Nonlinear Adaptive $H_\infty$ Control of Constrained Robotic Manipulators with Input Nonlinearity

Yoshihiko Miyasato

Abstract—The problem of constructing nonlinear adaptive $H_\infty$ control of constrained robotic manipulators with uncertain input nonlinearities such as dead-zone or backlash, is considered in this paper. In the proposed control scheme, adaptive inverse models are introduced to compensate effects of input nonlinearities, and the trajectory converges to the desired constrained trajectory, and the constraint force also follows the desired constraint one. The resulting control strategy is derived as a solution of certain $H_\infty$ control problem, where estimation errors of tuning parameters, errors of constraint forces and residual terms of the inverse models, are regarded as external disturbances to the process.

I. INTRODUCTION

Motion control problems of mechanical systems are divided into two categories, that is, free motion control and constrained motion control. Free motion control problems of mechanical systems are seen in the situations where there is no contact between controlled processes and environments, and have been studied extensively as basic control problems of mechanical systems [1], [2]. On the contrary, motion control problems of constrained mechanical systems are seen in the situations where there exists a contact between controlled processes and environments, and contact forces between end-effectors of mechanical systems and environments are generated. Compared with free motion control, constrained motion control has been a difficult problem, where not only constrained trajectory control but also simultaneous constraint force control should be considered [4], [5], [6], [7] [8], and the adaptive control version of that problem for mechanical systems with parametric uncertainties, is a difficult but important problem from the practical point of view.

For that control problem, in our previous study, we provided design methods of nonlinear adaptive $H_\infty$ control of constrained robotic manipulators based on the notion of inverse optimality [9]. In those approaches, estimation errors of tuning parameters in the adaptation mechanism and errors of constraint forces are regarded as external disturbances to the process, and the resulting control strategy is derived as a solution of corresponding $H_\infty$ control problems [10], [11], [12]. Asymptotic stability of tracking errors of constrained trajectories and the variables concerned with errors of constraint forces, are assured. Two approaches are deduced based on that policy, and it is shown that $L^2$ gains from those disturbances (errors of tuning parameters and constraint forces) to generalized outputs are prescribed by several design parameters, explicitly. The proposed control strategy contains a kind of nonlinear damping methodology, and thus, attains good convergence and transient property with less control efforts.

In the present work, we consider a more practical situation and present a design scheme of nonlinear adaptive $H_\infty$ control of constrained robotic manipulators with uncertain input nonlinearities such as dead-zone or backlash. Those actuator nonlinearities are often seen in mechanical connections, electric servo motors, hydraulic servo valves and other mechanical actuators. For the input nonlinearities, adaptive inverse approaches or high-gain feedback schemes have been proposed to compensate the effect of such nonlinearities in the related previous works [13], [14], [15], [16]. In the present manuscript, the adaptive inverse approaches including smooth approximations of nonlinearities, are employed, and the resulting control strategy is derived as a solution of certain $H_\infty$ control problem, where estimation errors of tuning parameters, errors of constraint forces and residual terms of the inverse models, are regarded as external disturbances to the process.

II. PROBLEM STATEMENT

Consider a robotic manipulator with $n$ degrees of freedom and with rotational joints described by

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + G(\theta) = \tau + f,$$

$$\tau = N(u),$$

where $\theta \in \mathbb{R}^n$ is a vector of joint angles, $M(\theta) \in \mathbb{R}^{n \times n}$ is a matrix of inertia, $C(\theta, \dot{\theta}) \in \mathbb{R}^{n \times n}$ is a matrix of Coriolis and centrifugal forces, $G(\theta) \in \mathbb{R}^n$ is a vector of gravitational torques, and $\tau \in \mathbb{R}^n$ is a vector of an input torque. $N(u)$ represents actuator characteristics such as dead-zone or backlash nonlinearities. It is assumed that the system parameters in $M(\theta)$, $C(\theta, \dot{\theta})$, $G(\theta)$ and the nonlinear characteristics $N(u)$ are unknown, and $\tau$ is not an actual control signal, and is unknown. Only, $\dot{\theta}$ and the actual input signal $u \in \mathbb{R}^n$ are assumed to be available for measurement. The trajectory $\theta$ of the robotic manipulator is subject to a constraint represented by a set of $m$ geometric equations (holonomic constraint and frictionless, $m < n$) such that

$$\Psi(\theta) = 0, \quad \frac{d}{dt}\Psi(\theta) = 0, \quad (\Psi \in \mathbb{R}^m),$$

