Delta_2(t)}^0 \sigma_{2,j} \left( \gamma_{a,j,2}(\phi_1(\theta)) \right) d\theta d\tau$$

where $\lambda_1$, $\lambda_2$ and $\sigma_{i,j}$ are given in (54), (55) and (46) with

$$\varpi_{i,0} = 1, \quad \varpi_{i,j} = \frac{1}{1 - b_j}, \quad j = 1, 2, ..., h + h_d.$$

**Proof:** The following inequality holds for (30) and (1):

$$D^+ V_c(t, \phi) \leq \sum_{i=1}^2 \alpha_i(V_i(\phi_i)) D^+ V_i(\phi_i, \phi_{3-i}, t)$$

$$+ \sum_{j=1}^h \frac{S_{1,j}}{1 - b_j} \sigma_{1,j} \left( \gamma_{a,j,1}(\phi_3-i(0)) \right)$$

$$- \frac{S_{1,j}}{1 - b_j} \left( 1 - \frac{d\Delta_1(t)}{dt} \right) \sigma_{1,j} \left( \gamma_{a,j,3-i}(\phi_3-i(-\Delta_j)) \right)$$

$$+ \sum_{j=h+1}^{h+h_d} \frac{S_{1,j}\Delta_1(t)}{1 - b_j} \sigma_{1,j} \left( \gamma_{a,j,3-i}(\phi_3-i(0)) \right)$$

$$- \frac{S_{1,j}}{1 - b_j} \left( 1 - \frac{d\Delta_1(t)}{dt} \right) \sigma_{1,j} \left( \gamma_{a,j,3-i}(\phi_3-i(\tau)) \right) d\tau$$

Combining (32) with Lemma 1 and using (27), we obtain the following for suitable functions $\bar{\sigma}_i$ and $\bar{\sigma}_{i,j} \in \mathcal{K}$.

$$D^+ V_c(t, \phi) \leq \sum_{i=1}^2 -\bar{\alpha}_i(M_{a,i}(\phi_i)) + S_{i,0} \bar{\sigma}_{i,0}(M_{a,3-i}(\phi_{3-i}))$$

$$+ \sum_{j=1}^h \frac{S_{i,j}}{1 - b_j} \bar{\sigma}_{i,j} \left( \gamma_{a,j,3-i}(\phi_{3-i}(0)) \right)$$

$$+ \sum_{j=h+1}^{h+h_d} \frac{S_{i,j}\Delta_1(t)}{1 - b_j} \bar{\sigma}_{i,j} \left( \gamma_{a,j,3-i}(\phi_{3-i}(0)) \right)$$

From Lemma 3, (18) and (2), it follows that there exist $\alpha_{cd} \in \mathcal{K}$ and a functional $M_a$ such that

$$\alpha_{cd}(\phi(0)) \leq V(t, \phi) \leq \pi_{cd}(M_a(\phi)), \quad \forall \phi \in \mathcal{C}$$

$$D^+ V_c(t, \phi) \leq -\alpha_{cd}(\phi(0)), \quad \forall \phi \in \mathcal{C}$$

and (3) hold for some $\alpha_{cd}$, $\pi_{cd} \in \mathcal{K}_\infty$. Hence, the Lyapunov-Krasovskii Theorem proves the claim.

There are a large number of studies on stability of systems with time-varying delays using Lyapunov-Krasovskii functionals (See, e.g., [5], [24], [9] and references therein). This paper shows how to extend the fundamentals of such ideas to cover essentially nonlinear interconnected systems consisting of iISS subsystems. The novelty of the Lyapunov-Krasovskii functional $V_c$ given by (30) lies in $\bar{\sigma}_{i,j}$’s which lead us to the small-gain condition (18) (See Appendix).

In the proof of Theorem 2, the function $\alpha_{cd}$ is obtained as a function of $|\phi(0)|$. Since $\alpha_{cd}$ is not guaranteed to be a function of $M_a(\phi)$, Theorem 2 proves 0-GAS based on the Lyapunov-Krasovskii Theorem instead of iISS and ISS.

**Remark 6:** The functional $V_c$ in (30) is time-dependent due to the presence of $\Delta_1(t)$. On the other hand, Definitions 3 and 4 require properties uniform in time $t$ since $V_c$ and $M_{a,i}$ are time-invariant. In order to succeed in verifying the iISS property for a larger class of time-varying systems, we need to develop a methodology of iISS Lyapunov-Krasovskii functionals which are allowed to be non-uniform in $t$.

