Linear Matrix Inequality Based Dynamic Output Feedback Sliding Mode Control for Uncertain Plants

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Abstract—The development of a synthesis methodology to design compensator-based sliding mode output feedback controllers using linear matrix inequalities is considered in this paper. The class of dynamical system dealt with belongs to the class of uncertain plants with matched, and mismatched uncertainties in polytopic form. The efficacy of the proposed design methodology is demonstrated by a numerical design and computer simulations.

I. INTRODUCTION

Variable Structure Control (VSC) is a robust nonlinear control method consisting of a switched control law and a decision function. Sliding Mode Control (SMC) is a type of VSC that rejects completely, when appropriately designed, a class of uncertainty known as matched uncertainty. The dynamics in the sliding mode will, however, be subject to variations from any mismatched uncertainty in the system. It is important to highlight that most of the early developments in SMC theory assumed that the state variables of the plant can be physically measured. This is an unrealistic assumption as in many practical engineering applications some states do not have physical meaning and hence they cannot be measured. This problem can be addressed by using either an observer-based controller or an output feedback controller. The former approach requires additional dynamics. Its main drawback is that, unless properly designed, the observer undermines the robustness properties of the state feedback control. The latter uses only measured plant output signals. These kinds of controllers can be static or dynamic in nature. Static output feedback control is the simplest approach since no further dynamics are needed. However, it may not be applied in some particular cases, e.g. if the Kimura-Davison conditions are not satisfied. In such situations, an appropriately dimensioned dynamic compensator is required in order to introduce extra dynamics and to increase the degrees of design freedom. This approach belongs to the class of dynamic output feedback controllers.

Most of the papers referenced in the technical literature have considered the class of Linear Time Invariant (LTI) systems either without uncertainties or with matched uncertainties. Only a few papers have been devoted to the class of systems with mismatched uncertainties. For instance, sliding mode static output feedback (SMSOF) control systems based on Linear Matrix Inequalities (LMIs) [2] have been developed by Choi [4] and Xiang et al. [8], whilst a dynamic output feedback variable structure controller has recently been proposed by Park et al. in [7].

Choi in [4] presented a static output feedback variable structure control synthesis approach based on LMIs. The proposed scheme corresponds to a high gain control law which may not be desirable in practical applications. As stated by Choi, this disadvantage can be overcome by means of a trade-off between complexity and control effort using a dynamic variable structure output feedback control law [4]. Xiang et al. in [8] developed an iterative LMI algorithm which neither requires a change of coordinates nor solves a static output feedback problem. Nevertheless, since the proposed control law is high gain then it is necessary to solve an optimization problem in order to design a sliding surface. The LMIs involved in the synthesis methodology are relatively complex. Furthermore, since the algorithm is iterative, its convergence depends on the initial conditions defined by the designer. The dynamic output feedback VSC proposed by Park et al. in [7] considers mismatched norm-bounded time-varying uncertainty and is of the same order as the plant. The design methodology consists of an iterative algorithm developed using the so-called cone complementary linearization algorithm for bi-convex problems.

In this paper, a compensator-based sliding mode output feedback controller is proposed for plants with matched and mismatched uncertainties. This controller belongs to the class of sliding mode dynamic output feedback controllers. The design methodology is based on an LMI framework. The proposed approach represents an extension to the work in [6] where only matched uncertainties were considered. In particular the existence and reaching problems are formulated from a polytopic perspective. The switching surface design problem for the augmented system, i.e. the uncertain plant and dynamic compensator, is recast in terms of LMIs as a static output feedback problem with mismatched uncertainties. In order to design the linear component, a polytopic LMI based approach is developed in which the mismatched uncertainties are considered whilst the synthesis of the nonlinear part takes into account matched uncertainties, disturbances and/or nonlinearities.

This paper is structured as follows: Section 2 defines the class of systems to be considered and presents the problem formulation. The proposed design approach for the Sliding Mode Dynamic Output Feedback Controller (SMDOFC) for
uncertain plants is described in Section 3. A numerical example in Section 4 demonstrates the new approach. Finally, Section 5 presents some concluding remarks.

Throughout this paper \(\|\cdot\|\) stands for the Euclidean norm of a vector and the induced spectral norm of a matrix. The index set \(I(e_1, e_2)\) is defined as \(I(e_1, e_2) = \{e_1, e_1 + 1, e_1 + 2, \ldots, e_2\}\) where \(e_1, e_2 \in \mathbb{Z}_+\) and \(e_1 < e_2\).

