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Abstract—This paper investigates the delay-dependent exponential stability problem of Takagi-Sugeno (TS) fuzzy Hopfield neural networks (HNNs) with time-varying delay. Based on a fuzzy Lyapunov-Krasovskii functional (LKF), some delay-dependent stability criteria guaranteeing the exponential stability of the fuzzy HNNs are devised by taking the relationship between the terms in the Leibniz-Newton formula into account. Since free weighting matrices are used to express this relationship and the appropriate ones are selected by means of linear matrix inequalities (LMIs), the criteria are less conservative than existing ones reported in the literature for delayed fuzzy neural networks. A simulation example is provided to illustrate the effectiveness of the developed method.

I. INTRODUCTION

Hopfield neural networks (HNNs) and their various generalizations have been successfully employed in many areas such as pattern recognition, associate memory and knowledge acquisition [1],[2]. Such applications of neural networks heavily depend on the dynamical behaviors of the networks. Therefore, stability analysis for neural networks has been investigated and a great number of approaches have been proposed [3]-[6]. Since time delays as a source of instability and bad performance always appear in many neural networks owing to the finite speed of information processing, the stability analysis for the delayed neural networks has received considerable attention. The existing results can be classified into two types: delay-independent criteria [3],[4] and delay-dependent criteria [5],[6]. The former is irrespective of the size of the delay and the latter is concerned with the size of the delay. It has been shown that the delay-dependent stability conditions are generally less conservative than the delay-independent ones, especially when the size of the delay is small.

In the past two decades, the fuzzy logic theory has provided an appealing and efficient approach to deal with the analysis and synthesis problems for complex nonlinear systems. Recently, the dynamic Takagi-Sugeno (TS) fuzzy model [7] has become a popular tool and has been employed in most model-based fuzzy analysis approaches. The main feature of the TS fuzzy model is that a nonlinear system can be approximated by a set of TS linear models. The overall fuzzy model of the system is achieved by fuzzy blending of the set of TS linear models. The stability issue of fuzzy control systems has been discussed in an extensive literature. Most of the existing results were obtained by using a single Lyapunov function (SLF) method [8]. However, the main drawback associated to this method is that an SLF must work for all linear models, which in general leads to a conservative result. To relax this conservatism, the fuzzy Lyapunov-Krasovskii functional (LKF), which directly includes the membership functions, has been proposed to derive the stabilization conditions for TS fuzzy control systems [9],[10].

Very recently, the TS fuzzy models are used to describe the delayed fuzzy neural networks [11]-[13]. In [11], the global exponential stability in the mean square for the stochastic fuzzy HNNs with time-varying delay was studied by using the Lyapunov-Krasovskii approach. In [12], the globally robustly asymptotically stable conditions were presented for the uncertain fuzzy bi-directional associative memories (BAM) neural networks with time-varying delays. In [13], the global exponential stability problem of TS fuzzy cellular neural networks with time-varying delay was investigated based on the Lyapunov functional theory and linear matrix inequality techniques. However, most of the existing results for the delayed fuzzy neural networks are dedicated to delay-independent conditions. Furthermore, the works in [11]-[13] are based on a single LKF which further increases the conservatism of the results.

Motivated by the above discussion, the aim of this paper is to study the delay-dependent exponential stability for the fuzzy HNNs with time-varying delay by using a fuzzy LKF approach. A free-weighting matrix method combining with LMI techniques is employed to derive some new delay-dependent exponential stability criteria for the fuzzy HNNs. In contrast to the existing methods [11],[12], the proposed method reduces the conservatism of the stability results from three main aspects. The first one is that a fuzzy LKF is employed to further reduce the conservatism of the results. The second is that neither any model transformation nor any bounding technique for bounding cross terms is needed in the derivation processes. The third is that the time derivative of time-varying delay must be smaller than one is released in the proposed scheme. Moreover, the derived conditions are expressed in terms of linear matrix inequalities (LMIs), which can be checked numerically very efficiently via the LMI toolbox.

Notations: Throughout this paper, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, and $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. $I$ denotes the identity matrix with appropriate dimensions and diag(·) denotes the diagonal matrix. $X^T$ and $X^{-1}$ denote respectively the transpose and
We use $X > 0$ ($X < 0$) to denote a positive- (negative-) definite matrix $X$. $\lambda_{\max}(X)$ ($\lambda_{\min}(X)$) denotes the maximum (minimum) eigenvalue of $X$. The symbol $*$ is used to denote a matrix which can be inferred by symmetry.

