Robust Switching-type $H_\infty$ Filtering for Time-Varying Uncertain Time-Delay Systems

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Abstract—This paper is concerned with the problem of $H_\infty$ filter design for a class of linear uncertain systems with time-varying delay. The uncertainty parameters are supposed to be time-varying, unknown, but bounded, which appear affinely in the matrices of system model. Our proposed robust $H_\infty$ filter is a switching-type filter, in which the filter parameters are tuned in a switching manner via a switching logic. Asymptotical stability and a prescribed $H_\infty$ performance of the filtering error systems are guaranteed. The resultant filter design conditions are less conservative than those of filter with fixed gains. A numerical example is given to illustrate the validity of the proposed design.

I. INTRODUCTION

Over the past few years, state estimation of dynamic systems has been extensively investigated [1], [6] and [15]. Compared with traditional Kalman filtering, the $H_\infty$ filtering approach possesses many advantages, such as no need for priori information on the external noises and insensitivity to uncertainty in dynamic model [16]. Hence, recently there has been substantial interest in the study of $H_\infty$ filtering problem [2], [9], [11], [13], [18], which is designed to make the $H_\infty$ norm of the system minimized.

On the other hand, time-delay phenomena often arises from biology, mechanics and economics intrinsically, and also appears in actuation and measurement. As is well known [8], the existence of time-delay degrades the control performance and may make the closed-loop stabilization very difficult. Recently, $H_\infty$ filtering results have been extended to linear systems with time-delays. Both delay-independent and delay-dependent results have been proposed [4], [5], [12], [14], [15], [17] and [19]. Some of these results deal with the so-called norm-bounded uncertainty, which is somewhat conservative in many application [7] and [10].

Recently, robust $H_\infty$ filtering linear time-delay systems with polytopic type uncertainties have been treated, based on parameter-independent Lyapunov function [4], [14], [17] or parameter-dependent Lyapunov function [5] using LMI methodologies. In fact, parameter-dependent Lyapunov method can reduce conservativeness compared with parameter-independent one when the uncertain parameters are time-invariant. Also parameter-dependent Lyapunov method can include the traditional quadratic stability approach as a special case if the time-varying parameters and their rate of variation are assumed to belong to a given convex-bounded polyhedral domain. However, while the uncertain parameters are time-varying and the bound of its derivative is unknown, only the parameter-independent Lyapunov function method can be applied.

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II. PROBLEM STATEMENT AND PRELIMINARIES

A. Problem Statement

Consider a linear uncertain system with time-varying delay described by

$$
\dot{x}(t) = A(\delta(t))x(t) + A_d(\delta(t))x(t - d(t)) + B_\omega \omega(t)
$$

$$
y(t) = C_2x(t) + D_21\omega(t) + C_23x(t - d(t))
$$

$$
z(t) = C_1x(t)
$$

$$
x(t) = \phi(t), \quad t \in [-d, 0]
$$

(1)

where $x(t) \in R^n$ is the state, $y(t) \in R^p$ is the measured output and $z(t) \in R^q$ is the output signal vector to be estimated, respectively. $\omega(t) \in R^r$ is the exogenous disturbance in $L_2[0, \infty)$. $\phi(t)$ is the given initial vector function that is continuous on the interval $[-d, 0]$. $d(t)$ is time-varying bounded delays satisfying

$$
d(t) \leq d < \infty, \quad 0 \leq d(t) \leq h < 1
$$

And

$$
A(\delta(t)) = A_0 + \sum_{i=1}^{N_0} \delta_i(t)A_i, \quad A_d(\delta(t)) = A_{d0} + \sum_{i=1}^{N_0} \delta_i(t)A_{di}
$$

$A_0, A_i, A_{d0}, A_{di}, B_\omega, C_1, C_2, C_3$ and $D_{21}$ are known constant matrices of appropriate dimensions. $\delta_i(t) (i = 1 \cdots N_0)$ are unknown time-varying uncertainty, which satisfy $\delta_i(t) \leq \delta_i \leq \delta$. Here $\delta_i$ and $\delta$ are known lower and upper bounds of $\delta_i(t)$, respectively. Since $C_2 \in R^{p \times r}$ and rank$(C_2) = p_1 \leq p$, then there exists a matrix $T_c \in R^{p_1 \times q'}$ such that rank$(T_c C_2) = p_1$. Furthermore, there exists a matrix $C_{cn}$ such that rank$\left[\begin{array}{c} T_c C_2 \\ C_{cn} \end{array}\right] = n$. Denote $T_c^{-1}$.

