Analysis of linear systems using truncated ellipsoids

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Abstract—The objective of this paper is to develop numerically simple and effective methods for system analysis using truncated ellipsoids. The paper studies linear time-invariant systems subject to persistent disturbance and state constraint. The maximal output under a set of initial conditions and the overshoot under a given step input are estimated. Attempts are also made to detect an invariant set, as large as possible, within state/output constraint. The results are based on the set invariance condition for the truncated ellipsoid, and a characterization of the set where the output possibly reaches a local extreme.

Keywords: maximal output, overshoot, state constraint, invariant set, truncated ellipsoid

I. INTRODUCTION

We consider the following linear system subject to persistent disturbance:

\[ \dot{x} = Ax + Bu + Ew, \quad y = Cx, \]

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is a step input, \( w \in \mathbb{R}^q \) is the persistent disturbance bounded by \( w^Tw \leq 1 \) and \( y \in \mathbb{R}^p \) is the output. Assume that \( A \) is Hurwitz. We examine analysis problems including estimating the maximal output and overshoot, and searching for an invariant set, as large as possible, within a certain state constraint.

Characterizing the maximal output and overshoot under step input and/or persistent disturbance is a traditional problem in control theory. It has been studied under different frameworks, such as the \( L_1 \) performance framework (e.g., [6]) and the invariant set framework (e.g., [1], [3], [4], [8]). The \( L_1 \) framework exactly characterizes the worst case maximal output under persistent disturbance for linear time-invariant systems. It allows for dynamic uncertainties, but it is not clear how the methods can be used for general parametric, possibly time varying, uncertainties. On the other hand, the invariant set framework usually provides an estimate for the maximal output by using various Lyapunov functions, and the methods can be readily extended to handle systems with general parametric uncertainties and time-varying nonlinearities that can be described with linear differential inclusions.

Two typical types of invariant sets are the invariant ellipsoids [4] and the invariant polytopes[1], [3], [8], corresponding to quadratic Lyapunov functions and polyhedral Lyapunov functions, respectively. The analysis methods resulting from quadratic functions can be conservative but are still widely used due to computational efficiency via LMI

technique. The methods based on invariant polytopes may theoretically yield non-conservative results, if the number of vertices is allowed to be arbitrarily large. However, for systems of order greater than 3, the number of vertices quickly grows out of reach for any numerical methods. In recent years, other types of non-quadratic Lyapunov functions have been developed for uncertain systems, constrained control systems and hybrid systems (see e.g.[5], [7], [9], [11], [12], [13]). The Lyapunov functions in these works pertain to or are composed from several quadratic functions. Thus they lead to optimization problems with matrix inequality constraints, usually a mixture of LMIs and BMDs.

An interesting invariant set is considered in [14] for systems with input and state constraint. The set is formed by cutting off parts of an ellipsoid with several pairs of hyperplanes, representing the state and input constraint. The resulting invariant set is called “semi-ellipsoidal set” in [14]. In this paper, we will call it a truncated ellipsoid. In [14], the truncated ellipsoid is used as a viability set, or an admissible set in [8]: if the initial condition \( x_0 \) starts from the set, the response \( x(t) \) will satisfy the input and state constraint for all \( t \geq 0 \).

The truncated ellipsoid is actually an intersection of an ellipsoid and a polytope. So part of its boundary is from an ellipsoid and the rest from a polytope. In terms of Lyapunov function, the truncated ellipsoid is the level set of a function of the form

\[ V(x) = \max \{ x^T P x, x^T C_1^T C_1 x, \ldots, x^T C_p^T C_p x \} \]

When \( P = 0 \), \( V(x)^{1/2} \) is a polyhedral function. Thus \( V(x) \) can be considered as the mix of a quadratic function and a polyhedral function.

In this paper, we will use \( V \) similar to that in (2) and invariant truncated ellipsoid to estimate a bound for the maximal output under step input and persistent disturbance, as well as to find an invariant set, as large as possible, within state constraint. Since the function \( V \) incorporates the structure of the output and the constraint, the resulting optimization problems involves only a few bilinear terms and the constraint becomes LMIs when one or two variables are fixed. Thus the computational burden is just a little heavier than the corresponding algorithm resulting from applying quadratic functions, but the improvement is significant, as will be demonstrated with examples. Some miscellaneous analysis problems were considered in [16] using truncated ellipsoid. This paper deals with more general problems in a systematic way.