II. MODEL DESCRIPTION AND PRELIMINARIES

The model of Hopfield neural networks with time-varying delay can be expressed as follows:

$$\dot{u}_i(t) = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t - \tau_j(t))) + J_i,$$

where $u_i(t)$ ($i = 1, 2, \ldots, n$) is the state variable of the $i$th neuron at time $t$; $d_i > 0$ represents the passive decay rate; $a_{ij}$ is the synaptic connection weight; $g_j(\cdot)$ is the activation function of the neuron; $J_i$ denotes the external input; $\tau_j(t)$ represents the time-varying delay of the $j$th neuron; $\theta_j$ is the premise variable vector. The model of Hopfield neural networks with time-varying delay can be expressed as follows:

$$\dot{\hat{x}}(t) = -C(t) \hat{x}(t) + \hat{A}(t) f(\hat{x}(t)),$$  \hspace{1cm} (10)

where $\hat{x}(t)$ is the state variable of the fuzzy system, $C(t)$ and $\hat{A}(t)$ are the constant matrices, $f(\cdot)$ is the activation function of the neuron.

The final output of the fuzzy system is inferred as follows:

$$\hat{y}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) [ -D_k x(t) + A_k f(x(t - \tau(t))) ],$$  \hspace{1cm} (5)

where $\mu_k(\theta(t)) = \frac{v_k(\theta(t))}{\sum_{j=1}^{p} v_j(\theta(t))}$, $v_k(\theta(t)) = \prod_{j=1}^{p} \eta_{jk}(\theta_j(t))$, in which $\eta_{jk}(\theta_j(t))$ is the grade of membership of $\theta_j(t)$ in $\eta_{jk}$. According to the theory of fuzzy sets, we have $v_k(\theta(t)) \geq 0, k = 1, 2, \ldots, r, \sum_{k=1}^{r} v_k(\theta(t)) > 0$, for all $t$. Therefore, it implies $\mu_k(\theta(t)) \geq 0, k = 1, 2, \ldots, r, \sum_{k=1}^{r} \mu_k(\theta(t)) = 1$, for all $t$.

Definition 1. The system (5) is said to be globally exponentially stable with a convergence $\lambda$ if there exist constants $\alpha > 0$ and $\nu \geq 1$ such that

$$\|x(t)\| \leq \nu \sup_{-\tau \leq \theta \leq 0} \|x(\theta)\| e^{-\alpha t} \quad \text{for all } t \geq 0.$$  \hspace{1cm} (8)

In order to confirm that the origin of (5) is globally exponentially stable, let $\hat{x}(t) = e^{\alpha t} \hat{x}(t)$ and the dynamics of (3) can be transformed into the following form:

\textbf{Plant Rule $k$:}

\textbf{IF} $\theta_i(t)$ is $n_i^k$ and \ldots and $\theta_p(t)$ is $n_p^k$ \textbf{THEN}

$$\hat{x}(t) = -(D_k - \alpha I) \hat{x}(t) + A_k \hat{f}(\hat{x}(t - \tau(t))),$$

$$\hat{f}(\hat{x}(t - \tau(t))) = \|\hat{f}(\hat{x}(t - \tau(t)))\|^2 \leq e^{2\alpha t} |\hat{f}(t - \tau(t))|^2,$$  \hspace{1cm} (3)

$$\hat{f}(\hat{x}(t - \tau(t))) = \|\hat{f}(\hat{x}(t - \tau(t)))\|^2 \leq e^{2\alpha t} |\hat{f}(t - \tau(t))|^2,$$  \hspace{1cm} (4)

where $\alpha > 0$. The final output of the fuzzy system is inferred as follows:

$$\hat{y}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) \left[ -C_k \hat{x}(t) + A_k \hat{f}(\hat{x}(t - \tau(t))) \right].$$  \hspace{1cm} (9)

For convenience, we set

$$\hat{C}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) C_k, \quad \hat{A}(t) = \sum_{k=1}^{r} \mu_k(\theta(t)) A_k,$$

then the system (9) can be rewritten as

$$\dot{\hat{x}}(t) = -\hat{C}(t) \hat{x}(t) + \hat{A}(t) \hat{f}(\hat{x}(t - \tau(t))).$$  \hspace{1cm} (10)

4297
To give our main results in the next section, we need the following lemmas.

**Lemma 1.** For any vectors $a, b \in \mathbb{R}^n$, the inequality

$$2a^Tb \leq a^TXa + b^TX^{-1}b$$

holds, where $X$ is any positive matrix (i.e., $X > 0$).