Assumption 1: System (1) is asymptotically stable.
For traditional robust filtering, the following form filter is usually used.
\[
\begin{align*}
\dot{\xi}_1(t) &= A_{Ff} \xi_1(t) + B_{Ff} y(t) \\
z_{F1}(t) &= C_{Ff} \xi_1(t)
\end{align*}
\] (2)
where \(A_{Ff} \in \mathbb{R}^{n \times n}, B_{Ff} \in \mathbb{R}^{n \times p} \) and \(C_{Ff} \in \mathbb{R}^{q \times n} \) are the filter parameter matrices to be determined. Here, we assume that the filter is of the same order as the system model.

Denote \( x_{ef}(t) = [x_T(t) \xi_1^T(t)]^T \) and \( z_{ef}(t) = z(t) - z_{F}(t). \) Then combining (2) with (1), the filtering error dynamic can be obtained as
\[
\begin{align*}
\dot{x}_{ef}(t) &= A_{ef} x_{ef}(t) + A_{efd} x_{ef}(t - d(t)) + B_{ef} \omega(t) \\
z_{ef}(t) &= C_{ef} x_{ef}(t)
\end{align*}
\] (3)
where
\[
A_{ef} = \begin{bmatrix}
A(\delta) & 0 \\
B_{Ff} C_2 & A_{Ff}
\end{bmatrix},
A_{efd} = \begin{bmatrix}
A_d(\delta) & 0 \\
B_{Ff} C_3 & A_{Ff}
\end{bmatrix},
\]
\[
B_e = \begin{bmatrix}
B_\omega \\
B_{Ff} D_{21}
\end{bmatrix},
C_{ef} = [C_1 - C_{Ff}].
\]

In this paper, the following robust filter with switching-type gains is considered.
\[
\begin{align*}
\dot{\xi}(t) &= A_F(\hat{\delta}(t))\xi(t) + B_F(\hat{\delta}(t))y(t) \\
z_F(t) &= C_F(\hat{\delta}(t))\xi(t)
\end{align*}
\] (4)
where \(\hat{\delta}(t)(i = 1 \ldots N_0)\) are the estimations of \(\delta(t),\) which will be obtained according to the designed switching laws. \(A_F(\delta) \in \mathbb{R}^{n \times n}, B_F(\delta) \in \mathbb{R}^{n \times p} \) and \(C_F(\delta) \in \mathbb{R}^{m \times n} \) have the following forms, that is
\[
\begin{align*}
A_F(\delta) &= A_{F0} + \sum_{i=1}^{N_0} \delta_i A_{Fi}, \\
B_F(\delta) &= B_{F0} + \sum_{i=1}^{N_0} \delta_i B_{Fi}, \\
C_F(\delta) &= C_{F0} + \sum_{i=1}^{N_0} \delta_i C_{Fi}
\end{align*}
\]
where \(A_{F0}, A_{Fi}, B_{F0}, B_{Fi}, C_{F0}, C_{Fi}\) are fixed parameter matrices to be designed. Here, the designed filter is of the same order as the system model.

Denote \( x_e(t) = [x_T(t) \xi_1^T(t)]^T \) and \( z_e(t) = z(t) - z_F(t). \) Applying the robust filter (4) to the system (1), it follows
\[
\begin{align*}
\dot{x}_e(t) &= A_e x_e(t) + A_{ed} x_e(t - d(t)) B_e \omega(t) \\
z_e(t) &= C_e x_e(t)
\end{align*}
\] (5)
where
\[
A_e = \begin{bmatrix}
A(\delta) & 0 \\
B_{Ff}(\delta) C_2 & A_{Ff}(\delta)
\end{bmatrix},
A_{ed} = \begin{bmatrix}
A_d(\delta) & 0 \\
B_{Ff}(\delta) C_3 & A_{Ff}(\delta)
\end{bmatrix},
B_e = \begin{bmatrix}
B_\omega \\
B_{Ff}(\delta) D_{21}
\end{bmatrix},
C_e = [C_1 - C_{Ff}].
\]

The purpose of this paper is to develop delay-dependent conditions for the existence of the robust switching-type \(H_\infty\) filter (4) for linear time-delay system (1). Specially, for given \(\gamma > 0,\) find a filter of the form (4) such that the corresponding error dynamics (5) is asymptotically stable and satisfies \(\|T_{z_{ef}}\|_\infty < \gamma\) under zero-initial conditions for any nonzero \(\omega(t) \in L_2[0, \infty]\) and all admissible uncertainties.