II. System Description and Problem Formulation

Consider an uncertain dynamical system described in state-space form \(\forall t \geq 0\) by

\[
\begin{align*}
\dot{x}(t) &= (A + \Delta A)x(t) + B\left(u(t) + \xi(t, x, u)\right), \\
y(t) &= Cx(t), \quad x_0 = x(0)
\end{align*}
\]

where \(x \in \mathcal{X} \subseteq \mathbb{R}^n\) is the state vector with \(\mathcal{X}\) an open set, \(u \in \mathcal{U} \subseteq \mathbb{R}^m\) is the control input vector with \(\mathcal{U}\) the set of all admissible control signals and \(y \in \mathcal{Y} \subseteq \mathbb{R}^p\) is the output vector with \(\mathcal{Y}\) the set of all measurable output signals. The uncertain vector function \(\xi(t, x, u) : \mathcal{X} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n\) represents the lumped sum of matched nonlinearities and/or uncertainties.

Throughout the paper the following is assumed:

A.1 The order of the system and the number of output and input signals satisfy \(n > p > m\).

A.2 The input and output matrices are both full rank, i.e. rank(\(B\)) = \(m\) and rank(\(C\)) = \(p\).

A.3 In the nominal triple \((A, B, C)\), rank(\(CB\)) = \(m\).

A similarity transformation exists such that the nominal triple \((A, B, C)\) in the new coordinates involving the state vector partition

\[
x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T
\]

where \(x_1 \in \mathbb{R}^{(n-m)}\) and \(x_2 \in \mathbb{R}^m\) has the following structure [5]:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & T \end{bmatrix}
\]

where \(A_{11} \in \mathbb{R}^{(n-m) \times (n-m)}\), \(A_{12} \in \mathbb{R}^{(n-m) \times m}\), \(A_{21} \in \mathbb{R}^{m \times (n-m)}\), \(A_{22} \in \mathbb{R}^{m \times m}\), \(B_2 \in \mathbb{R}^{m \times m}\) and \(T \in \mathbb{R}^{p \times p}\) are assumed to be known constant matrices. Furthermore, the matrix \(B_2\) is non-singular and \(T\) is orthogonal.

It is assumed that all matched components of the uncertainty associated with the state matrix have been merged into \(\xi(t, x, u)\). Commensurate with the structure of the state matrix \(A\) defined in (2), the uncertain state matrix \(\Delta A = A + \Delta A\) is taken to have the form

\[
\Delta A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ 0 & 0 \end{bmatrix}
\]

The matrix sub-blocks \(\Delta A_{11}\) and \(\Delta A_{12}\) represent structured uncertainties with bounded uncertain parameters. These parameters are bounded between two extreme real values as follows

\[
\Theta_i \in [\underline{\Theta}_i, \overline{\Theta}_i] \quad \text{for } i \in I(1, r)
\]

Define an associated hyper-rectangle in the parameter space \(\Theta \subseteq \mathbb{R}^r\). The uncertain parameter vector \(\Theta = [\Theta_1, \Theta_2, \ldots, \Theta_r]^T\) can describe either physical constant parameters or time dependent parameters, i.e. \(\Theta(t) : \mathbb{R}_+ \rightarrow \Theta\). In this work, for the sake of generality, uncertain time-varying parameters \(\Theta_i(t)\) with \(i \in I(1, r)\) are considered. Hence, the uncertain matrix sub-blocks in the uncertain state matrix (3) will hereafter be written as \(\Delta A_{11}(t)\) and \(\Delta A_{12}(t)\).

For the remainder of the paper, it is assumed that:

A.4 The matched uncertainty term is bounded by

\[
\|\xi(t, x, u)\| \leq k_1 \|u(t)\| + \phi(t, y(t)) + k_2
\]

where \(\phi(t, y(t))\) is a known function such that \(\phi : \mathbb{R}^+ \times \mathbb{R}^p \rightarrow \mathbb{R}_+\). Furthermore, \(0 \leq k_1 < 1\) and \(k_2 \in \mathbb{R}_+\).

The sliding surface \(\mathcal{S}\) is defined as follows

\[
\mathcal{S} = \{x \in \mathbb{R}^n : \sigma(t) = \Gamma y(t) = \Gamma Cx(t) = 0\}
\]

where \(\sigma(t) \in \mathbb{R}^m\) is the switching function and \(\Gamma \in \mathbb{R}^{m \times p}\) is the switching gain matrix to be designed. The sliding surface synthesis problem is the so-called existence problem.