**Lemma 2** (Schur Complement [26]). Given constant matrices $\Omega_1, \Omega_2, \Omega_3$, where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then $\Omega_1 + \Omega_3 \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix}
\Omega_1 & \Omega_3^T
\
\Omega_3 & -\Omega_2
\end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix}
-\Omega_2 & \Omega_3
\Omega_3^T & \Omega_1
\end{bmatrix} < 0.$$

**III. MAIN RESULTS**

In this section, we will derive some delay-dependent criteria for the global exponential stability of the delayed fuzzy system (10) based on a fuzzy LKF.

**Theorem 1.** For the system (10), suppose (H) hold. Given scalars $\tau$ and $\sigma$, the equilibrium point of system (10) is globally exponentially stable with a convergence rate $\alpha > 0$, if there exist a matrix $P = P^T > 0$, time-varying matrices $Q(t) = Q_1(t) \geq 0$, $R(s) = R(s) \geq 0$, $s \in [t - \tau(t), t]$, $\bar{X}_1(t)$, $\bar{Y}_1(t)$, $\bar{Z}_{lm}(t)$, $l \leq m = 1, 2, 3, 4$ and a positive constant $\eta$ satisfying the following inequalities:

$$\Psi(t) = \begin{bmatrix}
\Omega_{11}(t) & \Omega_{12}(t) & \Omega_{13}(t) & \Omega_{14}(t) \\
\ast & \Omega_{22}(t) & \Omega_{23}(t) & \Omega_{24}(t) \\
\ast & \ast & \Omega_{33}(t) & \Omega_{34}(t) \\
\ast & \ast & \ast & \Omega_{44}(t)
\end{bmatrix} < 0$$

$$\Phi(t, s) = \begin{bmatrix}
\hat{Z}_{11}(t) & \hat{Z}_{12}(t) & \hat{Z}_{13}(t) & \hat{Z}_{14}(t) \\
\ast & \hat{Z}_{22}(t) & \hat{Z}_{23}(t) & \hat{Z}_{24}(t) \\
\ast & \ast & \hat{Z}_{33}(t) & \hat{Z}_{34}(t) \\
\ast & \ast & \ast & \hat{Z}_{44}(t)
\end{bmatrix} \geq 0,$$

for all $t$, where

$$\Omega_{11}(t) = \hat{Q}(t) + \hat{X}_1(t) + \hat{X}_1^T(t) - \hat{Y}_1(t)\hat{C}(t)$$

$$\Omega_{12}(t) = -\hat{X}_1(t) + \hat{X}_1^2(t) - \hat{C}(t)\hat{Y}_1^2(t),$$

$$\Omega_{13}(t) = \hat{X}_1^T(t) + \hat{Y}_1(t)\hat{A}(t) - \hat{C}(t)\hat{Y}_1^T(t),$$

$$\Omega_{14}(t) = \hat{P} + \hat{X}_1^T(t) - \hat{Y}_1(t) - \hat{C}(t)\hat{Y}_1^T(t),$$

$$\Omega_{22}(t) = -(1 - \sigma)\hat{Q}(t) - \hat{X}_2(t) - \hat{X}_2^T(t)$$

$$+ \eta e^{2\alpha t}L^2,$$

$$\Omega_{23}(t) = -\hat{X}_1^T(t) + \hat{Y}_2(t)\hat{A}(t),$$

$$\Omega_{24}(t) = -\hat{X}_1^T(t) - \hat{Y}_2(t),$$

$$\Omega_{33}(t) = -\eta I + \hat{Y}_3(t)\hat{A}(t) + \hat{A}^T(t)\hat{Y}_3^T(t),$$

$$\Omega_{34}(t) = -\hat{Y}_3(t) + \hat{A}^T(t)\hat{Y}_3^T(t),$$

$$\Omega_{44}(t) = \tau \hat{R}(t) - \hat{Y}_4(t) - \hat{Y}_4^T(t).$$

**Proof.** Define the following free fuzzy weighting matrices:

$$\bar{X}_1(t) = \sum_{k=1}^{r} \mu_k(\theta(t))X_{ik}, \quad \bar{Y}_1(t) = \sum_{k=1}^{r} \mu_k(\theta(t))Y_{ik},$$

$$\bar{Z}_{lm}(t) = \sum_{k=1}^{r} \mu_k(\theta(t))Z_{lmk},$$

where $X_{ik} \in \mathbb{R}^{n \times n}$, $Y_{ik} \in \mathbb{R}^{n \times n}$ and $Z_{lmk} \in \mathbb{R}^{n \times n}$, $l \leq m = 1, 2, 3, 4, k \in \mathcal{S}.$