B. Preliminaries

**Lemma 1** [20]: Let \(x(t) \in \mathbb{R}^n\) be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices \(M_1, M_2 \in \mathbb{R}^{n \times n} \) and \(X = X^T > 0,\) and a scalar function \(h := h(t) \leq 0:\)
\[
- \int_{t-h}^t x^T(s) X x(s) ds \\
< 0
\] (6)

**Lemma 2**: Consider the closed-loop system described by (2). Then the following statements are equivalent:

(i) there exist symmetric positive-definite matrices \(P_a, Q, S,\) matrices \(M_1, M_2\) and a filter described by (3) such that for \(\delta_i \in [\delta_i, \delta_i]\)
\[
\Delta_0 = \begin{bmatrix}
P_a & P_a A_{ef} & \ddot{d} M_1^T + M_2 & 0 \\
0 & - (1 - h) Q & - M_2^T + M_2 & 0 \\
\ast & \ast & 0 & 0 \\
\ast & \ast & \ast & \ast
\end{bmatrix} < 0
\] (7)
\[
\Delta_0 = \begin{bmatrix}
H_0 & P_a & \ddot{d} M_1^T + M_2 & 0 \\
0 & - (1 - h) Q & - M_2^T + M_2 & 0 \\
\ast & \ast & 0 & 0 \\
\ast & \ast & \ast & \ast
\end{bmatrix} < 0
\]

where
\[
H_0 = \begin{bmatrix}
\Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\
0 & \Theta_5 & \Theta_6 & \Theta_7 \\
0 & \Theta_8 & \Theta_9 & \Theta_{10} \\
0 & \Theta_{11} & \Theta_{12} & \Theta_{13}
\end{bmatrix}
\]

(ii) there exist symmetric matrices \(Y, N, Q, S\) with \(0 \leq N < Y,\)
\[
Q = \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{12} & Q_{22}
\end{bmatrix},\] matrices \(M_1, M_2\) and a filter described by (3) with \(A_{Ff} = A_{F0}, B_{Ff} = B_{F0}\) and \(C_{Ff} = C_{F0}\) such that for \(\delta_i \in [\delta_i, \delta_i]\)
\[
V_{a1} := \begin{bmatrix}
\Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\
0 & \Theta_5 & \Theta_6 & \Theta_7 \\
0 & \Theta_8 & \Theta_9 & \Theta_{10} \\
0 & \Theta_{11} & \Theta_{12} & \Theta_{13}
\end{bmatrix}
\]

where
\[
\Theta_i = \begin{bmatrix}
\Theta_i & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{bmatrix},
\Theta_i = \begin{bmatrix}
\Theta_i & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{bmatrix},
\]

\[
\begin{bmatrix}
\ddot{d} M_1^T + M_2 & \ddot{d} M_2 & \ddot{d} A^T S & C_{F0}^T \\
0 & 0 & 0 & C_1 \\
0 & 0 & 0 & 0 \\
- \gamma^2 I & 0 & 0 & 0 \\
\ast & - S & 0 & 0 \\
\ast & \ast & - D S & 0 \\
\ast & \ast & \ast & - I
\end{bmatrix} < 0
\] (8)
Algorithm 1: Let $\gamma$ denotes the robust $H_{\infty}$ performance bound of the closed-loop system (3). Let $NA_{F} = \hat{A}_{F}$ and $NB_{F} = \hat{B}_{F}$, where $\eta = \gamma^2$. Then the resultant gains of robust filter (2) are $A_{F} = \hat{A}_{F} N^{-1}$, $B_{F} = \hat{B}_{F} N^{-1}$ and $C_{F}$. 