and $f$ is a constraint force which is expressed as

$$f = J(\theta)^T\lambda,$$

$$J(\theta) = \frac{\partial \Psi}{\partial \theta}, \quad (J(\theta) \in \mathbb{R}^{m \times n}),$$
where $\lambda$ is a Lagrangian multiplier. It is assumed that the constraint force is measured by a force sensor mounted at the end-effector of the system. The control objective is to synthesize a proper input signal $u$ such that the desired constrained trajectory $\theta_d(t)$ (differentiable on $t \in [0, \infty)$ and $\Psi(\theta_d) = 0$) and the desired constraint force $f_d$, respectively, for unknown system parameters in $M(\theta)$, $C(\theta, \dot{\theta})$, $G(\theta)$, unknown nonlinear characteristics $N(u)$ and unknown control torques $\tau$.

$$\theta \rightarrow \theta_d, \quad (\Psi(\theta) = \Psi(\theta_d) = 0),$$

$$f \rightarrow f_d.$$  

Typical examples of that control problem are grinding, polishing, inserting, deburring, and scribining, etc [3], where the end-effector of the mechanical system exerts a desired force to the environment as the controlled process moves along a prescribed constrained trajectory.

Robotic manipulators with rotational joints have the following properties [17].

*Properties of Robotic Manipulators* [17]

1) $M(\theta)$ is a bounded, positive definite, and symmetric matrix.

2) $M(\theta) - 2C(\theta, \dot{\theta})$ is a skew symmetric matrix.

3) The left-hand side of (1) can be written into the following form,

$$M(\theta)a + C(\theta, \dot{\theta})b + G(\theta) = \Omega_1(\theta, \dot{\theta}, a, b)^T \Phi_1,$$

where $\Omega(\dot{\theta}, \dot{\theta}, a, b)$ is a known function of $\theta$, $\dot{\theta}$, $a$, $b$, and $\Phi_1$ is an unknown system parameter.

### III. TRACKING CONTROL UNDER CONSTRAINT

First, we introduce the conventional adaptive control for constrained manipulators [7], where the control torque $\tau$ is assumed to be an actual input signal.

#### A. System Description Including Constraint

System descriptions of controlled processes which includes constraints implicitly, are to be obtained in the present section. The development of such descriptions is mainly owing to the previous study [4].

According to the dimension $m$ of the geometric constraint, the output $\theta$ is divided into $\theta^1$ and $\theta^2$, where

$$\theta = \begin{bmatrix} \theta^1 \\ \theta^2 \end{bmatrix}, \quad \theta^1 \in \mathbb{R}^{n-m}, \quad \theta^2 \in \mathbb{R}^m.$$  

Then, $J(\theta)$ is also described in the following decomposed form,

$$J(\theta) = \begin{bmatrix} \frac{\partial \Psi}{\partial \theta^1} & \frac{\partial \Psi}{\partial \theta^2} \end{bmatrix} = [J_1(\theta), J_2(\theta)],$$

$$J_1(\theta) \in \mathbb{R}^{m \times (n-m)}, \quad J_2(\theta) \in \mathbb{R}^{m \times m}.$$  

There is a proper partition such that $\det J_2(\theta) \neq 0$. Since the next relation holds,

$$0 = \frac{d}{dt} \Psi(\theta) = J(\theta) \dot{\theta} = J_1(\theta) \dot{\theta}^1 + J_2(\theta) \dot{\theta}^2,$$

$\dot{\theta}^2$ is represented by $\dot{\theta}^1$ such as

$$\dot{\theta}^2 = -J_2(\theta)^{-1}J_1(\theta) \dot{\theta}^1,$$

and it follows that $\dot{\theta}$ is represented by utilizing $\dot{\theta}^1$.

$$\dot{\theta} = L(\theta) \dot{\theta}^1,$$

$$L(\theta) = \begin{bmatrix} I_{n-m} \\ -J_2(\theta)^{-1} J_1(\theta) \end{bmatrix}.$$  

For $L(\theta)$, it is easily shown that the next relation holds.

$$L(\theta)^T J(\theta)^T = J_1(\theta)^T - J_1(\theta)^T = 0.$$  

By utilizing the property of $L(\theta)$, the system description which includes constraint implicitly, is deduced. The substitution of (14) and the next relation

$$\dot{\theta} = L(\theta) \dot{\theta}^1 + \dot{\theta}^1,$$

into (1) yields

$$M(\theta)L(\theta) \dot{\theta}^1 + M(\theta)L(\theta) \dot{\theta}^1 + \dot{\theta}^1 + \dot{\theta}^1 + G(\theta) = \tau + f.$$  

By multiplying $L(\theta)^T$ to above equation, the following representation is derived.