**Remark 7:** Theorem 2, in general, requires that state delays in individual subsystem $\Sigma_i$ are time-invariant. In fact, the inequality (8) with (28) implies that the property of $\Sigma_i$ with respect to $\phi_i$ is uniform in time $t$ since $V_i$ and $M_{a,i}$ are time-independent functional of $\phi_i$. Nevertheless, it is stressed that time-varying delays in the interconnection channels can be incorporated into the supply rates (28) of $\Sigma_i$, $i = 1, 2$, effectively, so that the selection of $V_i$’s and $M_{a,i}$’s can be independent of the interconnection delays.

**Remark 8:** The assumption (J1) in Theorem 2 can be relaxed by using a non-uniform small-gain condition as in the delay-free case [11].

**V. AN EXAMPLE**

Consider the interconnected system described by

$$\dot{x}_1(t) = -\frac{x_1^3(t)}{1 + x_1(t)^2} + x_1(t) \int_{-\Delta_1}^t x_2(\tau) d\tau + x_1(t) r_1(t)$$

$$\dot{x}_2(t) = -\gamma x_2(t) + x_2(t - \Delta_2) + \frac{1}{\Delta_3(t)} x_1(t - \Delta_3(t))^2$$

where $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, $r(t) = r_1(t) \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. The time-delays $\Delta_1, \Delta_2 > 0$ are time-invariant, while the time-varying delay $\Delta_3(t)$ is supposed to satisfy

$$0 \leq \Delta_3(t) \leq \Delta_3, \quad \frac{d\Delta_3(t)}{dt} \leq b_3 < 1, \quad \forall t \in \mathbb{R}_+$$

For $\phi_1 \in \mathcal{C}_1, \phi_2 \in \mathcal{C}_2$, let

$$V_1(\phi_1) = \log(1 + \phi_1(0)^2)$$

$$V_2(\phi_2) = \phi_2^2(0) + \int_{-\Delta_2}^0 \left( -\frac{\tau}{\Delta_2} + (1 + \epsilon_2) \frac{\tau + \Delta_2}{\Delta_2} \right) \phi_2(\tau)^2 d\tau$$

where $\epsilon_2 > 0$ has yet to be determined. Then, we obtain

$$D^+ V_1 \leq -(1 - \delta) \frac{\phi_1(0)^4}{(1 + \phi_1(0)^2)^2} + \Delta_1 \int_{-\Delta_1}^0 \phi_2(\tau)^2 d\tau + \frac{1}{\delta} r_1(t)^2$$

$$D^+ V_2 \leq -2(\gamma - 3 - \epsilon_2) \phi_2(0)^2 + \frac{\phi_1(-\Delta_3(t))^2}{1 + \phi_1(-\Delta_3(t))^2} - \frac{\epsilon_2}{\Delta_2} \int_{-\Delta_2}^0 \phi_2(\tau)^2 d\tau$$
Here, Young inequality and Cauchy-Schwarz inequality are used. We can choose the following functions and parameters:

\[ M_{a,1}(\phi_1) = \phi_1(0)^2, \quad M_{a,2}(\phi_2) = \phi_2(0)^2 + \int_{-\Delta_2}^{\phi_2(\tau)} d\tau \]

\[ \omega_1(s) = \tau_1(s) = \log(1 + s), \quad \tau_{a,1}(s) = s^2 \]

\[ \omega_2(s) = s, \quad \tau_{a,2}(s) = (1 + \epsilon_2)s, \quad \tau_{a,a}(s) = (1 + \Delta_2)s^2 \]

\[ \bar{\omega}_2(s) = s^2, \quad S_{1,0} = S_{1,2}, \quad S_{2,0} = S_{2,2} = 0, \quad h = h_d = 1 \]

\[ S_{1,2} = S_{2,1} = 1, \quad \alpha_1(s) = \frac{(1+\delta)s}{1+s}, \quad \sigma_1(s) = \frac{s}{1+\delta} \]

Thus, the dissipation inequality (8) with \( \rho_i \) in the form of (28) implies that the \( x_2 \) subsystem is ISS with respect to input \( x_1 \) and state \( x_2 \), while the \( x_1 \) subsystem is only iISS with respect to input \( x_2, r_1 \) and state \( x_1 \).

First, we suppose \( b_3 = 0 \), i.e., \( \Delta_3(t) \) is a constant. For the supply rates in the form of (16), there exist \( c_1, c_2 > 1 \) such that the small-gain condition (18) is satisfied if

\[ \gamma > \frac{3 + \Delta_1 - 2\Delta_2 \Delta_3}{2(1 - \Delta_1 \Delta_2)} \land \Delta_1 \Delta_2 < 1 \quad (37) \]

holds, which is independent of \( \Delta_3 \). Thus, the interconnected system (36) is iISS with respect to input \( r \) and state \( x \) if (37) is satisfied. Theorem 1 provides an iISS Lyapunov-Krasovskii functional as (19).