From Leibniz-Newton formula and (8), we have

$$\int_{t-\tau(t)}^{t} \hat{x}(s)ds = \hat{x}(t) - \hat{x}(t-\tau(t)), \quad (13)$$

$$e^{2\alpha t} \hat{x}^T(t) - \tau(t))L^2 \hat{x}(t-\tau(t))$$

$$- \hat{f}^T(\hat{x}(t-\tau(t)))\hat{f}(\hat{x}(t-\tau(t))) \geq 0. \quad (14)$$

Therefore, by considering (10) and (14), for some time-varying matrices $\bar{X}_1(t)$ and $\bar{Y}_1(t)$, $l = 1, 2, 3, 4$, we obtain

$$\gamma_1(t) = 2[\hat{x}^T(t)\hat{X}_1(t) + \hat{x}^T(t-\tau(t))\hat{X}_2(t)]$$

$$+ \hat{f}^T(\hat{x}(t-\tau(t)))\hat{X}_3(t) + \hat{x}^T(t)\hat{X}_4(t)$$

$$\times [\hat{x}(t) - \hat{x}(t-\tau(t)) - \int_{t-\tau(t)}^{t} \hat{x}(s)ds]$$

$$= 0, \quad (15)$$

$$\gamma_2(t) = 2[\hat{x}^T(t)\hat{Y}_1(t) + \hat{x}^T(t-\tau(t))\hat{Y}_2(t)]$$

$$+ \hat{f}^T(\hat{x}(t-\tau(t)))\hat{Y}_3(t) + \hat{x}^T(t)\hat{Y}_4(t)$$

$$\times [-C(t)\hat{x}(t) + A(t)\hat{f}(\hat{x}(t-\tau(t))) - \hat{x}(t)]$$

$$= 0. \quad (16)$$

Since (12) implies that

$$Z(t) = \begin{bmatrix}
\bar{Z}_{11}(t) & \bar{Z}_{12}(t) & \bar{Z}_{13}(t) & \bar{Z}_{14}(t) \\
\ast & \bar{Z}_{22}(t) & \bar{Z}_{23}(t) & \bar{Z}_{24}(t) \\
\ast & \ast & \bar{Z}_{33}(t) & \bar{Z}_{34}(t) \\
\ast & \ast & \ast & \bar{Z}_{44}(t)
\end{bmatrix} \geq 0,$$

we have the following inequality:

$$\tau \xi^T(t)Z(t)\xi(t) - \int_{t-\tau(t)}^{t} \xi^T(t)\xi(t)ds \geq 0, \quad (17)$$

where $\xi(t) = \begin{bmatrix} \hat{x}^T(t) \hat{x}(t-\tau(t)) \hat{f}(\hat{x}(t-\tau(t))) \hat{x}^T(t) \end{bmatrix}^T$.

Consider the following fuzzy LKF for the system (10):

$$V(t) = \hat{x}^T(t)P\hat{x}(t) + \int_{t-\tau(t)}^{t} \hat{x}^T(s)Q(s)\hat{x}(s)ds$$

$$+ \int_{t-\tau(t)}^{t} \hat{x}^T(s)\bar{R}(s)\hat{x}(s)dsd\beta, \quad (18)$$

where $Q(s) = \sum_{k=1}^{r} \mu_k(\theta(s))Q_k$ and $\bar{R}(s) = \sum_{k=1}^{r} \mu_k(\theta(s))R_k$ are fuzzy weighting matrices which include the membership functions, $0 < P = P^T \in \mathbb{R}^{n \times n}, 0 < Q_k = Q_k^T \in \mathbb{R}^{n \times n}, 0 < R_k = R_k^T \in \mathbb{R}^{n \times n}, k \in \mathcal{S}.$ As discussed in [10], a single matrix $P$ is used in the first term of
(18) instead of a fuzzy weighting matrix \( \sum_{k=1}^{r} \mu_k(\theta(t)) P_k \). The reason of such construction is to avoid the difficulty of determining the upper bound of \( |\mu_k(\theta(t))|, \ k \in S \) in the stability analysis.

