Semiglobal Robust Output Regulation with Generalized Immersion

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Abstract—The semiglobal robust output regulation problem is solved in this paper for a class of nonlinear systems that do not satisfy the standard conditions for the existence of a linear internal model, but admit a so-called “generalized immersion.” It is shown how the obstacle given by the presence of the exosystem dynamics in the generalized immersion mapping can be overcome by resorting to a recently developed framework for time-varying internal model design.

I. PROBLEM FORMULATION

In this paper, we consider a prototypical robust output regulation problem for systems of the form

\[
\begin{align*}
\dot{w} &= Sw, \\
\dot{z} &= f(z,e,w,\mu), \\
\dot{e} &= h(z,e,w,\mu) + b(\mu)[u - c(w,\mu)]
\end{align*}
\]

(1)

with exosystem state \( w \in \mathbb{R}^p \), plant state in the error-coordinates \( x = (z,e) \in \mathbb{R}^{n-1} \times \mathbb{R}^p \), control input \( u \in \mathbb{R} \), regulated error \( e \in \mathbb{R} \), and unknown parameters \( \mu \in \mathcal{P} \), where \( \mathcal{P} \) is a given compact set in \( \mathbb{R}^p \). The vector fields \( f(x,e,w,\mu) \) and \( h(x,e,w,\mu) \) are smooth and satisfy \( f(0,0,w,\mu) = 0 \), \( h(0,0,w,\mu) = 0 \) for all \( w \in \mathbb{R}^p \). Moreover, \( b(\mu) \geq 0 \) for all \( \mu \in \mathcal{P} \). The eigenvalues of the known matrix \( S \) are all simple and lie on the imaginary axis. The semiglobal robust regulation problem is stated as follows: Problem 1.1: Given arbitrary compact sets \( \mathcal{K}_x \subset \mathbb{R}^n \), \( \mathcal{K}_w \subset \mathbb{R}^p \), determine a dynamic error-feedback controller

\[
\dot{\xi} = F(\xi,e), \quad u = H(\xi,e)
\]

(2)

with state \( \xi \in \mathbb{R}^r \), and a compact set \( \mathcal{K}_\xi \subset \mathbb{R}^r \) such that all the trajectories of the closed-loop system (1)-(2) originating from any initial conditions \((x_0,z_0,\xi_0) \in \mathcal{K}_x \times \mathcal{K}_z \times \mathcal{K}_\xi\) are bounded and satisfy limit \( t \to \infty e(t) = 0 \) for all \( \mu \in \mathcal{P} \). Without loss of generality, we henceforth assume that \( \mathcal{K}_w \) is an invariant set for \( \dot{w} = Sw \).

Assumption 1.1: There exists a smooth, positive definite function \( V_0(z,w,\mu) \) such that

\[
\begin{align*}
\alpha_0(\|z\|) &\leq V_0(z,w,\mu) \leq \alpha_0(\|z\|) \\
\frac{\partial V_0}{\partial z} f(z,0,w,\mu) + &\frac{\partial V_0}{\partial w} Sw \leq -\alpha_0(\|z\|)
\end{align*}
\]

(3)

for all \( z \in \mathbb{R}^{n-1}, w \in \mathcal{K}_w, \) and \( \mu \in \mathcal{P}, \) where \( \alpha_0, \alpha_0 \) are class-\(K_\infty\) functions satisfying \( \alpha_0(s) \geq a_0 s^2, \alpha_0(s) \leq \alpha_0(s) \geq a_0 s^2 \) for all \( s \in [0, r_0] \), and for some positive numbers \( \alpha_0, \alpha_0, a_0, \) and \( r_0 \).

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In the classical internal-model based approach (that is, in the spirit of [1]), the solvability of Problem 1.1 relies upon the possibility of embedding in the controller an internal model of the exosystem with output

\[
\mu = 0, \quad \dot{w} = Sw, \quad y_w = c(w,\mu)
\]

(4)

which can be accomplished if the system (3) can be immersed into a detectable nonlinear system of the form

\[
\dot{\eta} = \Phi(\eta), \quad y_\eta = \Gamma(\eta).
\]

(5)

In the literature, the nonlinear function \( c(w,\mu) \) is usually assumed to satisfy the following property, which ensures the existence of an immersion into an LTI system:

Property 1.1: There exists an integer \( q \) and real numbers \( a_0, \cdots, a_{q-1} \) such that \( a_0 c(w,\mu) + a_1 L_{Sc}(w,\mu) + \cdots + a_{q-1} L_{Sc}^{q-1}(w,\mu) + L_{Sc}^q(c(w,\mu)) = 0 \) for all \( \mu \in \mathcal{P} \).