Notation We use \( \text{co} S \) to denote the convex hull of a set \( S \).
II. CONDITION OF SET INVARIANCE FOR INTERSECTION OF ELLIPSOIDS

Consider the following system:
\[ \dot{x} = Ax + Ew, \]
where \( x \in \mathbb{R}^n, w \in \mathbb{R}^q \). Assume that \( A \) is Hurwitz and \( w(t) \), \( w(t) \leq 1 \) for all \( t \). A set \( S \) is said to be invariant for this system if every \( x(0) \in S \) implies \( x(t) \in S \) for all \( t \geq 0 \) and all possible \( w(\cdot) \). Invariant ellipsoids for such a system are characterized via a matrix inequality (in [4]), which becomes an LMI when one scalar variable is fixed. This matrix inequality is used to evaluate several input-state and input-output properties including the maximal output and the overshoot. As acknowledged in [4], the estimation of these quantities can be conservative. In this paper, we would like to use the intersection of several ellipsoids as invariant set to reduce the conservatism.

The intersection of ellipsoids can be described as a level set of the pointwise maximum of a family of quadratic functions:
\[ V(x) = \max\{x^TP_jx : j = 1, 2, \ldots, J\}, \]
where \( P_j = P_j^T \geq 0 \). This function has been used in [7], [9], [11] as a Lyapunov function to analyze robust stability and performance under finite energy disturbances for linear differential inclusions and saturated systems. In [7], [9], [11], \( V \) is simply called the max function. Denote the one level set of \( V \) as
\[ L_V = \{x \in \mathbb{R}^n : x^TP_jx \leq 1, j = 1, \ldots, J\}, \]
and denote its boundary as \( \partial L_V \).

For \( P = P^T \geq 0 \), denote \( E(P) = \{x \in \mathbb{R}^n : x^TPx \leq 1\} \). If \( P > 0 \), then \( E(P) \) is an ellipsoid; if \( P \geq 0 \) and the rank of \( P \) is one, i.e., \( P = C^TC \) for a certain row vector \( C \), then \( E(P) = E(C^TC) \) is the region between two hyperplanes \( Cx = \pm 1 \), which is unbounded. We may regard \( E(P) \) for a singular \( P \) as a degenerated ellipsoid. To unify all the cases, we may simply call \( L_V \) the intersection of ellipsoids, allowing some of the ellipsoids to be degenerated.

For the special case where \( P_1 > 0 \) and the rest \( P_j = C_j^TC_j, j = 2, \ldots, J \), all have rank one, \( L_V \) is part of the ellipsoid \( E(P_1) \) after truncated by the planes \( C_jx = \pm 1 \). For simplicity, we call such an \( L_V \) a truncated ellipsoid. It is actually the intersection of an ellipsoid and a polytope.

The set \( L_V \) is invariant for (3) if and only if \( \dot{x} \) points inward of \( L_V \) at each \( x \in \partial L_V \), for all possible \( w, w^Tw \leq 1 \). Since \( V \) is not everywhere differentiable, we need to use directional derivative to describe this property. A general result about the directional derivative of this type of functions can be found in [11].

For a function \( V(x) \), the one sided directional derivative is defined ([15], page 213) with respect to two variables: \( x \) and a vector \( \zeta \) specifying the direction of motion. In particular, the one-sided directional derivative of \( V \), at \( x \) along \( \zeta \) is defined as
\[ \dot{V}(x; \zeta) := \lim_{\Delta t \to 0_+, \Delta t > 0} \frac{V(x + \zeta \Delta t) - V(x)}{\Delta t}. \]
For \( x \in \mathbb{R}^n \), let
\[ I_{\text{max}}(x) := \{j : x^TP_jx \geq x^TP_kx \, \forall k\}. \]
Then by [11], the directional derivative of \( V \) at \( x \) along \( \zeta \) is
\[ \dot{V}(x; \zeta) = \max\{2x^TP_j\zeta : j \in I_{\text{max}}(x)\}. \]
With directional derivative, the set \( L_V \) is invariant if and only if,
\[ \dot{V}(x; Ax + Ew) \leq 0 \quad \forall x \in \partial L_V, w^Tw \leq 1. \]
In what follows, we give a condition for the invariance of \( L_V \) in terms of some bilinear matrix inequalities.