III. ROBUST $H_{\infty}$ SWITCHING-TYPE FILTER DESIGN

Theorem 1: Consider the filtering error system (5), and let $\gamma > 0$, $\hat{d} > 0$ and $0 < h < 1$ be given scalars. If there exist positive definite matrices $N, Y, Q_{11}, Q_{22}, S$ with $0 < N < Y$ and matrices $Q_{12}, M_{1}, M_{2}, A_{F}, B_{F}, B_{0}, P, C_{F}, F, i = 1 \cdots N_{0}$ such that for $\delta_{i}(t), \dot{\delta}_{i}(t) \in [\hat{\delta}_{i}, \bar{\delta}_{i}]$ the following matrix inequalities hold:

$$
\dot{T}_{i} = T_{i} \quad \dot{T}_{d} = T_{d} \quad \dot{M}_{i} = M_{i} \quad \dot{M}_{d} = M_{d} \\
N_{i} \leq 0 \\
C_{i} \leq 0 \\
S \leq 0 \\
-I \leq 0
$$

with

$$
A_{F} = A_{F_0} + \sum_{i=1}^{N_{0}} \delta_{i} A_{F_i}, \quad B_{F} = B_{F_0} + \sum_{i=1}^{N_{0}} \delta_{i} B_{F_i}
$$

$$
T_{1} = \Psi_{1} + \Psi_{1}^{T} \\
\Psi_{1} = Y A_{i} \delta_{i} N B_{F} \delta_{i} C_{2} + M
$$

$$
T_{2} = -N A_{F} (\delta_{i}) - A_{i} (\delta_{i}) N + C_{i}^{T} B_{F} \delta_{i} N
$$

$$
T_{3} = N A_{F} (\delta) + (N A_{F} (\delta))^T + Q_{22}
$$

$$
T_{4} = Y A_{i} (\delta_{i}) - N B_{F} (\delta_{i}) C_{3} - M_{i} + \sum_{i=1}^{N_{0}} (\delta_{i} - \dot{\delta}_{i})
$$

$$
T_{5} = -N A_{F} (\delta) + N B_{F} (\delta) C_{3}
$$

then the filter error system (5) is asymptotically stable with an $H_{\infty}$ disturbance attenuation level $\gamma$.

Proof: Choose the following Lyapunov-Krasovskii functional

$$
V(t) = V_{1}(t) + V_{2}(t) + V_{3}(t).
$$

where

$$
V_{1}(t) = \int_{t-d(t)}^{t} x_{e}(\tau) \dot{P} x_{e}(\tau) d\tau,
$$

$$
V_{2}(t) = \int_{t-d(t)}^{t} x_{e}(\tau) \dot{Q} x_{e}(\tau) d\tau,
$$

$$
V_{3}(t) = \int_{t-d(t)}^{t} \dot{x}_{H}(\tau) \dot{S} x_{H}(\tau) d\tau
$$

with $H = [I \quad 0], Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} > 0$, and $S > 0$. Then the derivative of $V(t)$ along any trajectory of the filtering error system (5) is given by

$$
\dot{V}_{1}(t) = 2x_{e}(t) P \dot{x}_{e}(t)
$$

$$
\dot{V}_{2}(t) \leq x_{e}(t) Q x_{e}(t) - (1-h) x_{e}(t-d(t)) Q x_{e}(t-d(t))
$$

and

$$
\dot{V}_{3}(t) \leq d_{H} \int_{t-d(t)}^{t} \dot{H}_{H} Q \dot{H}_{H} d\tau
$$
with \( \eta_0 = [x^T(t), x^T(t-d(t)), \omega^T(t)] \).

Using Lemma 2, it follows

\[
- \int_{t-d(t)}^t \dot{x}^T(s) S \dot{x}(s) \, ds
\]

\[
\leq \left[ x(t) \right]^T \left( M_1^T + M_1 - M_2^T - M_2 \right) x(t) + \left[ x(t) \right]^T M_2^T S^{-1} [M_3 \ M_2] x(t) \]

\[
+ \left[ x(t) \right]^T M_2^T S^{-1} [M_3 \ M_2] x(t) \quad (14)
\]