With (14)-(17), the time derivative of \( V(t) \) along the trajectory of (10) is computed as

\[
\dot{V}(t) = 2\bar{x}^T(t)P\dot{x}(t) + \tilde{x}^T(t)\tilde{Q}(t)\tilde{x}(t) - (1-\tau(t))\tilde{x}^T(t)(t-\tau(t))\tilde{Q}(t-\tau(t))\tilde{x}(t-\tau(t)) + \tau\tilde{x}^T(t)\bar{R}(t)\tilde{x}(t) - \int_{t-\tau}^{t} \tilde{x}^T(s)\bar{R}(s)\tilde{x}(s)ds \leq 2\bar{x}^T(t)P\dot{x}(t) + \tilde{x}^T(t)\tilde{Q}(t)\tilde{x}(t) + \tau\tilde{x}^T(t)\bar{R}(t)\tilde{x}(t) - \int_{t-\tau}^{t} \tilde{x}^T(s)\bar{R}(s)\tilde{x}(s)ds + \gamma_1(t) + \eta_2\tilde{x}^T(t)(t-\tau(t))L^2\tilde{x}(t-\tau(t)) + \gamma_2(t) - \eta_2\tilde{x}^T(t)(t-\tau(t))\tilde{f}(\tilde{x}(t-\tau(t))) \geq \xi^T(t)\Psi(t)\xi(t) - \int_{t-\tau}^{t} \xi^T(s)\Phi(t,s)\xi(s),
\]

(19)

where \( \xi^T(t) = \left[ \xi^T(t), \tilde{x}^T(s) \right] \).

Obviously, \( \Psi(t) < 0 \) and \( \Phi(t,s) \geq 0 \) as defined in (11) and (12) imply that \( \dot{V}(t) < 0 \) for any \( \tilde{x}(t) \neq 0 \). It follows that

\[
V(t) \leq V(0).
\]

According to (18), we have

\[
V(0) = \dot{x}^T(0)P\dot{x}(0) + \int_{-\tau(0)}^{0} \tilde{x}^T(s)\tilde{Q}(s)\tilde{x}(s)ds + \int_{-\tau}^{0} \int_{-\tau}^{0} \tilde{x}^T(s)\bar{R}(s)\tilde{x}(s)dsd\beta, \leq \lambda_{\max}(P)\|\dot{x}\|^2 + \lambda_{\max}(Q_M)\int_{-\tau(0)}^{0} \tilde{x}^T(s)\tilde{x}(s)ds + \lambda_{\max}(R_M)\int_{-\tau}^{0} \int_{-\tau}^{0} \tilde{x}^T(s)\tilde{x}(s)dsd\beta,
\]

(21)

where \( Q_M = \sup_{-\tau \leq s \leq 0} Q(s), R_M = \sup_{-\tau \leq s \leq 0} \bar{R}(s) \) and \( \dot{x} = \sup_{-\tau \leq s \leq 0} \|\tilde{x}(s)\| \).

It follows from Lemma 1 that

\[
\dot{x}^T(s)\tilde{x}(s) = \left[ -\bar{C}(s)\tilde{x}(s) + \tilde{A}(s)\tilde{f}(\tilde{x}(s)) \right]^T \times \left[ -\bar{C}(s)\tilde{x}(s) + \tilde{A}(s)\tilde{f}(\tilde{x}(s)) \right],
\]

\[
\leq 2 \left[ \dot{x}^T(s)C^T(s)\bar{C}(s)\tilde{x}(s) + \tilde{f}^T(s)\bar{A}(s)\tilde{x}(s) \right] \bar{A}(s)\tilde{f}(\tilde{x}(s)) + \bar{A}(s)\tilde{f}(\tilde{x}(s)) \right] \bar{A}(s)\tilde{f}(\tilde{x}(s)) \right] \|\tilde{x}\|^2, \leq \lambda_{\max}(C_MC_M) + \lambda_{\max}(A^T_MA_M)\lambda_{\max}(L^2) \|\tilde{x}\|^2,
\]

(22)

where \( C_M = \sup_{-\tau \leq s \leq 0} \bar{C}(s) \) and \( A_M = \sup_{-\tau \leq s \leq 0} \bar{A}(s) \).