$$M_1(\theta) \dot{\theta}^1 + C_1(\theta, \dot{\theta}) \dot{\theta}^1 + G_1(\theta) = L(\theta)^T \tau,$$

$$M_1(\theta) = L(\theta)^T M(\theta) L(\theta),$$

$$C_1(\theta, \dot{\theta}) = L(\theta)^T (M(\theta) \dot{\theta} + C(\theta, \dot{\theta}) L(\theta),)$$

$$G_1(\theta) = L(\theta)^T G(\theta).$$

The system description (19) does not contain constraint force nor geometric constraint, explicitly. Then, for given $\tau$, constrained trajectories $\theta^1$, $\theta^2$ and $\dot{\theta}$ are computed from (19), and $\dot{\theta}^2$, $\dot{\theta}^2$ and $\dot{\theta}^2$ are also derived by considering (3), (14), (17). Finally, the constraint force $f$ is computed from the relation $f = M(\theta) \dot{\theta} + C(\theta, \dot{\theta}) \dot{\theta} + G(\theta) - \tau$.

#### B. Adaptive Control Under Constraint

We introduce the conventional adaptive control for constrained manipulators [7]. Define the following signals.

$$\dot{\theta}^1 = \dot{\theta}^1 - \dot{\theta}^1_d \in \mathbb{R}^{n-m},$$

$$\dot{\theta}^2 = \dot{\theta}^2 - \dot{\theta}^2_d \in \mathbb{R}^m,$$

$$\dot{\theta}^1_d = -\Lambda \dot{\theta}^1 \in \mathbb{R}^{n-m},$$

$$\dot{\nu} = \dot{\theta}^1 - \dot{\theta}^1_d = \dot{\theta}^1_L + \dot{\theta}^1 + \Lambda \dot{\theta}^1 \in \mathbb{R}^{n-m},$$

$$\ddot{f} = f - f_d \in \mathbb{R}^m.$$  

($\Lambda \in \mathbb{R}^{(n-m) \times (n-m)}; \Lambda = \Lambda^T > 0$), where $\theta^1_d$ is a subset of elements in $\theta_d$ which corresponds to $\theta^1$. $\mu$ is a variable to handle the force control part, and is synthesized from $\ddot{f}$ such as

$$\ddot{\mu} = -\kappa \mu - \kappa \ddot{f}, \quad (\mu \in \mathbb{R}^n),$$

($\kappa > 0$).

Also $\sigma$ and $\nu$ are introduced as follows:

$$\sigma \equiv Ls + \mu = \dot{\theta} - \nu \in \mathbb{R}^n,$$

$$\nu \equiv \dot{\nu} - \mu \in \mathbb{R}^n.$$  

For $\sigma$ and $\nu$, we obtain the following relations.

$$\dot{\sigma} = Ls + \dot{\sigma} - \kappa (\mu + \ddot{f}),$$

$$\dot{\nu} = L\dot{\theta}^1_d + \dot{\theta}^1_L + \kappa (\mu + \ddot{f}).$$  

2001
The substitution of above relations into (1) yields
\[ M(\theta) \dot{\sigma} + C(\theta, \dot{\theta}) \sigma + M(\theta) \dot{\nu} + C(\theta, \dot{\theta}) \nu + G(\theta) = M(\theta) \dot{\sigma} + C(\theta, \dot{\theta}) \sigma + \Omega_1(\theta, \dot{\sigma}, \dot{\nu}, \nu)^T \dot{\Phi}_1 = \tau + f. \] (34)

This corresponds to the error equation of the traditional adaptive control [18]. For that error system, the control input is synthesized such as
\[ \tau = -K(\dot{\sigma} - f_d) + \alpha \dot{\bar{J}} + \Omega_1^T \dot{\Phi}_1, \] (35)
\[ \Omega_1 = \Omega_1(\theta, \dot{\sigma}, \dot{\nu}, \nu), \] (36)
\[ (K \in \mathbb{R}^{n \times n} : K = K^T > 0, \quad \alpha > 0), \]
where \( \dot{\Phi}_1 \) is a current estimate of \( \dot{\Phi}_1 \), and is tuned by the following adaptive law.
\[ \ddot{\Phi}_1 = -K_1 \dot{\Phi}_1, \quad (K_1 = K^T > 0). \] (37)

Then, the error equation becomes
\[ M(\theta) \dot{\sigma} + C(\theta, \dot{\theta}) \sigma = -K(\dot{\sigma} - f_d) + \alpha \dot{J} + \Omega_1^T \dot{\Phi}_1, \] (38)
\[ \dot{\Phi}_1 = \dot{\Phi}_1 - \dot{\Phi}_1. \] (39)