Let

\[
\Gamma \mathcal{T} = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix}
\]

where \(\Gamma_1 \in \mathbb{R}^{(m-p) \times (m-p)}\), \(\Gamma_2 \in \mathbb{R}^{m \times m}\) and \(\det(\Gamma_2) \neq 0\). Define

\[
C_1 = \begin{bmatrix} 0 \left( (p-m) \times (n-p) \right) \end{bmatrix} + I_{(p-m)}
\]

and the gain matrix \(K \in \mathbb{R}^{m \times (p-m)}\) as

\[
K = \Gamma_2^{-1} \Gamma_1
\]

During a sliding motion, \(\sigma(t) = 0\), and hence \(x_2(t) = -KC_1x_1(t)\). Moreover, considering the null space dynamics, it follows that

\[
\dot{x}_1(t) = (\dot{A}_{11}(t) - \dot{A}_{12}(t)KC_1)x_1(t)
\]

where the matrices \(\dot{A}_{11}(t) = (A_{11} + \Delta A_{11}(t))\) and \(\dot{A}_{12}(t) = (A_{12} + \Delta A_{12}(t))\).

From [5], the switching gain matrix \(\Gamma\) is parameterised straightforwardly from (7) and (9) as follows

\[
\Gamma = \Gamma_2 \begin{bmatrix} K & I_m \end{bmatrix} \mathcal{T}^T
\]

The matrix \(\Gamma_2\) in (11) corresponds to a scaling of the switching matrix \(\Gamma\). In this paper, it is assumed that \(\Gamma_2 = B_2^{-1}\) in order to obtain \(\Gamma_2 = \mathcal{B}_1\). This assumption is useful when designing the control law.

In some particular cases, the existence problem for systems given by (1) with matched and mismatched uncertainties cannot be solved, as the following Bilinear Matrix Inequality is not feasible

\[
\begin{bmatrix} \dot{A}_{11}(t) - \dot{A}_{12}(t)KC_1 + \dot{A}_{11}(t) - \dot{A}_{12}(t)KC_1 \end{bmatrix}^T P_1 + P_1 \begin{bmatrix} \dot{A}_{11}(t) - \dot{A}_{12}(t)KC_1 \end{bmatrix} < 0
\]

An approach to overcome the problem is to design a compensator which introduce additional dynamics and consequently provides further degrees of freedom for synthesising a sliding surface.

Therefore, the first phase of the problem to be addressed consists of the design of a dynamic compensator given by

\[
\dot{x}_c(t) = \Xi x_c(t) + \Psi y(t)
\]

where \(\Xi \in \mathbb{R}^{n \times q}\) and \(\Psi \in \mathbb{R}^{n \times p}\), and a hyperplane in the augmented state space \(\mathcal{X}_a \subseteq \mathbb{R}^{n+q}\).

\[
\mathcal{X}_a = \{x_a \in \mathbb{R}^{n \times q} : \sigma_a(t) = C_1 x_a(t) + \Gamma Cx(t) = 0\}
\]

where \(x_a = [x_1^T x_2^T]^T\) is the augmented state vector, \(\sigma_a(t) \in \mathbb{R}^m\) is the augmented switching function, whilst \(\Gamma_a \in \mathbb{R}^{m \times q}\) and \(\Gamma \in \mathbb{R}^{m \times p}\) are components of the augmented switching
gain matrix $\Gamma_a$ to be synthesised.

The control law synthesis for the uncertain dynamical system (1) is dealt with in the second phase of the problem to be solved. Such a control law has to guarantee that the sliding surface $\mathcal{S}_a$ is reached in finite time from any initial point $x(t_0) \notin \mathcal{S}_a$ in the augmented state space $\mathcal{S}_a$ and the sliding motion takes place thereafter. This control law design problem is the so-called reachability problem.

III. SLIDING MODE DYNAMIC OUTPUT FEEDBACK CONTROL

The SMDOFC design approach developed in this work considers a polytopic formulation for the existence and reachability problems and employs LMI methods.