Then \( A_e \) and \( A_{ed} \) can be written as

\[
A_e = A_{ea} + A_{eb}, \quad A_{ed} = A_{e1} + A_{e2}
\]

where

\[
A_{ea} = \begin{bmatrix} A(\delta) & 0 \\ B_F(\delta) C_2 & A_F(\delta) \end{bmatrix},
\]

\[
A_{eb} = \sum_{i=1}^{N_0} \left[ \begin{array}{cc} (\hat{\delta}_i - \delta_i) & 0 \\ B_F C_2 & A_F \end{array} \right],
\]

\[
A_{e1} = \begin{bmatrix} 0 \\ B_F(\delta) C_3 \end{bmatrix}, \quad A_{e2} = \sum_{i=1}^{N_0} \left[ \begin{array}{cc} (\hat{\delta}_i - \delta_i) & 0 \\ B_F C_3 & 0 \end{array} \right]
\]

Let \( P \) be of the following form,

\[
P = \begin{bmatrix} Y & -N \\ -N & N \end{bmatrix}
\]

with \( 0 < N < Y \), which implies \( P > 0 \). From (1), it follows

\[
T_c C_2 x = T_c[y - D_{21} \omega - C_3 x(t-d(t))] \quad (16)
\]

Thus

\[
x = T_{cn} C_{cn} x = N_1 y - N_2 \omega + N_3 x(t-d(t)) \quad (17)
\]

with

\[
N_1 = T_{cn} \begin{bmatrix} T_c \\ 0 \end{bmatrix}, \quad N_2 = T_{cn} \begin{bmatrix} T_c D_{21} \\ 0 \end{bmatrix}, \quad N_3 = T_{cn} \begin{bmatrix} 0 \\ C_{cn} \end{bmatrix}
\]

Furthermore, we have

\[
P A_{ea} = \begin{bmatrix} Y A(\delta) - N B_F(\delta) C_2 & -N A_F(\delta) \\ -N A(\delta) + N B_F(\delta) C_2 & N A_F(\delta) \end{bmatrix}
\]

and

\[
P A_{eb} = \sum_{i=1}^{N_0} \left[ \begin{array}{cc} (\hat{\delta}_i - \delta_i) & -N B_F C_3 \\ N B_F C_3 & -N A_F \end{array} \right]
\]

which follows

\[
[x^T \ \xi^T] P A_{eb} [x^T \ \xi^T]^T
\]

\[
= \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \left\{ -x^T N B_F C_2 x - x^T N A_F \xi \\
+ \xi^T N B_F C_2 x + \xi^T N A_F \xi \right\}
\]

By (16) and (17), it is easy to see

\[
x^T N B_F C_2 x
\]

\[
= (\omega^T N_2^T - x^T N_3^T + x^T (t-d(t)) N_B F_1 \times (C_2 x + D_{21} \omega + C_3 x(t-d(t))) + x^T N B_F D_{21} \omega - y^T N_1^T N_B F y + y^T N B_F C_3 x(t-d(t)) - x^T N A_F \xi - (y^T N_1^T - \omega^T N_2^T + x^T N_3^T - x^T (t-d(t)) N_B F_1 \xi \xi^T N B_F C_2 x = x^T N B_F (y - D_{21} \omega - C_3 x(t-d(t)))
\]

Thus,

\[
x^T P A_{eb} x = \eta^T A_{P e} \eta + \eta^T B_{P e} \omega + U
\]

\[
+ \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \omega^T N_2^T N_B F D_{21} \omega
\]

where

\[
\eta^T = \begin{bmatrix} x^T(t) \\ \xi^T(t) \\ x^T(t-d(t)) \end{bmatrix}, \quad \xi^T(t-d(t)) = \xi^T \quad (15)
\]

\[
A_{P e} = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \quad B_{P e} = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) F_i
\]

with

\[
\Psi_{11} = \begin{bmatrix} -N_1^T N_B F C_2 & -N_2^T N_A F_1 \\ 0 & 0 \end{bmatrix}, \quad \Psi_{12} = \begin{bmatrix} -N_1^T N_B F C_3 + N B F C_3 & 0 \\ -N F C_3 & 0 \end{bmatrix},
\]

\[
\Psi_{21} = \begin{bmatrix} N_1^T N_B F C_2 & N_1^T N A F_1 \\ 0 & 0 \end{bmatrix}, \quad \Psi_{22} = \begin{bmatrix} N_1^T N_B F C_3 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
F_1 = \begin{bmatrix} C_3^T B^T_F C_2 N_2 + N B F D_{21} - N_2^T N_B F D_{21} \\ A^T F_2 N_2 - N B F D_{21} \end{bmatrix}, \quad F_2 = \begin{bmatrix} C_3^T B^T_F C_2 N_2 + N_2^T N_B F D_{21} \\ 0 \end{bmatrix}
\]