**Proposition 1:** The level set \( L_V \) is invariant for system (3) if there exist \( \lambda_{jk} \geq 0, \beta_j \geq 0, j, k = 1, \ldots, J \), such that
\[ \begin{bmatrix} M_j & P_j E \\ E^TP_j & -\beta_j I \end{bmatrix} \leq 0, j = 1, \ldots, J, \]
where \( M_j = A^TP_j + P_jA - \sum_{k=1}^J \lambda_{jk}(P_k - P_j) + \beta_j P \).

When \( J = 1 \), (6) reduces to one matrix inequality which is the one appears in [4]. It becomes an LMI when \( \beta_1 \) is fixed. For the general case, we need to fix \( J \times J \) scalar variables \( \lambda_{jk}, \beta_j \), to make the \( J \) inequalities LMIs. (Notice that \( \lambda_{jk} \) has no effect when \( j = k \).)

In the next two sections, we will use the condition for the invariance of the intersection of ellipsoids to address several performance analysis problems. We will choose some of the \( P_j \)'s as \( P_j = C_j^TC_j \), where \( C_j \) is from the output matrix \( C \) or a certain state constraint \( |C_r| \leq 1 \). By doing so, we incorporate the structure of the output and/or constraint into the Lyapunov function \( V \). Moreover, with some algebraic manipulation, we can reduce the number of bilinear terms, so that the matrix inequalities become LMIs when a few variables are fixed. If we define the optimal value of the performance index as the function of these few variables, we can use Matlab function such as “fminbnd” or “fminsearch” to optimize these variables. Thus the problem is reduced to a low dimensional optimization via a certain LMI solvers.

III. ESTIMATION OF OUTPUT BOUND AND OVERSHOOT

In this section, we use the truncated ellipsoid and the max function to estimate the bound of output due to a set of initial conditions and a step input, respectively.

A. Output bound under a set of initial conditions

Consider the system (3) with the output \( y = Cx \) where \( C \) is a row vector. If there are several output channels, we may consider each one separately. Assume that the initial condition belongs to a set \( X_0 = \{x(0) \in \mathbb{R}^n : x(0) = x_0\text{ for all }x_0 \in X_0\} \). Our objective is to estimate an upper bound of \( |y(t)| \) for all possible \( w(\cdot), w^Tw(t)w(t) \leq 1 \) and \( x(0) \in X_0 \). This will be
achieved by using a truncated ellipsoid, which is the 1-level set

\[ V(x) = \max\{x^T(C'C/\gamma^2)x, x^T(P/\gamma^2)x\}. \]

If \( L_V \) is invariant and contains \( X_0 \), then we have \( x^T(t)C'Cx(t) \leq \gamma^2, \) i.e., \( |y(t)| \leq \gamma \) for all \( t \). The condition for invariance of \( L_V \) follows from Proposition 1 by taking \( P_1 = C'C/\gamma^2, P_2 = P/\gamma^2 \): there exist \( \lambda_1, \lambda_2 \geq 0, \beta_1, \beta_2 \geq 0 \), such that

\[
\begin{bmatrix}
M_1 & C'C \beta_1 \\
E'C'C & -\beta_1 \gamma^2
\end{bmatrix} \leq 0,
\]

\[
\begin{bmatrix}
M_2 & PE \\
P & -\beta_2 \gamma^2
\end{bmatrix} \leq 0,
\]

where

\[
M_1 = A^T C^T C + C'TCA - \lambda_1 (P - C'TC) + \beta_1 C'TC,
\]

\[
M_2 = A^T P + PA - \lambda_2 (C'TC - P) + \beta_2 P.
\]

The objective is to minimize \( \gamma^2 \) subject to the above inequalities and that \( X_0 \subset L_V \), i.e., \( x_0^TPx_0 \leq \gamma^2, x_0^TC'Cx_0 \leq \gamma^2, i = 1, \ldots, K \). It seems that we need to fix 4 variables, \( \lambda_1, \lambda_2, \beta_1, \beta_2 \) to make all the constraint LMs. However, with a change of variables, we can turn the problem into a standard "gevp" problem by fixing two variables. Let \( \alpha_1 = 1/\lambda_1, \alpha_2 = \lambda_2, \beta_1 = \beta_1 \gamma^2/\lambda_1 \) and \( \beta_2 = \beta_2 \gamma^2 \), the two matrix inequalities can be rearranged as:

\[
\begin{bmatrix}
\beta_1 C'TC & 0 \\
0 & 0
\end{bmatrix} \leq \gamma^2 \begin{bmatrix}
-\alpha_1 (A'C'C + C'TCA) + P - C'TC & \alpha_1 C'TC \\
\alpha_1 E'C'C & \beta_1
\end{bmatrix}
\]