Here we define positive functions \( V_0, V_1 \)
\[ V_0 = \frac{1}{2} \sigma^T M(\theta) \sigma + \frac{(1 + \alpha)}{2\kappa} \| \mu \|^2, \] (40)
\[ V_1 = V_0 + \frac{1}{2} \dot{\Phi}_1^T \Gamma_1 \ddot{\Phi}_1, \] (41)
and take the time derivative of \( V_1 \) along the trajectory of the manipulator.
\[ \dot{V}_1 = -\sigma^T K(\dot{\sigma} - (1 + \alpha)) \| \mu \|^2 \leq 0, \] (42)
where (16) is considered. Then it follows that \( \sigma, \mu \in \mathcal{L}^2 \cap \mathcal{L}^\infty \) and that \( \dot{\Phi}_1 \in \mathcal{L}^\infty \). By considering the following relation
\[ s = (L^TL)^{-1}L^T(\sigma - \mu), \] (43)
it is shown that \( s \in \mathcal{L}^2 \cap \mathcal{L}^\infty \), if \( (L^TL)^{-1}L^T \in \mathcal{L}^\infty \). Furthermore, by considering the next relation
\[ s = \dot{\theta} + \Lambda \ddot{\theta} \in \mathcal{L}^2 \cap \mathcal{L}^\infty, \] (44)
we obtain \( \dot{\theta} \in \mathcal{L}^\infty \) and \( \ddot{\theta} \to 0 \). Also, by seeing \( \dot{\theta} = L \ddot{\theta} \) and \( \Psi(\dot{\theta}) = 0 \), it follows that \( \dot{\theta} = -\dot{\theta}_d \), \( \ddot{\theta} = \dot{\theta} \in \mathcal{L}^\infty \) and \( \dot{\theta} \to 0 \), if \( \dot{\theta} \in \mathcal{L}^\infty \) implies \( L \in \mathcal{L}^\infty \). Furthermore, \( \dot{\theta}_1 = \dot{\theta} = \Lambda \ddot{\theta} \in \mathcal{L}^\infty \). Hence, it is shown that \( \dot{\theta} \in \mathcal{L}^\infty \) implies \( (1 + \alpha)I + \kappa \dot{\Phi}_1 \in \mathcal{L}^\infty \). Additionally, \( \nu = \dot{\theta}_d - \mu \in \mathcal{L}^\infty \). Next, we consider constraint force. Since it holds that \( \lambda = \lambda - \lambda_d \in \mathcal{L}^\infty \) when \( (1 + \alpha)I + \kappa \dot{\Phi}_1 \in \mathcal{L}^\infty \), we consider \( \nu \). When \( (1 + \alpha)I + \kappa \dot{\Phi}_1 \) is non-singular, \( (\dot{\Phi}_1) \) is a current estimate of \( \dot{\Phi}_1 \) [7], it follows that \( \dot{f} = J^T \lambda \in \mathcal{L}^\infty \), and that \( f = J^T \lambda \in \mathcal{L}^\infty \). Then it is shown that \( \tau \in \mathcal{L}^\infty \), and that \( \dot{\sigma}, \mu \in \mathcal{L}^\infty \). It suggests that \( \sigma, \mu \to 0 \).

Then, we obtain the next theorem.

**Theorem 1** The adaptive control system is uniformly bounded, if the following conditions 1) ~ 3) are satisfied.

1) \( (L^TL)^{-1}L^T \in \mathcal{L}^\infty \).
2) \( \dot{\theta} \in \mathcal{L}^\infty \) implies \( L \in \mathcal{L}^\infty \).
3) \( (1 + \alpha)I + \kappa \dot{\Phi}_1 \) is non-singular.

Furthermore, \( \dot{\theta}, \sigma, \mu \) converge to zero asymptotically.
\[ \lim_{t \to \infty} \dot{\theta}(t) = 0, \quad \lim_{t \to \infty} \sigma(t) = 0, \quad \lim_{t \to \infty} \mu(t) = 0. \] (45)

**Remark** Many practical constraints satisfy the conditions 1) and 2) directly.

### IV. INPUT NONLINEARITY AND INVERSE CHARACTERISTIC

Next, we consider the case where the manipulator is preceded by input nonlinearities such as dead-zone or backlash [13], [14], [15], [16].
\[ \tau = [\tau_1, \ldots, \tau_n]^T \]
\[ = N(u) = [N_1(u_1), \ldots, N_n(u_n)]^T. \] (46)