On the other hand, we have

\[
P A_{e1} = \begin{bmatrix} Y A(\delta) - N B_F(\delta) C_3 & 0 \\ -N A_F(\delta) + N B_F(\delta) C_3 & 0 \end{bmatrix}
\]

and

\[
P A_{e2} = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \begin{bmatrix} -N B_F C_3 & 0 \\ N B_F C_3 & 0 \end{bmatrix}
\]

which follows

\[
x^T P A_{eb} x = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \left\{ -x^T N B_F C_2 x - x^T N A_F \xi \\
+ \xi^T N B_F C_2 x + \xi^T N A_F \xi \right\}
\]

By (16) and (17), it is easy to see

\[
x^T N B_F C_2 x
\]

\[
= (\omega^T N_2^T - x^T N_3^T + x^T (t-d(t)) N_B F_1 \times (C_2 x + D_{21} \omega + C_3 x(t-d(t))) + x^T N B_F D_{21} \omega - y^T N_1^T N_B F y + y^T N B_F C_3 x(t-d(t)) - x^T N A_F \xi - (y^T N_1^T - \omega^T N_2^T + x^T N_3^T - x^T (t-d(t)) N_B F_1 \xi \xi^T N B_F C_2 x = x^T N B_F (y - D_{21} \omega - C_3 x(t-d(t)))
\]

Thus,

\[
x^T P A_{e2} x = \eta^T A_{P e} \eta + \eta^T B_{P e} \omega + U
\]

\[
+ \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \omega^T N_2^T N_B F D_{21} \omega
\]
\[ A_{\Phi e} = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, B_{\Phi e} = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i) \begin{bmatrix} F_3 \\ F_4 \end{bmatrix} \]

with

\[ \Phi_{11} = \begin{bmatrix} -N_0^T N B F_i C_2 + N B F_i C_2 & 0 \\ -N B F_i C_2 & 0 \end{bmatrix}, \Phi_{12} = \begin{bmatrix} N_0^T N B F_i C_3 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{21} = \begin{bmatrix} N_0^T N B F_i C_2 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{22} = \begin{bmatrix} N_0^T N B F_i C_3 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ F_3 = \begin{bmatrix} C_2 T B_{F_1} N N_2 + N B F_i D_{21} - N_0^T N B F_i D_{21} \\ -N B F_i D_{21} \end{bmatrix}, F_4 = \begin{bmatrix} C_2 T B_{F_1} N N_2 + N_0^T N B F_i D_{21} \end{bmatrix} \]

Then from the derivative of \( V(t) \) along the closed-loop system (5), it follows

\[ \dot{V}_1(t) \leq \begin{bmatrix} \eta \\ \omega \end{bmatrix}^T \Omega_0 \begin{bmatrix} PB_e + F_1 & F_2 \\ F_2 & F_4 \end{bmatrix} \begin{bmatrix} \eta \\ \omega \end{bmatrix} + 2(U + \bar{U}) \]

where

\[ \Omega_0 = \begin{bmatrix} \Gamma_1 & \Theta_1 \\ \Theta_2 & \Theta_3 \end{bmatrix} = \begin{bmatrix} C_0^T & 0 \\ 0 & C_0 \end{bmatrix} \begin{bmatrix} \Phi_1 \end{bmatrix} + \begin{bmatrix} \Phi_2 \end{bmatrix} + \begin{bmatrix} \Phi_3 \end{bmatrix} + \begin{bmatrix} \Phi_4 \end{bmatrix} \]

\[ \Phi_1 = 2 \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i)(N_0^T N B F_i D_{21} + D_{21}^T B_{F_1} N N_2) - \gamma^2 I \]

with

\[ \Gamma_1 = PA_{\Phi e} + A_{\Phi e}^T P + \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i)\Phi_{11} + \Phi_{12} + \Phi_{21}^T + \Phi_{22}^T \]

\[ \Phi_2 = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i)\Phi_{12} + \Phi_{21}^T + \Phi_{22}^T \]

\[ \Phi_3 = \sum_{i=1}^{N_0} (\hat{\delta}_i - \delta_i)\Phi_{22} + \Phi_{22}^T \]