(7)

\[
\begin{bmatrix}
\beta_2 P & 0 \\
0 & 0
\end{bmatrix} \leq \gamma^2 \begin{bmatrix}
-A^T P - PA + \alpha_2 (C'TC - P) & PE \\
E'TP & \beta_2
\end{bmatrix}
\]

(8)

Note that the two matrices in (7) are linear with respect to all variables and the two matrices in (8) are linear in \( P \) for fixed \( \alpha_2 \) and \( \beta_2 \). When \( \alpha_2, \beta_2 \) are fixed, the minimal \( \gamma \) can be obtained by solving a "gevp" problem. If we define the minimal \( \gamma \) for the "gevp" problem as a function of \( \alpha_2 \) and \( \beta_2 \), \( \gamma_1(\alpha_2, \beta_2) \), we may use "fminsearch" in Matlab to find the minimal \( \gamma_1 \) over \( \alpha_2, \beta_2 \in [0, \infty) \).

Also note that the optimization problem reduces to the corresponding problem in [4] if \( \alpha_1 = \alpha_2 = \beta_1 = 0 \).

In the absence of disturbance, i.e., \( w = 0 \) or \( E = 0 \), we can take \( \beta_1 = \beta_2 = 0 \) and the two inequalities reduce to

\[
\alpha_1 (C'TCA + A'TC'C) \leq P - C'TC,
\]

\[
PA + A^T P \leq \alpha_2 (C'TC - P)
\]

which become LMs when \( \alpha_2 \) is fixed.

We use two simple examples to demonstrate the improvement.

**Example 1:** Consider a second-order system with

\[
A = \begin{bmatrix}
0 & 1 \\
-0.1 & -1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad x_0 = \begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]

The maximal output estimated with invariant ellipsoid is 1.2252. Using the truncated ellipsoid, we obtain a smaller bound as 0.9161. The actual value for the maximal output is 0.8374.

**Example 2:** Consider a third-order system with disturbance, where

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-3 & -2 & -4
\end{bmatrix}, \quad E = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix},
\]

and \( x_0 = 0 \). The bound on the output obtained via invariant ellipsoid is 0.9023. The bound obtained via the truncated ellipsoid (constraints (7) and (8)) is 0.6789. The actual maximal output for this case equals the \( L_1 \) norm of the system, which is \( \int_0^\infty |Ce^{At}|dt = 0.6 \).

**B. Estimation of the maximal output and overshoot under step input**

Consider the system

\[
\dot{z} = Az + Bu + Eu, \quad y = Cz, \quad z_0 = 0,
\]

where \( z \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R} \) is a scalar output, \( u \in \mathbb{R}^m \) is a step input with final value \( u_f \) and \( w \) is the disturbance bounded by \( w^Tw \leq 1 \). Assume \( A \) is Hurwitz and \( w \) is piecewise continuous. We’d like to estimate the maximal \( y \) that will be reached during the transient response. To do this, we shift the origin to the steady state value of \( z \) for \( w = 0 \) by defining \( x = z + A^{-1}Bu_f \). Then

\[
\dot{x} = Ax + Eu, \quad y = Cx - CA^{-1}Bu_f, \quad x_0 = A^{-1}Bu_f.
\]

Let \( y_1(t) = Cx(t) \) and \( y_\infty \) be the steady state value of the output \( y \) in case of \( w = 0 \), i.e., \( y_\infty = -CA^{-1}Bu_f \). For simplicity, assume \( y_\infty > 0 \). Then \( y_1(0) = -y_\infty < 0 \). Denote

\[
y_{1,\text{max}} = \sup\{y_1(t) : t \geq 0, w^Tw \leq 1\}.
\]

It is clear that \( y_{1,\text{max}} \geq 0 \) since \( y_1(\infty) = 0 \) with \( w = 0 \). If \( y_{1,\text{max}} > 0 \), then the original output \( y \) has an overshoot equaling this value and the maximal value of \( y \) is \( y_{1,\text{max}} + y_\infty \). Unlike the problem of estimating the maximal absolute value of the output \( |y(t)| \) in Section III-A, we intend to estimate the maximal value of \( y_1(t) \).