A. Dynamic Compensator-based Sliding Surface Design

Consider the dynamic compensator given in (13) and let

$$\mathbf{\Psi T} = [\mathbf{\Psi}_1 \quad \mathbf{\Psi}_2]$$

(15)

where $\mathbf{\Psi}_1 \in \mathbb{R}^{p \times (p-m)}$ and $\mathbf{\Psi}_2 \in \mathbb{R}^{q \times m}$.

Consequently, the dynamic compensator (13) considering the state vector partition used in (2) can be written as follows

$$\dot{x}_e(t) = \Xi x_e(t) + \Psi_1 C_1 x_1(t) + \Psi_2 x_2(t)$$

(16)

where $C_1 \in \mathbb{R}^{(p-m) \times (n-m)}$ is defined in (8).

Defining the gain matrix $\mathbf{K}_c \in \mathbb{R}^{m \times q}$ as

$$\mathbf{K}_c \triangleq \Gamma_2^{-1} \Gamma_c$$

(17)

the reduced order sliding motion associated with $\mathcal{S}_a$ is given by

$$\dot{x}_e(t) = (\dot{\mathbf{A}}_{11} - \hat{\mathbf{A}}_{12} \mathbf{K} \mathbf{C}_1) x_1(t) - \hat{\mathbf{A}}_{12}(t) \mathbf{K}_e x_e(t)$$

(18)

$$\dot{\mathbf{x}}_e(t) = (\mathbf{\Psi}_1 - \mathbf{\Psi}_2 \mathbf{K}) C_1 x_1(t) + (\Xi - \mathbf{\Psi}_2 \mathbf{K}) \dot{x}_e(t)$$

(19)

The aim is to find $\Xi, \mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{K}$ and $\mathbf{K}_c$ so that

$$\Phi(t) = \left[ \begin{array}{c} \dot{\mathbf{A}}_{11} - \hat{\mathbf{A}}_{12}(t) \mathbf{K} \mathbf{C}_1 \\ (\mathbf{\Psi}_1 - \mathbf{\Psi}_2 \mathbf{K}) \mathbf{C}_1 \\ (\Xi - \mathbf{\Psi}_2 \mathbf{K}) \end{array} \right]$$

(20)

is stable.

The problem of designing the compensator gains $\mathbf{\Psi}_1, \mathbf{\Psi}_2,$ and $\Xi$ as well as the gain matrices $\mathbf{K}$ and $\mathbf{K}_c$ can be written in a static output feedback as follows:

$$\Phi(t) = \mathcal{A}(t) - \mathcal{B}(t) \mathcal{H} \mathcal{C}$$

(21)

where

$$\mathcal{A}(t) = \begin{bmatrix} \hat{\mathbf{A}}_{11}(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}(t) = \begin{bmatrix} \hat{\mathbf{A}}_{12}(t) & 0 \\ \mathbf{\Psi}_2 & -\mathbf{I}_q \end{bmatrix}$$

(22)

$$\mathcal{C} = \begin{bmatrix} \mathbf{C}_1 & 0 \\ 0 & \mathbf{I}_q \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} \mathbf{K}_c & \mathbf{K} \\ \mathbf{\Psi}_1 & \Xi \end{bmatrix}$$

(23)

Let $\mathcal{P}$ be a polyhedral closed convex subset: a so-called polytope. Let $\Pi(t) \in \mathcal{P} \subset \mathbb{R}^{(n+p-2m+2q) \times (n+2q)}$ be an uncertain matrix of the form

$$\Pi(t) = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}(t) \end{bmatrix}$$

(24)

then $\mathcal{P}$ is the convex hull $\mathcal{P} = \text{Co}\{\Pi_1, \Pi_2, \ldots, \Pi_N\}$ defined as follows

$$\mathcal{P} = \left\{ \sum_{j=1}^{N} \mu_j \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}(t) \end{bmatrix} : \sum_{j=1}^{N} \mu_j = 1, \mu_j \geq 0 \text{ for } j \in I(1,N) \right\}$$

(25)

where $N$ is the number of vertices of $\mathcal{P}$, and $\mu_j$ with $j \in I(1,N)$ are the polytopic coordinates of $\Pi$. The vertices of $\mathcal{P}$ in (25) are given by $C$ defined in (23) and

$$\mathcal{A}_j = \mathcal{A}_0 + \sum_{i=1}^{r} \theta_i \Delta \mathcal{A}_j \quad \text{for } j \in I(1,N=2^r)$$