(Dead-zone)
\[ \tau_i = N_i(u_i) \]
\[ = DZ_i(u_i) \equiv \begin{cases} m_{ri}(u_i - b_{ri}) & (u_i \geq b_{ri}) \\ 0 & (b_{li} \leq u_i \leq b_{ri}) \\ m_{li}(u_i - b_{li}) & (u_i \leq b_{li}) \end{cases} \] (47)

An inverse characteristic of dead-zone is written as follows:
\[ u_i = DZ_i^{-1}(\tau_i) = \frac{\tau_i + m_{ri}b_{ri}}{m_{ri}} \sigma_i(\tau_i) + \frac{\tau_i + m_{li}b_{li}}{m_{li}} \sigma_i(\tau_i), \] (48)
\[ \sigma_i(\tau) = \begin{cases} 1 & (\tau > 0) \\ 0 & (\tau \leq 0) \end{cases}. \] (49)

Also, the following representation is given for an input torque \( \tau_i \).
\[ \tau_i = m_{ri}(u_i - b_{ri})\sigma_i(\tau_i) + m_{li}(u_i - b_{li})\sigma_i(\tau_i). \] (50)

(Backlash)
\[ \tau_i = N_i(u_i) = BL_i(u_i) \equiv \begin{cases} m_i(u_i - b_{ri}) & (u_i > 0 \& \tau_i = m_i(u_i - b_{ri})) \\ m_i(u_i - b_{li}) & (u_i < 0 \& \tau_i = m_i(u_i - b_{li})) \end{cases} \] (51)

An inverse characteristic of backlash is written by
\[ u_i = BL_i^{-1}(\tau_i) = \begin{cases} \frac{\tau_i + m_{ri}b_{ri}}{m_{ri}} & (\dot{\tau}_i > 0 \& u_i = \frac{\tau_i + m_{ri}b_{ri}}{m_{ri}}) \\ \frac{\tau_i + m_{li}b_{li}}{m_{li}} & (\dot{\tau}_i < 0 \& u_i = \frac{\tau_i + m_{li}b_{li}}{m_{li}}) \end{cases} \] (52)

Next, we construct estimation schemes for inverse characteristics of the input nonlinearities. The rigorous inverse models of input nonlinearities include non-smooth functions such as \( \sigma_i(\tau_i), \sigma_i(\tau_i), \sigma_i(\tau_i), \sigma_i(\tau_i) \). However, these may not be adequate for controller design. Hence, approximate inverse models which include smooth functions [14], [15] are employed in the estimation schemes.

(Inverse model of dead-zone)
An estimation scheme for the inverse characteristic of dead-zone is given by
where \( \tau_{d_1} \) is an ideal input torque. Then, the following relation is deduced for \( \tau_{d_1} \).

\[
\tau_{d_1} = \hat{\phi}_{2i}^T \omega_{2i} + \epsilon_i,
\]

(Inverse model of backlash)

An estimation scheme for the inverse characteristic of backlash is given by

\[
u_i = \hat{N}_i^{-1}(\tau_{d_1}) = B L_i^{-1}(\tau_{d_1})
\]

where \( \tau_{d_1} \) is an ideal input torque. Then, the following relation is deduced for \( \tau_{d_1} \).

\[
\tau_{d_1} = \hat{\phi}_{2i}^T \omega_{2i},
\]

(Effect of input scheme)

The difference between \( \tau_i = N_i(u_i) \) and \( \tau_{d_1} = \hat{N}_i(u_i) \) (\( \tau_{d_1} \) is deduced from \( u_i = \hat{N}_i^{-1}(\tau_{d_1}) \)) is evaluated by the following relation.

\[
\tau_i - \tau_{d_1} = (\hat{\phi}_{2i} - \omega_{2i}^T \omega_{d_1} + d_i,
\]

where \( \hat{\phi}_{2i} \) is a true value of \( \hat{\phi}_{2i} \), and \( \epsilon_i = 0 \) for \( \hat{N}_i^{-1} = B L_i^{-1} \). Then, for \( \hat{\phi}_{2i} \) satisfying

\[
0 < \hat{m}_{ri}, \hat{m}_{ti} < \infty, \quad |(m_r \hat{b}_{r_i})|, \quad |(m_t \hat{b}_{t_i})| < \infty,
\]

the total residual term \( d_i \) satisfies the next inequality [14], [15].