Suppose that we have an invariant set \( L_V \) including \( x_0 \) for (11). The maximal value of \( y_1(t) \) for all \( t \geq 0 \) can be estimated by evaluating the maximal \( Cx \) over the entire \( L_V \). This might be too conservative since it is the same as the maximal \( |Cx| \) over \( L_V \) (a symmetric set), which is probably reached at \( t = 0 \) with the negative value \( y_1(0) = -y_\infty \). Since \( y_{1,\text{max}} \) is clearly reached at a certain \( t > 0 \) instead of \( t = 0 \), we can restrict our attention to a subset of \( L_V \), where a local extremal of \( y_1 \) is possible in the presence of \( w \).

**Proposition 2:** Let \( V(x) = \max\{x^TPx : j = 1, \ldots, J\} \) and suppose that \( L_V \) is an invariant set for (11) that includes \( x_0 \). If there exist \( \alpha_j \geq 0, j = 1, \ldots, J, \alpha_w \geq 0 \) and \( \alpha_c \in \mathbb{R} \) such that

\[
M_1 - \alpha_c A'C'CCE - \alpha_c E'C'CCE - \alpha_w I \leq 0,
\]

(12)
where

\[ M_1 = C^TC - \sum_{j=1}^{J} \alpha_j P_j - \alpha_c A^TC^CA, \]

then \( y_1(t_c)^2 \leq \alpha_1 + \cdots + \alpha_J + \alpha_w \) for every \( t_c > 0 \) where a local extreme of \( y_1 \) is reached. Thus \( y_{1,\text{max}} \leq \alpha_1 + \cdots + \alpha_J + \alpha_w \).

**Remark 1:** The main idea in Proposition 2 is to exclude \( t = 0 \), since \( |y_1(0)| \) is usually the maximal value of \( |y_1(t)| \) but \( y_1(0) \) is not the maximal value of \( y_1(t) \). If we extend the time to \( t < 0 \), \( y_1(0) \) may not be a local extreme of \( y_1(t) \) since there may exist no \( w \) such that \( C(Ax_0 + Ew) = 0 \).

**Remark 2:** The matrix inequality (12) can be replaced with

\[
\begin{bmatrix}
M_2 & -\alpha_c F'C
\
-\alpha_c E'C & -\alpha_w I
\end{bmatrix} \leq 0,
\]

where

\[ M_2 = C^TC - \sum_{j=1}^{J} \alpha_j P_j - \alpha_c (F'CA + A'C^TF) \]

and \( F \in \mathbb{R}^{1 \times n} \) is any row vector. This can be shown by following the same procedure as the proof of Proposition 2. Instead of using \( (Ax + Ew)^TC(Ax + Ew) = 0 \) as a consequence of \( C(Ax + Ew) = 0 \), we can use \( x^TF'C(Ax + Ew) = 0 \). This would introduce an additional variable \( F \) for optimization. We may pick \( F = C \) for simplicity. The resulting matrix inequality would be linear in \( A \) and the result can be extended to handle linear differential inclusions.

Combining the above discussion and the condition for the invariance of the set \( L_V \) in Section II, we can formulate an optimization problem to estimate a bound on the local extrema of \( y_1 \):

\[
\alpha^* = \inf \{ \alpha_1 + \cdots + \alpha_J + \alpha_w \}
\]

s.t. (6), (12)

\[
x_0^TP_jx_0 \leq 1; \ j = 1, \cdots, J,
\]

\[
\alpha_w \geq 0, \ \alpha_j, \beta_j \geq 0, \ \lambda_{jk} \geq 0, \ j, k = 1, \cdots, J,
\]

\[
P_j = P_j^T > 0, \ j = 1, \cdots, J,
\]

where (6) ensures that \( L_V \) is an invariant set. Then all the local extrema of \( y_1 \) and \( y_{1,\text{max}} \) are bounded by \( \sqrt{\alpha^*} \).