(26)

$$\mathcal{B}_j = \mathcal{B}_0 + \sum_{i=1}^{r} \theta_i \Delta \mathcal{B}_j \quad \text{for } j \in I(1,N=2^r)$$

(27)

where

$$\mathcal{A}_0 \triangleq \begin{bmatrix} \hat{\mathbf{A}}_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta \mathcal{A}_j \triangleq \begin{bmatrix} \Delta \hat{\mathbf{A}}_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

(28)

$$\mathcal{B}_0 \triangleq \begin{bmatrix} \hat{\mathbf{A}}_{12} & 0 \\ 0 & -\mathbf{I}_q \end{bmatrix}, \quad \Delta \mathcal{B}_j \triangleq \begin{bmatrix} \Delta \hat{\mathbf{A}}_{12} & 0 \\ 0 & 0 \end{bmatrix}$$

(29)

The following is assumed:

A.5 The triples $(\mathcal{A}_j, \mathcal{B}_j, \mathcal{C})$ for $j \in I(1,N)$ is stabilisable and detectable for all admissible uncertainties in the hyper-rectangle $\Theta$.

The reduced-order system (18)-(19) is output feedback stabilisable if there exists a positive definite Lyapunov matrix $P_1 = P_1^T \in \mathbb{R}^{(n+q-m) \times (n+q-m)}$ and a gain matrix $\mathcal{K}$ such that

$$(\mathcal{A}_j - \mathcal{B}_j \mathcal{C})^T P_1 + P_1 (\mathcal{A}_j - \mathcal{B}_j \mathcal{C}) < 0$$

(30)

for $j \in I(1,N=2^r)$.

The vertices of the polytope $\mathcal{P}$ are said to be simultaneously stabilised by the gain matrix $\mathcal{K}$ if (30) holds.

As the synthesis of the gain $\mathcal{K}$ corresponds to a static output feedback problem for the system triple $(\mathcal{A}(t), \mathcal{B}(t), \mathcal{C})$ any available LMI approach could be employed. In this work, the non-iterative LMI-based algorithm proposed by Benton and Smith in [1] is re-formulated in the context of the existence problem for the SMDOFC as follows:

Step 1: Define $N$ vertices of the polytope $\mathcal{P}$.

Step 2: Define a degree of stability such that

$$\mathcal{A}_{aj} \neq \mathcal{A}_j + \alpha I$$

Step 3: Solve the following optimization problem

$$\min \quad \text{trace}(Q_{sf})$$

s.t.

$$Q_{sf} - I > 0$$

$$Q_{sf} \mathcal{A}_{aj} + \mathcal{A}_{aj} Q_{sf} + Y_{sf}^T P_f + P_f Y_{sf} < 0$$

Step 4: Set $\mathcal{K}_{sf} = Y_{sf} Q_{sf}^{-1}$.

Step 5: Solve the LMI feasibility problem

$$\min \quad \text{find } \varepsilon \text{ and } P_1$$

s.t.

$$P_1 > I, \varepsilon > 0$$

$$\left( \mathcal{A}_{aj} + \mathcal{B}_j \mathcal{K}_{sf} \right)^T P_1 + P_1 \left( \mathcal{A}_{aj} + \mathcal{B}_j \mathcal{K}_{sf} \right) < 0$$

$$\mathcal{A}_{aj}^T P_1 + P_1 \mathcal{A}_{aj} - \varepsilon \mathcal{C}^T \mathcal{C} < 0$$

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Step 6: Solve the following LMI problem

\[
\begin{aligned}
\text{find } &\mathcal{K} \\
\text{s.t. } &\left( \mathcal{A}_{\alpha j} - \mathcal{B}_j \mathcal{K} \mathcal{C} \right)^T P_1 + P_1 \left( \mathcal{A}_{\alpha j} - \mathcal{B}_j \mathcal{K} \mathcal{C} \right) < 0
\end{aligned}
\]

where \( j \in I(1, N = 2^r) \).

The feasibility problem formulated in the step 6 can be replaced by an optimization problem involving the minimization of a norm defined by the designer.