Furthermore, from (11) and (16) we can obtain that

\[ \dot{V}(t) + z^T(t)z(t) - \gamma^2 \omega^T(t)\omega(t) \leq \begin{bmatrix} \eta \\ \omega \end{bmatrix}^T \Omega_1 \begin{bmatrix} \eta \\ \omega \end{bmatrix} + 2(U + \bar{U}) \]

where

\[ \Omega_1 = \begin{bmatrix} \Omega_2 & \Pi \\ \Pi^T & \Phi_1 \end{bmatrix} = \begin{bmatrix} PB_e + F_1 & F_2 \\ F_2 & F_4 \end{bmatrix} \begin{bmatrix} \alpha \\ P_0 \end{bmatrix} \]

\[ + \hat{\delta} A_{\Phi e}^T S \begin{bmatrix} A_0 & A_d & 0 \\ 0 & B_{\omega} \end{bmatrix} \]

with

\[ \Omega_2 = \begin{bmatrix} Q + \begin{bmatrix} M_1^T + M_1 & 0 \\ 0 & 0 \end{bmatrix} & -M_2^T + M_2 & 0 \\ 0 & 0 & \Theta_0 \end{bmatrix} + \Omega_0 \]

\[ \Pi = \hat{\delta} \begin{bmatrix} M_1^T \\ M_1^T \\ 0 \end{bmatrix} S^{-1} \begin{bmatrix} M_1 & 0 & M_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \]

\[ \Theta_0 = -(1 - h)Q + \begin{bmatrix} -M_2^T + M_2 & 0 \\ 0 & 0 \end{bmatrix} \]

The design condition \( \dot{V}(t) + z^T(t)z(t) - \gamma^2 \omega^T(t)\omega(t) \leq 0 \) is reduced to

\[ \Omega_1 < 0 \quad (19) \]

and

\[ U + \bar{U} \leq 0. \quad (20) \]

Since \( y \) and \( \xi \) are available on line, the switching law can be chosen as (10). So (20) can be achieved.

Notice that

\[ PB_e = \begin{bmatrix} Y B_{\omega} - N B F_i (\hat{\delta}) D_{21} \\ -N B_{\omega} + N B F_i (\hat{\delta}) D_{21} \end{bmatrix} \]

It is easy to see \( \Omega_1 < 0 \) is equivalent to

\[ \Omega_4 = \begin{bmatrix} \Omega_2 & \Pi \\ \Pi^T & \Phi_1 \end{bmatrix} = \begin{bmatrix} PB_e + F_1 & F_2 \\ F_2 & F_4 \end{bmatrix} \begin{bmatrix} \alpha \\ P_0 \end{bmatrix} \]

\[ + \hat{\delta} A_{\Phi e}^T S \begin{bmatrix} A_0 & A_d & 0 \\ 0 & B_{\omega} \end{bmatrix} \]

\[ < 0 \quad (21) \]

If (9) holds, which implies \( \Omega_4 < 0 \). Thus it follows \( \Omega_1 < 0 \).

Together with the switching laws (10), we can get \( \dot{V}(t) \leq 0 \). Furthermore, we have

\[ \dot{V}(t) + z^T(t)z(t) - \gamma^2 \omega^T(t)\omega(t) \leq 0. \]

Integrate the above-mentioned inequalities from 0 to infinity on both sides, we obtain

\[ V(\infty) - V(0) + \int_0^\infty z^2(t)dt \leq \gamma^2 \int_0^\infty \omega(t)\omega(t)dt. \]

which implies that the \( H_{\infty} \) disturbance attenuation of the closed-loop system (5) is no larger than \( \gamma \) holds.

\[ \square \]

**Theorem 2:** If the condition in Lemma 1 holds for the closed-loop system (3) with traditional robust filter (2), then the condition in Theorem 1 holds for the closed-loop system (5) with robust switching-type filter (4).