For the special case where \( V(x) = x^TPx \), the constraints (6), (12) reduce to

\[
\begin{bmatrix}
A^TP + PA + \beta_1 P & PE \\
E^TP & -\beta_1 I
\end{bmatrix} \leq 0,
\]

\[
\begin{bmatrix}
C'C - \alpha_1 P - \alpha_c A'C^TCA & -\alpha_c A'C^TCE \\
-\alpha_c E'C^TCA & -\alpha_c E'C^TCE - \alpha_w I
\end{bmatrix} \leq 0.
\]

When \( \alpha_1 \) and \( \beta_1 \) are fixed, the constraint becomes LMIs. So we can use “minsearch” to perform a two dimensional optimization on \( \alpha_1 \) and \( \beta_1 \). In the absence of disturbance \( (E = 0) \), with a change of variable, \( \alpha_1 P \rightarrow P \), the optimization problem can be further reduced to

\[
\inf_{P > 0, \alpha_c \alpha} \alpha,
\]

s.t. \( C'C - P - \alpha_c A'C^TCA \leq 0; \)

\( A^TP + PA \leq 0; \)

\( x_0^TPx_0 \leq \alpha. \)

**Example 3:** A second-order system is described as

\[
\dot{z} = \begin{bmatrix}
0 & 1 \\
-3 & -1
\end{bmatrix} z + \begin{bmatrix}
1 \\
1
\end{bmatrix} u + \begin{bmatrix}
0.1 \\
-0.1
\end{bmatrix} w,
\]

\( y = [1 \ 0]z, \ z(0) = 0. \)

Under a unit step input, the steady state output for \( w = 0 \) is \( y_\infty = 0.6667 \). When transformed to the state \( x = z + A^{-1}B \), we have \( x_0 = \begin{bmatrix}
-0.6667 \\
1
\end{bmatrix} \) and \( y_1(0) = -0.6667 \).

We first consider the case where \( w = 0 \). With a quadratic function, a bound for the maximal \( y_1(t) \) is obtained as 0.5971 (89.5%). With \( V(x) = \max\{x^TP_1x, x^TP_2x\} \), the bound is reduced to 0.5268 (79%). The resulting invariant sets are plotted in Fig. 1, where the outer curve is the boundary of the invariant ellipsoid from quadratic function and the inner closed curve in dash-dotted curve is the boundary of the invariant set as the intersection of two ellipsoids. The horizontal dotted line is \( CAx = 0 \), where the local extreme of \( y_1 \) is obtained. The actual overshoot determined from simulation is 0.3718 (55.8%). Fig. 1 plots the trajectory starting from the initial condition \( x_0 \). It should be noted that

![Fig. 1. Two invariant sets for estimating overshoot for \( w = 0 \).](image-url)
plotted in Fig. 2 with dash-dotted, dashed and solid curves, respectively. It should be mentioned that the maximal $y_1$ due
to $w$ only is $y_{1,w,\text{max}}(\infty) = 0.1278$, which is much greater
than the increase of output bound (0.4288-0.3718=0.057)
and the increase of the estimate of output bound (0.5757-
0.5268=0.0498). It is expected that the estimate of
$x$ where $x$ is constrained within a given bound. For simplicity, assume
that the bound for each output is 1, i.e., $|y_i(t)| \leq 1$ for all
$i = 1, 2, \ldots, p$. Denote the corresponding state constraint set
as
$$X_e = \{x \in \mathbb{R}^n : |C_i x| \leq 1, i = 1, \ldots, p\}.$$ The problem of determining the largest admissible set is
usually converted into one of finding the largest invariant
set inside $X_e$. A simple solution is to find a maximal
invariant ellipsoid inside the constraint set $X_e$, which can be
formulated as an LMI problem. To reduce the conservatism,
the recent work [14] proposed an interesting invariant set as
the intersection of $X_e$ and an invariant ellipsoid. Condition
for the invariance of the intersection is derived as nonlinear
matrix inequalities and the size of the invariant set is
maximized with a modified Newton’s method.

Here we consider the same type of invariant set as in [14].
It is actually the 1-level set of the Lyapunov function
$$V(x) = \max \{x^T P x, x^T C_j^T C_j x : j = 1, 2, \ldots, p\}.$$ As we mentioned earlier, the 1-level set $L_V = \{x \in \mathbb{R}^n : V(x) \leq 1\}$ is formed by truncating the ellipsoid $E(P)$ with
planes $C_i x = \pm 1$ and thus lies within the state constraint $X_e$.

In this paper, we will take a quite different approach to
equip the invariance of $L_V$ as compared to the method in
[14]. An important relaxation is that we don’t require the
ellipsoid $E(P)$ to be invariant. Instead, we directly give a
condition for the invariance of $L_V$ by using Proposition 1,
which can be considered as the result of the $S$ procedure.
Moreover, our method can be used to handle persistent
disturbances.