B. Control Law Synthesis

Consider the following system augmented matrices

\[
\mathcal{A}_\Delta(t) = \begin{bmatrix} \mathcal{X} & \Psi \mathcal{C} \\ 0 & \mathcal{A}_\Delta(t) \end{bmatrix}, \quad \mathcal{B}_\Delta = \begin{bmatrix} I_y \\ 0 \end{bmatrix}
\]

If a switching gain matrix \( \Gamma_\alpha = [\Gamma_c \Gamma] \) exists such that the sliding dynamics (18)-(19) is stable, then a nonsingular change of coordinates \( x \rightarrow \mathcal{T}x \) exists such that the triple \((\mathcal{A}_\Delta(t), \mathcal{B}_\Delta, \mathcal{C}_\Delta)\) above can be transformed into

\[
\begin{aligned}
\dot{x}_\Delta(t) &= \mathcal{A}_\Delta(t) x_\Delta(t) + \mathcal{B}_\Delta u(t) \\
\end{aligned}
\]

\[
\Gamma_\alpha \mathcal{C}_\Delta = \begin{bmatrix} 0 & I_m \end{bmatrix}^T
\]

with \( \mathcal{T} \in \mathbb{R}^{(p+q) \times (p+q)} \) such that \( \det(\mathcal{T}) \neq 0 \). The structure of \( \Gamma_\alpha \mathcal{C}_\Delta \) follows since by construction \( \Gamma_\alpha \mathcal{C}_\Delta \mathcal{B}_\Delta = \Gamma \mathcal{C} \mathcal{B} = I_m \).

Define

\[
\hat{\Pi}(t) = \begin{bmatrix} \mathcal{A}_\Delta(t) & \mathcal{B}_\Delta \\ \mathcal{C}_\Delta & 0 \end{bmatrix}
\]

consequently, the corresponding polytope \( \mathcal{P} \) is defined as follows

\[
\mathcal{P} = \left\{ \sum_{j=1}^{N} \mu_j \begin{bmatrix} \mathcal{A}_\Delta(t) \\ \mathcal{B}_\Delta \\ \mathcal{C}_\Delta \end{bmatrix} : \sum_{j=1}^{N} \mu_j = 1 \right\}
\]

where the vertices are given by

\[
\begin{aligned}
\mathcal{A}_\Delta(t) &= \mathcal{A}_\Delta(t) + \sum_{j=1}^{N} \theta_j \Delta \mathcal{A}_\Delta(t) \bigg|_{\theta_j = (\mu_j, \bar{\Pi}_j)} \\
&= \begin{bmatrix} \mathcal{A}_{\Delta 11j} & \mathcal{A}_{\Delta 12j} \\ \mathcal{A}_{\Delta 21j} & \mathcal{A}_{\Delta 22j} \end{bmatrix}
\end{aligned}
\]

for \( j \in I(1, N = 2^r) \) and \( \mathcal{B}_\Delta \) and \( \mathcal{C}_\Delta \) are defined in (34) and (35) respectively.

The sliding mode dynamics are represented by a convex combination of

\[
\mathcal{A}_{\Delta 11j} = \mathcal{A}_\Delta - \mathcal{B}_j \mathcal{K} \mathcal{C} \quad \text{for } j \in I(1, N)
\]

which are Hurwitz by design, and in turn

\[
\mathcal{A}_{\Delta 11}(t) = \mathcal{A}(t) - \mathcal{B}(t) \mathcal{K} \mathcal{C}
\]

is stable by the convexity property of the polytope \( \mathcal{P} \) defined in (37).

Consider the control law

\[
u(t) = u_L(t) + u_{NL}(t) \quad (41)
\]

with the linear component \( u_L(t) \) of the form

\[
u_L(t) = -\mathcal{G} \mathcal{y}_a(t) \quad (42)
\]

where \( \mathcal{G} \in \mathbb{R}^{m \times (p+q)} \) and \( \mathcal{y}_a(t) = [x_1^T(t) \ y^T(t)]^T \in \mathbb{R}^{(p+q)} \) is the augmented output vector; the nonlinear component given by

\[
u_{NL}(t) = \begin{cases} -\rho(t) \mathcal{P}_{21}^{-1} \frac{\Gamma_a \mathcal{y}_a(t)}{\|\Gamma_a \mathcal{y}_a(t)\|} & , \text{if } \Gamma_a \mathcal{y}_a(t) \neq 0 \\ 0 & , \text{otherwise} \end{cases} \quad (43)
\]

where

\[
\rho(t, y(t), u(t)) = \frac{k_1 \|u_L(t)\| + \phi(t, y(t)) + k_2 + \eta}{(1-k_1)}
\]