\[
|d_i| < \infty.
\]
is a stabilizing signal derived from \( H_\infty \) control criterion. A positive function \( V_3 \) is defined by
\[
V_3 = V_0 + \frac{1}{2} \tilde{\Phi}_T^2 \Gamma^{-1} \tilde{\Phi}_2.
\]
(80)
The time derivative of \( V_3 \) is evaluated as follows:
\[
\dot{V}_3 \leq (1 + \alpha)(\sigma - \mu)^T \tilde{f} + \sigma^T \Gamma \tilde{\Phi}_2
\]
\[
-(1 + \alpha)\|\mu\|^2 + \sigma^T (d + v),
\]
(81)
where the tuning law of \( \tilde{\Phi}_2(t) \) is the same as (77). By considering (81), a virtual system is introduced.
\[
\frac{d}{dt} \begin{bmatrix} \sigma \\ \mu \end{bmatrix} = \begin{bmatrix} -M^{-1} C \sigma \\ -\kappa \mu \end{bmatrix} + \begin{bmatrix} M^{-1} (1 + \alpha) \\ -\kappa I \end{bmatrix} \tilde{f}
\]
\[
+ \begin{bmatrix} M^{-1} \Omega_1^T \\ 0 \end{bmatrix} \tilde{\Phi}_2 + \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} d + \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} v.
\]
(82)
(82) is rewritten into the next form.
\[
\frac{d}{dt} x = f(x) + g_{11} \tilde{f} + g_{12} \tilde{\Phi}_1 + g_{13} d + g_2 v,
\]
\[
x \equiv [\sigma^T, \mu^T]^T.
\]
(83)
(84)
We are to stabilize the above system via a control input \( v \) by utilizing \( H_\infty \) criterion, where \( \tilde{f}, \tilde{\Phi}_1, \) and \( d \) are regarded as external disturbances to the process [12]. For that purpose, we introduce the following Hamilton-Jacobi-Isaacs (HJI) equation
\[
\frac{\partial}{\partial t} V + \mathcal{L}_f V
\]
\[
+ \frac{1}{4} \sum_{i=1}^3 \frac{\|L_{g_{1i}} V\|^2}{\gamma_i^2} - \mathcal{L}_{g_2} V R^{-1} (\mathcal{L}_{g_2} V)^T \bigg\} + q(x) \leq 0,
\]
(85)
where the solution \( V \) is given by \( V = V_0 \). \( q(x) \) and \( R \) are a positive function and a positive definite matrix, respectively, and those are derived from HJI equation based on inverse optimality for the given solution \( V \) and the positive constants \( \gamma_i \) \((i = 1 \sim 3)\). The substitution of the solution \( V = V_0 \) into HJI equation (85) yields
\[
-(1 + \alpha)\|\mu\|^2 + \frac{1}{2\gamma_1^2} (1 + \alpha)^2 \|\sigma - \mu\|^2
\]
\[
+ \frac{1}{4\gamma_2^2} \sigma^T \Omega_1^T \Omega_1 \sigma + \frac{1}{4\gamma_3^2} \|\sigma\|^2 - \frac{1}{4} \sigma^T R^{-1} \sigma
\]
\[
+ q(x) \leq 0.
\]
(86)