Next, in order to illustrate Theorem 2 in the case of time-varying \( \Delta_3(t) \) with \( b_3 > 0 \), we consider the interconnected system (36) for \( r_1(t) \equiv 0 \). Taking \( \delta = 0 \), we have the supply rates in the form of (28). There exist \( c_1, c_2 > 1 \) satisfying the small-gain condition (18) if

\[ \gamma > \frac{3(1-b_3) + \Delta_1 - 2\Delta_2 \Delta_3}{2(1 - b_3 - \Delta_1 \Delta_2)} \land \Delta_1 \Delta_2 < 1 - b_3 \quad (38) \]

holds. Theorem 2 asserts that the system (36) is 0-GAS if (38) is satisfied. A Lyapunov-Krasovskii functional is (30).

VI. CONCLUSIONS

This paper has investigated the problem of establishing stability of interconnected nonlinear systems involving discrete as well as distributed time-delays. It has been shown that Lyapunov-Krasovskii functionals establishing stability of the interconnections can be constructed when small-gain conditions are satisfied. This paper has improved the previous result [12] of the authors by generalizing supply rates of individual subsystems to deal with discrete delays and distributed delays in a unified manner. Lyapunov-Krasovskii functionals are tailored in order to handle the generalized supply rates. This paper has also covered time-varying delays in communication channels.

This paper has employed time-invariant functionals \( V_i \) for individual subsystems although time-varying functionals \( V_i \) have more potential to cope with fully time-varying networks. Indeed, Section IV aims mainly at time-varying delays only in the channels connecting time-invariant subsystems. When we deal with internally time-varying subsystems, time-invariant \( V_i \)'s can fulfill (8) with (28) only when properties of the subsystems are uniform in \( t \). Recently, Lyapunov-Krasovskii type characterizations of non-uniform in time ISS of nonlinear retarded systems have been derived in [15]. Therefore, our further research includes extending the result in this paper to more general stability which is allowed to be non-uniform in \( t \).

APPENDIX

Theorem 3: If there exists an ISS (iISS) Lyapunov-Krasovskii functional \( V_{cl} \) for (1), then the system (1) is ISS (iISS, respectively) with respect to input \( r \) and state \( x \).

This theorem was originally proved for time-invariant systems in [20]. The arguments given there remain true since Definition 3 and Definition 4 use time-invariant functional \( V_{cl} \) and require uniform properties in time \( t \). The following lemmas are extension of lemmas proving in [12] to the cases of distributed and time-varying delays.

Lemma 1: For \( i = 1, 2, \) consider

\[ \alpha_i, \sigma_{ij}, \epsilon_i \in \mathcal{K}, \quad \sigma_{r,i} \in \mathcal{P}_0, \quad j = 0, 1, ..., h + h_d \]

\[ S_{ij} \in \{0, 1\}, \quad w_{ij} \in \{1, \infty\}, \quad j = 0, 1, ..., h + h_d \]

\[ 0 \leq \Delta_j(t) \leq \Delta_i, \quad \forall t \in \mathbb{R}_+, \quad j = 1, 2, ..., h + h_d \]

with non-negative integers \( h, h_d \), and non-decreasing continuous functions \( \lambda_i : \mathbb{R}_+ \to \mathbb{R}_+ \). Assume that

\[ \lim_{s \to \infty} \alpha_i(s) < \infty \Rightarrow \lim_{s \to \infty} \lambda_i(s) < \infty \]

holds. If functionals \( V_i, M_{a,i} : C_i \to \mathbb{R}_+ \) satisfy

\[ \alpha_i(M_{a,i}(\phi_i)) \leq V_i(\phi_i) \leq \max(\alpha_i(M_{a,i}(\phi_i)), \rho_i), \quad \forall \phi_i \in C_i, \]

for some \( \alpha_i, \bar{\alpha}_i \in \mathcal{K}_+ \), then it holds that

\[ \lambda_i(V_i(\phi_i)) \left\{ -\alpha_i(M_{a,i}(\phi_i)) + \sum_{j=0}^{h+h_d} S_{ij} \sigma_{ij}(w_{ij}) \right\} \]

\[ \left( \sum_{j=0}^{h+h_d} S_{ij} \right) + \sum_{j=0}^{h+h_d} S_{ij} \int_{\Delta_j(t)}^{0} \sigma_{ij}(v_{ij}(s)) ds + \sigma_{r,i}(z_i) \]