Thus

\[
V(0) \leq \lambda_{\max}(P)\|\dot{x}\|^2 + \tau\lambda_{\max}(Q_M)\|\dot{x}\|^2 + \eta_2\lambda_{\max}(Q_M)\|\dot{x}\|^2 + \lambda_{\max}(A^T_MA_M)\lambda_{\max}(L^2) \|\tilde{x}\|^2 \leq \lambda_{\max}(P)\|\dot{x}\|^2.
\]

(23)

Therefore, we have

\[
\lambda_{\min}(P)\|\dot{x}\|^2 \leq \lambda\|\dot{x}\|^2.
\]

(25)

Furthermore, from \( \dot{x}(t) = e^{\alpha t}x(t) \), we can conclude the following result:

\[
\|x(t)\| \leq \sqrt{\frac{\lambda}{\lambda_{\min}(P)}} \|e^{-\alpha t}\| \|\dot{x}\|. \leq \left[ \Omega_{11}(t) \Omega_{12}(t) \Omega_{13}(t) \Omega_{14}(t) \right] \begin{bmatrix} \tau \tilde{X}_1(t) \\ \tau \tilde{X}_2(t) \\ \tau \tilde{X}_3(t) \\ \tau \tilde{X}_4(t) \end{bmatrix} < 0, \]

(27)

for all \( t \), where \( \Omega_{ij}(t), i \leq j = 1, 2, 3, 4 \) are the same as in (11).
Proof. In the following, we take
\[
\begin{bmatrix}
\bar{Z}_{11}(t) & \bar{Z}_{12}(t) & \bar{Z}_{13}(t) & \bar{Z}_{14}(t) \\
\ast & \bar{Z}_{22}(t) & \bar{Z}_{23}(t) & \bar{Z}_{24}(t) \\
\ast & \ast & \bar{Z}_{33}(t) & \bar{Z}_{34}(t) \\
\ast & \ast & \ast & \bar{Z}_{44}(t)
\end{bmatrix}
\]
and \(\Gamma_{ij}, km, i \leq j = 1, 2, 3, 4\) are the same as in (31).

Obviously, (28) ensures \(\Phi(t, s) \geq 0\) because
\[
\begin{bmatrix}
\bar{X}_1(t) & \bar{X}_2(t) & \bar{X}_3(t) & \bar{X}_4(t) \\
\ast & \bar{R}^{-1}(s) & \bar{R}^{-\frac{1}{2}}(s) & \bar{R}^{-\frac{1}{2}}(s) \\
\ast & \ast & \bar{R}^{-\frac{1}{2}}(s) & \bar{R}^{-\frac{1}{2}}(s) \\
\ast & \ast & \ast & \bar{R}^{2}(s)
\end{bmatrix}
\begin{bmatrix}
\bar{X}_1(t)R^{-\frac{1}{2}}(s) \\
\bar{X}_2(t)R^{-\frac{1}{2}}(s) \\
\bar{X}_3(t)R^{-\frac{1}{2}}(s) \\
\bar{X}_4(t)R^{-\frac{1}{2}}(s)
\end{bmatrix}^T \geq 0.
\]

According to Lemma2, the inequality (11) with the choice of (28) is equivalent to (27). The proof is completed.

Remark 1. It is noted that restricting \(\bar{R}(s) > 0\), \(s \in [t - \tau(t), t]\) for the stability condition of Theorem 1 may increase the conservatism of the overall condition. However, we can obtain a LMI-based exponential stability condition with fewer number of involved decision variables which definitely accelerates computation from Corollary 1.