In order to obtain \( q(x) \) and \( R \), we consider the following relation (87) which is a sufficient condition for the above inequality (86).
\[
-(1 + \alpha)\|\mu\|^2 + \frac{1}{2\gamma_1^2} (1 + \alpha)^2 (\|\sigma\|^2 + \|\mu\|^2)
\]
\[
+ \frac{1}{4\gamma_2^2} \sigma^T \Omega_1^T \Omega_1 \sigma + \frac{1}{4\gamma_3^2} \|\sigma\|^2 - \frac{1}{4} \sigma^T R^{-1} \sigma
\]
\[
+ q(x) \leq 0.
\]
(87)
Then, \( q(x) \) and \( R \) satisfying (87) are given as follows:
\[
q(x) = \frac{1}{4} \sigma^T K_R \sigma + (1 + \alpha) \left( 1 - \frac{1 + \alpha}{2\gamma_1^2} \right) \|\mu\|^2,
\]
(88)
\[
R = \frac{\left( 2(1 + \alpha)^2 \right)}{\gamma_1^2} I + \frac{\Omega_1^T \Omega_1}{\gamma_2^2} I + \frac{1}{\gamma_3^2} I + K_R
\]
\[
K_R > 0.
\]
(89)
(90)
In order that \( q(x) \) is a positive function, \( \alpha \) and \( \gamma_1 \) should satisfy the next relation.
\[
\gamma_1^2 > \frac{1 + \alpha}{2}.
\]
(91)
By utilizing \( R, v \) is deduced as a solution for the corresponding \( H_\infty \) control problem.
\[
v = -\frac{1}{2} R^{-1} (\mathcal{L}_{g_2} V)^T = -\frac{1}{2} R^{-1} \sigma
\]
\[
- \frac{1}{2} \left\{ \frac{2(1 + \alpha)^2}{\gamma_1^2} I + \frac{\Omega_1^T \Omega_1}{\gamma_2^2} I + \frac{1}{\gamma_3^2} I + K_R \right\} \sigma.
\]
(92)
By considering HJI equation, the time derivative of \( V_3 \) is evaluated as follows:
\[
\dot{V}_3 \leq \left( v + \frac{1}{2} R^{-1} \sigma \right)^T R \left( v + \frac{1}{2} R^{-1} \sigma \right) - v^T R v
\]
\[
- \gamma_1^2 \left\| \tilde{f} - \left( 1 + \alpha \right) \frac{2\gamma_1^2}{2\gamma_1^2} \right\|^2 + \gamma_1^2 \left\| \tilde{f} \right\|^2
\]
\[
- \gamma_2^2 \left\| \tilde{\Phi}_1 - \frac{2\gamma_2^2}{2\gamma_2^2} \right\|^2 + \gamma_2^2 \left\| \tilde{\Phi}_1 \right\|^2
\]
\[
- \gamma_3^2 \left\| d - \frac{2\gamma_3^2}{2\gamma_3^2} \right\|^2 + \gamma_3^2 \left\| d \right\|^2 - q(x).
\]
(93)
The tuning law of \( \tilde{\Phi}_1 \) is the same as (76). Then, the positive function \( V_2 \) satisfies the next relation.
\[
V_2 \leq \left( v + \frac{1}{2} R^{-1} \sigma \right)^T R \left( v + \frac{1}{2} R^{-1} \sigma \right) - v^T R v
\]
\[
- \gamma_1^2 \left\| \tilde{f} - \left( 1 + \alpha \right) \frac{2\gamma_1^2}{2\gamma_1^2} \right\|^2 + \gamma_1^2 \left\| \tilde{f} \right\|^2
\]
\[
- \gamma_2^2 \left\| \tilde{\Phi}_1 - \frac{2\gamma_2^2}{2\gamma_2^2} \right\|^2 + \gamma_2^2 \left\| \tilde{\Phi}_1 \right\|^2
\]
\[
- \gamma_3^2 \left\| d - \frac{2\gamma_3^2}{2\gamma_3^2} \right\|^2 + \gamma_3^2 \left\| d \right\|^2 - q(x).
\]
(94)
From the evaluation of \( V_2 \) and \( V_3 \), we obtain the next theorem.

**Theorem 3** The adaptive control system is uniformly bounded under the same conditions 1) \sim) 3) (Theorem 1), and \( \theta \) and \( \bar{\mu} \) converge to residual regions defined by \( \|(\theta^T, \bar{\mu}^T)\| \sim \gamma_1^2 \gamma_2^2 \gamma_3^2 \lambda_{\min}(K_R)^{-1} \). Also, \( v \) is an optical control solution which minimizes the following cost functional.
\[
J = \sup_{\tilde{f}, \tilde{\Phi}_1, d \in \mathbb{L}_2} \left\{ \int_0^t (q + v^T R v) d\tau + V_3(t) \right\}
\]
\[
- \gamma_2^2 \int_0^t \left\| \tilde{f} \right\|^2 d\tau - \gamma_2^2 \int_0^t \left\| \tilde{\Phi}_1 \right\|^2 d\tau - \gamma_3^2 \int_0^t \left\| d \right\|^2 d\tau \right\}.
\]
(95)
Additionally, the next inequality holds for any finite \( t \).
\[
\int_0^t (q + v^T R v) d\tau + V_3(t) \leq \gamma_1^2 \int_0^t \left\| \tilde{f} \right\|^2 d\tau
\]
\[
+ \gamma_2^2 \int_0^t \left\| \tilde{\Phi}_1 \right\|^2 d\tau + \gamma_3^2 \int_0^t \left\| d \right\|^2 d\tau + V_3(0).
\]
(96)

**Remark** It is shown that the \( \mathbb{L}_2 \) gains from the disturbances \( \tilde{f}, \tilde{\Phi}_1, d \) to the generalized output \( \sqrt{q + v^T R v} \) are prescribed by positive constants \( \gamma_1, \gamma_2, \gamma_3 \). However, \( \mathbb{L}_2 \) gain \( \gamma_1 \) is restricted by the control parameters \( \alpha \) (91).
\[ V_3 = \sigma^T \Omega_1^T \tilde{\Phi}_1 + \sigma^T d + \sigma^T v - (1 + \alpha)\|\mu\|^2. \quad (97) \]

From the above relation, we introduce the following virtual system.