**PROPOSITION:** Let \( \mathcal{P} \) be a Lyapunov matrix partitioned as follows

\[
\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} > 0 \quad (44)
\]

where \( \mathcal{P}_{11} \in \mathbb{R}^{(n+q) \times (n+q)} \) is the Lyapunov matrix in (30) calculated by means of the Benten and Smith algorithm, and \( \mathcal{P}_{22} \in \mathbb{R}^{m \times m} \). Let the gain matrix \( \mathcal{G} \) be parametrised as

\[
\mathcal{G} = \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \end{bmatrix} \mathcal{T}^{-1} \quad (45)
\]

where \( \mathcal{G}_1 \in \mathbb{R}^{m \times (p+q-m)} \) and \( \mathcal{G}_2 \in \mathbb{R}^{m \times m} \) such that the following matrix inequality holds for \( j \in I(1, N) \)

\[
\mathcal{A}_j^T \mathcal{P} + \mathcal{P} \mathcal{A}_j < 0 \quad (46)
\]

where

\[
\mathcal{A}_j = \begin{bmatrix} \mathcal{A}_{\Delta 11j} & \mathcal{A}_{\Delta 12j} \\ \mathcal{A}_{\Delta 21j} & \mathcal{A}_{\Delta 22j} \end{bmatrix} \quad (47)
\]

Then the control law in (41) guarantees a sliding motion on the surface \( \mathcal{P} \) inside the sliding patch

\[
\Omega = \left\{ (\hat{x}_1 \in \mathbb{R}^{n+q-m}, \hat{x}_2 \in \mathbb{R}^m) : \|\hat{x}_1\| < \eta \gamma^{-1} \right\} \quad (48)
\]

where \( \eta > 0 \) is a design scalar and

\[
\gamma = \max_{j \in I(1, N=2^r)} \left\{ \left\| \mathcal{P}_2 (\mathcal{A}_{\Delta 11j} - \mathcal{G}_1 \mathcal{C} \mathcal{B} \mathcal{C} \mathcal{B} \mathcal{A}_{\Delta 22j} - \mathcal{G}_2) \right\| \right\}
\]

**PROOF:** Consider the Lyapunov function \( V(t) = \hat{x}_1^T(t) \mathcal{P} \hat{x}_1(t) \).

Since \( \mathcal{P} \) has the partition shown in (45) then \( \mathcal{P} \mathcal{B}_\Delta = \left( \Gamma \mathcal{C}_\Delta \right)^T \mathcal{P}_2 \). After manipulations involving (43), (44) and (47), it can be shown \( V < 0 \forall \hat{x}(t) \neq 0 \). Therefore, the closed-loop system is quadratically stable.

Partition the state vector \( \hat{x}(t) \) as \( [ \hat{x}_1^T(t) \ \hat{x}_2^T(t) ]^T \). Using matrix inequality (47) with the previous partition, the following quadratic form can be obtained

\[
\hat{x}_1^T(t) \left( (\mathcal{A}_{\Delta 22j} - \mathcal{G}_2)^T \mathcal{P}_2 + \mathcal{P}_2 (\mathcal{A}_{\Delta 22j} - \mathcal{G}_2) \right) \hat{x}_2(t) < 0
\]

for \( j \in I(1, N = 2^r) \).
Consider the Lyapunov function \( \hat{V}(t) = \hat{x}^T(t)P_2 \hat{x}(t) \). Its derivative along the closed-loop trajectories is given by

\[
\hat{V}(t) = \sum_{j=1}^{N} \mu_j \left( 2\hat{x}^T(t)P_2 (\Delta \hat{a}_{21j}) - \hat{G}_1 \hat{x}(t) \right) + \hat{x}^T(t) \left( (\Delta \hat{a}_{22j})^T - \hat{G}_2 \right) P_2 \left( \Delta \hat{a}_{21j} - \hat{G}_1 \right) \hat{x}(t) + 2 \hat{x}^T(t) P_2 \left( u(t)_{\text{NL}} + \xi(t, x, u) \right)
\]

where \( \Delta \hat{a}_{21j} \in \mathbb{R}^{m \times (n+q-p)} \) and \( \Delta \hat{a}_{21j} \in \mathbb{R}^{m \times (p-m)} \).