**Proof:** The proof can be easily obtained from Theorem 1, so we omit it here.

**Algorithm 2:** Let \( \gamma \) denotes the robust \( H_{\infty} \) performance bound of the closed-loop system (5). Let \( N A_{F_0} = \tilde{A}_{F_0}, N A_{F_1} = \tilde{A}_{F_1}, N B_{F_0} = B_{F_0}, N B_{F_1} = B_{F_1}, \) and \( N B_{F_0} = B_{F_0} \) and \( N B_{F_1} = B_{F_1}. \)

\[ \min \eta \text{ s.t. } 0 < N < Y \text{ and } \eta = \gamma^2, \]

where \( \eta = \gamma^2. \) Then the resultant gains of robust switching-type filter (4) are \( A_{F_0} = \tilde{A}_{F_0} N^{-1}, A_{F_1} = \tilde{A}_{F_1} N^{-1}, B_{F_0} = B_{F_0} N^{-1}, B_{F_1} = B_{F_1} N^{-1}, C_{F_0} = C_{F_0} \) and \( C_{F_1}, i = 1 \cdots N_0. \)
IV. Numerical Example

Consider the following linear time-delay system (1) with time-varying uncertainty satisfying

\[
A(\delta) = \begin{bmatrix}
-2 & 5 \\
-1 & -4
\end{bmatrix} + \delta_1(t) \begin{bmatrix}
1 & 0 \\
0 & 0.5
\end{bmatrix} + \delta_2(t) \begin{bmatrix}
0 & 1 \\
0.1 & 0.1
\end{bmatrix}
\]

\[
A_2(\delta) = \begin{bmatrix}
-0.1 & 0.4 \\
0.2 & 0.3
\end{bmatrix} + \delta_1(t) \begin{bmatrix}
0.2 & 0.1 \\
0.05 & 0
\end{bmatrix}
\]

\[
+ \delta_2(t) \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix}
\]

\[
C_1 = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix}, \quad B_\infty = \begin{bmatrix}
0 & 1 \\
0 & 2
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
3 & 0 \\
1 & 0
\end{bmatrix}
\]

\[
C_3 = \begin{bmatrix}
0 & 1 \\
2 & -1
\end{bmatrix}, \quad D_{21} = \begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix}, \quad x(0) = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

with \( \delta_1(t) = 0.5\cos(t), \delta_2(t) = \sin(t) \) and \( d(t) = \frac{1}{2}\sin(t) + \frac{1}{2} \). Here we chose \( T_o = 1 \).

Using Matlab LMI toolbox [3], Algorithm 1 and Algorithm 2, we get the \( H_\infty \) performance index is 1.1179 with the robust switching-type filter while that of traditional robust filter with fixed gains is 1.3077. Just as the theory has proved the robust switching-type \( H_\infty \) filter design method is less conservative than traditional robust filter with fixed gains.

The simulations are carried out with the following disturbance \( \omega(t) = [\omega_1(t) \quad \omega_2(t)]^T \), where

\[
\omega_1(t) = \omega_2(t) = \begin{cases}
1, & 1 \leq t \leq 2 \ (\text{seconds}) \\
0, & \text{otherwise}
\end{cases}
\]

Figure 1 and Figure 2 are the responses curves of system states with the robust switching-type \( H_\infty \) filter and traditional robust \( H_\infty \) filter with fixed gains, respectively. It is easy to see that our robust switching-type \( H_\infty \) filter has more disturbance attenuation ability than that of traditional robust filter with fixed gains as theory has proved.

V. Conclusions

This paper has proposed a novel robust \( H_\infty \) filter design procedure for linear uncertain systems with time-varying delay. The uncertainty parameters are supposed to be time-varying, unknown, but bounded, which appear affinely in the matrices of system model. A new switching-type filter is established based on LMI method and switching laws to guarantee asymptotic stability and a prescribed \( H_\infty \) performance level of the error systems for all admissible uncertainties. The derived design conditions are less conservative than those of the corresponding filter with fixed gains, which has also been demonstrated by an illustrative example.

Fig. 1. Response curve of the first state with robust switching-type filter (solid) and traditional robust filter with fixed gains (dashed).

Fig. 2. Response curve of the second state with robust switching-type filter (solid) and traditional robust filter with fixed gains (dashed).

REFERENCES