\[ \frac{d}{dt} \begin{bmatrix} \sigma \\ \mu \end{bmatrix} = \begin{bmatrix} -M^{-1}C \sigma \\ -K\mu \end{bmatrix} + \begin{bmatrix} M^{-1} \Omega_1^T \\ 0 \end{bmatrix} \tilde{\Phi}_1 \\
+ \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} d + \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix} v. \quad (98) \]

The virtual process is rewritten in the next form.

\[ \frac{d}{dt} x = f(x) + g_{11} \tilde{\Phi}_1 + g_{12} d + g_2 v. \quad (99) \]

We are to stabilize the virtual system via the control input \( v \) by utilizing \( H_\infty \) control criterion, where \( \tilde{\Phi}_1 \) and \( d \) are regarded as external disturbances to the process. Similarly to the previous section, for HJI equation

\[ \frac{\partial}{\partial t} V + L_f V + \frac{1}{4} \sum_{i=1}^2 \frac{\|L_{g_i} V\|^2}{\gamma_i^2} - L_{g_2} V R^{-1} (L_{g_2} V)^T + q(x) \leq 0, \quad (100) \]

together with the solution \( V = V_0 \), or for the following equivalent relation

\[ -(1 + \alpha)\|\mu\|^2 + \frac{1}{4\gamma_1^2} \sigma^T \Omega_1^T \Omega_1 \sigma + \frac{1}{4\gamma_2^2} \|\sigma\|^2 \]
\[ -\frac{1}{4} \sigma^T R^{-1} \sigma + q(x) \leq 0, \quad (101) \]

\[ q(x), R \] and the optimal solution \( v \) are given as follows:

\[ q(x) = \frac{1}{4} \sigma^T K_R \sigma + (1 + \alpha)\|\mu\|^2, \quad (102) \]

\[ R = \left( \frac{\Omega_1^T \Omega_1}{\gamma_1^2} + \frac{1}{\gamma_2^2} I + K_R \right)^{-1}, \quad (103) \]

\[ K_R = K_R^T > 0, \quad (104) \]

\[ v = -\frac{1}{2} R^{-1} \sigma = -\frac{1}{2} \left( \frac{\Omega_1^T \Omega_1}{\gamma_1^2} + \frac{1}{\gamma_2^2} I + K_R \right) \sigma. \quad (105) \]

**Theorem 4** The adaptive control system is uniformly bounded under the same conditions 1) ~ 3) (Theorem 1), and \( \theta \) and \( \hat{\mu} \) converge to residual regions defined by \( \|Q^T, \hat{\mu}^T\| \sim \gamma_2^2, \lambda_{\text{min}}(K_R)^{-1} \). Furthermore, \( v \) is an optimal control solution which minimizes the following cost functional.

\[ J = \sup_{\hat{\Phi}_1, d \in L^2} \left\{ \int_0^t (q + v^T R v) d\tau + V_3(t) \right\} - \gamma_1^2 \int_0^t \|\hat{\Phi}_1\|^2 d\tau - \gamma_2^2 \int_0^t \|d\|^2 d\tau \}. \quad (106) \]

Additionally, the next inequality holds for any finite \( t \).

\[ \int_0^t (q + v^T R v) d\tau + V_3(t) \leq \gamma_1^2 \int_0^t \|\hat{\Phi}_1\|^2 d\tau + \gamma_2^2 \int_0^t \|d\|^2 d\tau + V_3(0). \quad (107) \]

**Remark** The \( L^2 \) gains from the disturbances \( \hat{\Phi}_1 \) and \( d \) to the generalized output \( \sqrt{q} + v^T R v \) are prescribed by positive constants \( \gamma_1, \gamma_2 \). However, \( L^2 \) gain from \( f \) to the generalized output \( \sqrt{q} + v^T R v \) is not prescribed in the present control scheme.

**VIII. CONCLUDING REMARKS**

Design methodologies of nonlinear adaptive \( H_\infty \) control for constrained robotic manipulators with uncertain input nonlinearities, are proposed, where tracking control of constrained trajectories and control of constraint forces are considered. The adaptive inverse approaches are employed to compensate the effect of the input nonlinearities. The resulting control strategy is derived as a solution of certain \( H_\infty \) control problem, where estimation errors of tuning parameters, errors of constraint forces and residual terms of the inverse models, are regarded as external disturbances to the process. Two approaches are deduced based on that policy, and it is shown that \( L^2 \) gains from those disturbances to generalized outputs are prescribed by several design parameters, explicitly. Although the effectiveness of the proposed methodology was assured in the several simulation examples, the experimental verification is left in our future research.

**REFERENCES**