Then, choose any

\[
\gamma > \max_{j \in (1,N)} \left\{ \| \Delta \hat{a}_{21j} \| \right\}
\]

and solve the following LMI problem:

\[
\begin{align*}
\min \quad & \xi \\
\text{s.t.} \quad & \begin{bmatrix}
\hat{G}_1 \\
\hat{G}_2 \\
\end{bmatrix} \begin{bmatrix}
\Delta \hat{a}_{21j} \\
\Delta \hat{a}_{22j} \\
\end{bmatrix} \leq \gamma \\
\begin{bmatrix}
P_2 \Delta \hat{a}_{21j} - \hat{G}_1 \hat{x} \\
\Delta \hat{a}_{22j} \\
\end{bmatrix} \leq 0
\end{align*}
\]

for \( j \in (1,N) \) with \( L_1 \equiv P_2 \hat{G}_1 \) and \( L_2 \equiv P_2 \hat{G}_2 \).

The Lyapunov inequality (52) with (53) depends affinely on the matrix variables \( P_1, P_2, L_1 \) and \( L_2 \). Therefore, an LMI problem can be formulated in order to design a gain matrix \( G \) such that

\[
\|G\| < \zeta
\]

and

\[
\|P_2 \Delta \hat{a}_{21j} - \hat{L}_1 \| < \gamma
\]

Inequality (54) has to be formulated in terms of the matrix variables \( L_1 \) and \( L_2 \): by considering \( P_2 \) from (45) and the parameterisation of \( G \) given in (46), after straightforward computation it follows

\[
\|G\| \leq \left\| \begin{bmatrix} L_1 & L_2 \end{bmatrix} T^{-1} \right\|
\]

By ensuring by design

\[
\left\| \begin{bmatrix} L_1 & L_2 \end{bmatrix} T^{-1} \right\| < \zeta
\]

inequality (54) is satisfied.

The poles of \( \Delta \hat{a}_j \) for \( j \in (1,N) \) can be placed in an convex region of the complex plane \( C \) established by the designer as in [3]. In this paper, a convex region characterised by the intersection of the disk \( D(c_n, r_d) \) centered at \((-c_n, 0)\) with radius \( r_d \) and a half-plane \( H(h) \) delimited by a vertical line at \((-h, 0)\) are considered.

In order to formulate an optimization problem for synthesising the gain matrix \( G \), consider a partition of \( \Delta \hat{a}_{21j} = [ \Delta \hat{a}_{211j} \quad \Delta \hat{a}_{212j} ] \) for \( j \in (1,N) \) (58)

where \( \Delta \hat{a}_{211j} \in \mathbb{R}^{m \times (n+q-p)} \) and \( \Delta \hat{a}_{212j} \in \mathbb{R}^{m \times (p-m)} \).

The root loci of \( (\hat{A}_{11j}, \hat{A}_{12j}, \hat{C}_{1j}) \) for \( j \in (1,2) \), considering the extreme values of the mismatched parameter \( \theta(t) \), are shown in Fig. 1 and Fig. 2 respectively. Fig. 2 demonstrates that the reduced-order system is not static output feedback stabilisable. Therefore, a dynamical compensator is required to solve the SMC problem.

Defining \( \Psi_2 = 1 \) and \( \Gamma_2 = 1 \), the LMI approach proposed in this paper generates the following matrix

\[
\Psi_2 = \begin{bmatrix}
-5.8603 & 4.6965 \\
-4.7174 & 3.0759
\end{bmatrix}
\]

which determines the compensator and the switching gain matrix \( \Gamma \).
The convex region is defined through $c_n = 0$, $r_d = 5$ and $h = 0.10$. The gain matrix $G$ designed using the LMI method developed in this paper is given by

$$
G = \begin{bmatrix}
25.3915 & -23.5923 & 6.8839 \\
\end{bmatrix}
$$

(64)

where $P_2 = 1.0000$. The nonlinear part of the control law can be straightforwardly computed from the matched uncertainty $\xi(t, x, u)$.

Computer simulations were carried out using the initial condition $[1 \ -1 \ 0]^T$. Time evolution of the output signals $y(t)$ and the unmeasurable state variable $x_1(t)$ is shown in Fig. 3. The designed SMDOFC stabilises the plant (62) in spite of the mismatched uncertain parameter $\theta(t)$. Fig. 4 depicts the corresponding control signal whilst Fig. 5 shows the switching function.

**V. CONCLUSION**

This paper has described an LMI design framework for SMDOFCs. Plants with both matched and mismatched uncertainties and partial state information can be dealt with. The SMDOFC is compensator-based and represents an alternative when SMSOF cannot be applied.

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**REFERENCES**