By Proposition 1, a sufficient condition for $L_V$ to be
invariant is: there exist $\lambda_{jk} \geq 0, \beta_j \geq 0, j, k = 0, \ldots, p$ such that
$$M_0 P E + E^T P \beta_j I \leq 0, $$
$$M_j P_j E + \beta_j I \leq 0, j = 1, \ldots, p,$$
where
$$M_0 = A^T P + PA - \sum_{k=1}^p \alpha_{0k} P_k + \left(\sum_{k=1}^p \lambda_{0k}\right) P + \beta_0 P,$$
$$M_j = A^T P_j + P_j A - \lambda_{0j} (P - P_j) - \sum_{k=1}^p \lambda_{jk} (P_k - P_j) + \beta_j P_j,$$
and $P_j = C_j^T C_j$. The above become LMIs when $\Sigma_{k=1}^p \lambda_{0k}, \beta_0$
and $\lambda_{0j}$ are fixed. In the absence of $w$ or $E = 0$, we can set
$\beta_j = 0$. With some manipulation, the condition can be further
reduced to: there exist $a_j > 0, b_{jk} > 0, \alpha_0 \geq 0, \alpha_j \geq 0,$
$j, k = 1, 2, \ldots, p$, such that $\Sigma_{j=1}^p \alpha_j = \alpha_0$, and
$$A^T P + PA \leq -\alpha_0 P + \sum_{j=1}^p \alpha_j C_j^T C_j,$$
$$a_j (A^T C_j C_j + C_j^T C_j) \leq P - C_j^T C_j,$$
$$+ \sum_{k=1}^p b_{jk} (C_k^T C_k - C_j^T C_j), \quad j = 1, 2, \ldots, p. \quad (17)$$
The invariant set $L_V$ can be maximized with respect to
certain shape reference set $X_R$ such that $\eta X_R \subset L_V$ for
the maximal $\eta$. The inclusion condition $\eta X_R \subset L_V$ can
be stated as LMIs if $X_R$ is a polytope or an ellipsoid. For
example, consider $X_R = \text{co}\{x_i : i = 1, 2, \ldots, K\}$. Then
$$\eta X_R \subset L_V \text{ if and only if}$$
$$x_i^T P x_i \leq 1/\eta^2, \quad x_i^T C_j^T C_j x_i \leq 1/\eta^2,$$
$$\forall i = 1, \ldots, K, j = 1, \ldots, J. \quad (18)$$
An optimization problem can be formulated to maximize $\eta$
satisfying (16), (17) and (18). Note that all the conditions in
(16) are LMIs and the condition (17) is LMI for a fixed $\alpha_0$.

The method can be easily extended to linear differential
inclusions by duplicating the matrix inequalities for each vertex
matrix $A_k$, with respective coefficients $a_{ik}, b_{ijk}, \alpha_{0k}, \alpha_{jk}$.
This is because $L_V$ is a convex set. It is invariant for the
linear differential inclusion if and only if it is invariant for
each vertex system.
Example 4: Consider a third-order system in [14]:
\[ \dot{x} = (A - BK)x, \]
where
\[
A = \begin{bmatrix}
-100 & 1 & -2 \\
1 & -1 & 0 \\
0 & 1 & 0
\end{bmatrix},
B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
and \( K = [0.360 \ -0.053 \ -0.671] \). The state constraint set is
\[ X_c = \{ x \in \mathbb{R}^3 : |x_i| \leq 1, i = 1, 2, 3, |Kx| \leq 1 \}. \]

The truncated ellipsoid obtained in [14] is \( S_1 = X_c \cap \{ x : x^TPx \leq 1 \} \), where
\[
P = \begin{bmatrix}
0.130 & 0.127 & -0.030 \\
0.127 & 0.228 & 0.198 \\
-0.030 & 0.198 & 1.007
\end{bmatrix}.
\]

The surface of the set \( S_1 \) is plotted in Fig. 3.

The truncated ellipsoid obtained in this paper is \( S_2 = X_c \cap \{ x : x^TP_1x \leq 1 \} \) where
\[
P_1 = \begin{bmatrix}
0.0342 & 0.0679 & 0.0109 \\
0.0679 & 0.1368 & 0.0638 \\
0.0109 & 0.0638 & 0.8427
\end{bmatrix}.
\]

The surface of the set \( S_2 \) is plotted in Fig. 4.

Using digital integration, the volume of \( S_1 \) and \( S_2 \) are estimated via optimization problems whose constraints become LMIs when a few scalar variables are fixed.

**REFERENCES**