\[ \leq -\sigma_i(M_{a,i}(\phi_i)) + \sum_{j=0}^{h+h_d} S_{ij} \sigma_{ij}(w_{ij}) \]

\[ \left( \sum_{j=0}^{h+h_d} S_{ij} \right) + \sum_{j=0}^{h+h_d} S_{ij} \int_{\Delta_j(t)}^{0} \sigma_{ij}(v_{ij}(s)) ds + \sigma_{r,i}(z_i) \]

\[ \forall \phi_i \in C_i, \quad \forall w_{ij}, z_i \in \mathbb{R}_+, \quad \forall v_{ij} \in \mathbb{C}^0([-\Delta_i, 0]; \mathbb{R}_+) \]

where \( \alpha_i, \bar{\alpha}_i \in \mathcal{K} \) and \( \sigma_{r,i} \in \mathcal{P}_0 \) are

\[ \alpha_i(s) = \delta \left( 1 - \frac{1}{\tau_i} \right) \lambda_i(\alpha_i(s)) \alpha_i(s) \]

\[ \sigma_{ij}(s) = \left\{ \begin{array}{ll} \lambda_i(\theta_{ij}(s)) \sigma_{ij}(s) & \text{if } \lim_{v \to \infty} \alpha_i(v) \geq \frac{N_i}{\tau_i} \beta_{ij}(s) \\ \lim_{v \to \infty} \lambda_i(\alpha_i(v)) \sigma_{ij}(s) & \text{otherwise} \end{array} \right. \]

\[ \sigma_{r,i}(s) = \left\{ \begin{array}{ll} \lambda_i(\theta_{r,i}(s)) \sigma_{r,i}(s) & \text{if } \lim_{v \to \infty} \alpha_i(v) \geq \tau_{r,i} \sigma_{r,i}(s) \\ \lim_{v \to \infty} \lambda_i(\alpha_i(v)) \sigma_{r,i}(s) & \text{otherwise} \end{array} \right. \]

\[ \theta_{ij}(s) = \sigma_i \circ \alpha_i^{1} \circ N_i \tau_i \beta_{ij}(s), \quad j = 0, 1, ..., h + h_d \]

\[ \beta_{ij}(s) = \sigma_i \circ \alpha_i^{1} \circ \tau_{r,i} \sigma_{r,i}(s), \quad j = h + 1, h + 2, ..., h + h_d \]

\[ \theta_{r,i}(s) = \sigma_i \circ \alpha_i^{1} \circ \tau_{r,i} \sigma_{r,i}(s), \quad N_i = \sum_{k=0}^{h+h_d} S_{ik,k} \]
Furthermore, $\sigma_{t,i}, \sigma_{t,k} \in K_{\infty}$ holds if $\alpha_{1}, \alpha_{2} \in K_{\infty}$.

The existence of $k_{1}, k_{2}, \tau_{1}, \psi \in \mathbb{R}$ and $\mu_{1}, \alpha_{1} \in K$ fulfilling (56)-(62) is guaranteed when (18) and $(H1) \vee (H2) \vee (H3)$ hold. Note that (42) is fulfilled by (54) and (55) with the help of (60)-(61) when $(H1) \vee (H2) \vee (H3)$ holds.

**Lemma 3:** In the case of $\sigma_{t,i}(s) \equiv 0$, $i = 1, 2$, the assumption $(H1) \vee (H2) \vee (H3)$ in Lemma 2 can be replaced by $(J1)$.

**Lemma 4:** Given a locally Lipschitz functional $V_{i}: C_{i} \to \mathbb{R}_{+}$ and a continuous function $\lambda_{i} : \mathbb{R}_{+} \to \mathbb{R}_{+}$, let $W_{i} : C_{i} \to \mathbb{R}_{+}$ be a continuous functional defined as $W_{i}(\phi_{i}) = \int_{0}^{\phi_{i}} \lambda_{i}(s) ds$. Then,

\[
D^{+} W_{i}(\phi_{i}, \phi_{3-i}, r_{i}, r_{t}) \leq \lambda_{i}(V_{i}(\phi_{i})) D^{+} V_{i}(\phi_{i}, \phi_{3-i}, r_{i}, r_{t}),
\]

where

\[
D^{+} W_{i}(\phi_{i}, \phi_{3-i}, r_{i}, t) = \limsup_{h \to 0^{+}} \frac{W_{i}(\phi_{i}^{h}) - W_{i}(\phi_{i})}{h}, \\
D^{+} V_{i}(\phi_{i}, \phi_{3-i}, r_{i}, t) = \limsup_{h \to 0^{+}} \frac{V_{i}(\phi_{i}^{h}) - V_{i}(\phi_{i})}{h}.
\]

**REFERENCES**