Theorem 2. For the system (10), suppose (H) hold. Given scalars \(\tau\) and \(\sigma\), the equilibrium point of system (10) is globally exponentially stable with a convergence rate \(\alpha > 0\), if there exist matrices \(P = P^T > 0\), \(Q_k = Q_k^T \geq 0\), \(R_k = R_k^T > 0\), \(X_k, Y_k, l = 1, 2, 3, 4, k \in \mathcal{S}\) and a positive constant \(\eta\) satisfying the following LMIs:
\[
\Xi_{pgkm} < 0, \quad \rho, \varrho, k \in \mathcal{S}, \quad l = 1, 2, 3, 4, \quad \eta > 0. \tag{29}
\]

where
\[
\Xi_{pgkm} = \begin{bmatrix}
Q_k + \Gamma_{11,km} & \Gamma_{12,km} & -Q_k \rho + \Gamma_{22,k} & \ast \\
\ast & -Q_k \rho + \Gamma_{22,k} & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{bmatrix}, \tag{31}
\]

and
\[
\begin{align*}
\Gamma_{11,km} &= X_k + X_k^T - Y_k C_k + C_k Y_k^T, \\
\Gamma_{12,km} &= -X_k + X_k^T - C_k Y_k^T, \\
\Gamma_{13,km} &= X_k^T + Y_k A_k - C_k Y_k^T, \\
\Gamma_{14,km} &= X_k^T - Y_k - C_k Y_k^T, \\
\Gamma_{22,km} &= -X_k - X_k^T + \eta \sigma^2 \tau L^2, \\
\Gamma_{23,km} &= -X_k - Y_k^T + A_k Y_k^T, \\
\Gamma_{24,km} &= -Y_k + A_k Y_k^T, \\
\Gamma_{33,km} &= Y_k A_k + A_k Y_k^T, \\
\Gamma_{34,km} &= -Y_k^T + A_k Y_k^T, \\
\Gamma_{44,m} &= -Y_m - A_m Y_m^T, \\
\Gamma_{ij,km}, i \leq j &= 1, 2, 3, 4, \text{ are the same as in (31).}
\end{align*}
\]

Proof. The inequality (27) can be rewritten as
\[
\sum_{\rho=1}^{r} \sum_{k=1}^{r} \sum_{m=1}^{r} \mu_{\rho} (\theta(t - \tau(t))) \mu_{\varphi} (\theta(s)) \mu_{k} (\theta(t)) \Xi_{pgkm} < 0, \quad s \in [t - \tau(t), t]. \tag{32}
\]

According to the Theorem 2.2 in [8], with \(\Xi_{pgkm}\) given in (31), if the conditions (29) and (30) hold, then (32) is fulfilled. Therefore, it follows from Corollary 1 that the equilibrium of the system (10) is globally exponentially stable with the convergence rate \(\alpha\).

Remark 2. The number of LMIs in Theorem 2 is \(r^3 + r^2(r - 1)\), which may lead to a great computational effort. However, we can get a trade-off between the increase of conservatism and the reduction of number in LMIs by choosing certain matrices. For instance, by taking \(Q_k = Q\) or \(R_k = R\), \(k \in \mathcal{S}\), the number of LMIs is \(r^2 + r^2(r - 1)\). If we take \(Q_k = Q\) and \(R_k = R\), \(k \in \mathcal{S}\), then the number is \(r + r(r - 1)\), which would be greatly reduced. Therefore, we have the following corollary.

Corollary 2. For the system (10), suppose (H) hold. Given scalars \(\tau\) and \(\sigma\), the equilibrium point of system (10) is globally exponentially stable with a convergence rate \(\alpha > 0\), if there exist matrices \(P = P^T > 0\), \(Q = Q^T \geq 0\), \(R = R^T > 0\), \(X_k, Y_k, l = 1, 2, 3, 4, k \in \mathcal{S}\) and a positive constant \(\eta\) satisfying the following LMIs:
\[
\Xi_{kk} < 0, \quad k \in \mathcal{S}, \quad \Xi_{mk} < 0, \quad k, m \in \mathcal{S}, \quad k \neq m. \tag{33}
\]

where
\[
\Xi_{km} = \begin{bmatrix}
Q + \Gamma_{11,km} & \Gamma_{12,km} & -Q \rho + \Gamma_{22,k} & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{bmatrix}, \tag{35}
\]

and \(\Gamma_{ij,km}, i \leq j = 1, 2, 3, 4\) are the same as in (31).
V. CONCLUSION

In this paper, some delay-dependent exponential stability criteria are established for fuzzy Hopfield neural networks with time-varying delay by using the free-weighting matrix approach, the Leibniz-Newton formula and the Lyapunov method. A fuzzy LKF instead of a single LKF is employed to derive the proposed delay-dependent results, which reduce the conservatism of the results. The stability conditions are presented in terms of linear matrix inequalities. Finally, a numerical example is provided to illustrate the effectiveness of the derived results.

REFERENCES