However, the only case where the above property is guaranteed to hold is that the nonlinear function \( c(w,\mu) \) is a polynomial in \( w \), with \( \mu \)-dependent coefficients. In the next section, we show how Assumption 1.1 can be relaxed by defining linear time-varying immersions.

II. GENERALIZED IMMERSION

It was shown in [2] that for systems which do not admit an immersion of the form (4), it is still possible to obtain a so-called generalized immersion if the function \( c(w,\mu) \) satisfies the following property:

Property 2.1: There exists an integer \( q \) and smooth functions \( a_0(w), \cdots, a_{q-1}(w) \) such that, for all \( w \in \mathbb{R}^p \) and \( \mu \in \mathcal{P}, \) \( a_0(w)c(w,\mu) + a_1(w)L_{Sc}(w,\mu) + \cdots + a_{q-1}(w)L_{Sc}^{q-1}(w,\mu) + L_{Sc}^q(c(w,\mu)) = 0 \).

Property 2.1 is found to hold for more general classes of nonlinear functions, including sinusoidal, exponential or rational terms, (see [2]). Property 2.1 implies that the exosystem admits a generalized immersion into the system

\[
\dot{w} = Sw, \quad \dot{\eta} = \Phi_p(w,\eta), \quad y_\eta = \Gamma_p(\eta),
\]

(6)

where the pair \( (\Phi_p(w),\Gamma_p) \) is in phase-variable form with \( w \)-dependent coefficient \( a_i(w), i = 1, \cdots, q \).

While the above system is dependent on \( w \), and thus not implementable as such, it suffices to notice that, since \( w(t) = e^{St}w_0 \), one can rewrite the \( \eta \)-dynamics as \( \dot{\eta} = \Phi_p(t,\sigma)\eta \), with \( \Phi_p(t,\sigma) := \Phi_p(e^{St}w_0) \) and \( \sigma = w_0 \) plays the role of a vector of unknown parameters. Therefore, the generalized immersion is trivially transformed into a time-varying parameter-dependent immersion similar to the one considered in [3] for periodic systems. Moreover, one
can always rewrite system (4) in observer canonical form
\[ \dot{\tau}(w, \mu) = \Phi_o(t, \sigma)\tau(w, \mu), \quad c(w, \mu) = \Gamma_o\tau(w, \mu) \]

**Assumption 2.1:** There exists a re-parametrization \( \sigma \mapsto \theta \in \mathbb{R}^e \) such that the coefficients of the matrix \( \Phi_o(t, \sigma) \) depends linearly on \( \theta \).

### III. REGULATOR DESIGN

The robust regulator consists of the parallel connection 
\( u = u_{st} + u_{im} \) of a high-gain stabilizer \( u_{st} = -ke, \) \( k > 0, \) and a parameterized internal model unit of the form (see [3])
\[ \dot{\xi} = (F + G(t)H(\hat{\theta}))\xi - kG(t)e, \quad u_{im} = H(\hat{\theta})\xi. \]

Note the explicit dependence on time occurs through the known matrix exponential \( e^{St} \). It can be shown that there exists a parameterized family of almost-periodic smooth pairs \((\Sigma(t, w, \mu), H(\theta))\), satisfying
\[ \frac{\partial \Sigma}{\partial t} + \frac{\partial \Sigma}{\partial w}Sw = [F + G(t)H(\theta)]\Sigma(t, w, \theta) \]
\[ c(w, \mu) = H(\theta)\Sigma(t, w, \theta), \]
for all \( \mu \in \mathcal{P} \) and all \( t \geq t_0 \geq 0 \). The proof then follows along the same lines of [3, Lemma 1.2]. The main result can be easily proved by combining the results of [4, Proposition 5.1] and applying LaSalle's invariance principle:

**Theorem 3.1:** There exists \( k^* > 0 \) such that for all \( k \geq k^* \) and all \( \gamma > 0 \) the adaptive controller
\[ \dot{\xi} = (F + G(t)H(\hat{\theta}))\xi - kG(t)e \quad \hat{\theta} = \gamma \xi + \dot{\theta} \quad u = H(\hat{\theta})\xi - ke \]
solves the semiglobal robust output regulation problem for the class of systems under consideration.

**Example:** Consider the error system
\[
\begin{align*}
\dot{w} &= Sw \\
\dot{\xi} &= -z^3 + r_1ew_1 \\
\dot{e} &= a\sin(e) - z^2 + b(u - c(w, \mu)),
\end{align*}
\]
where \( S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), and \( c(w, \mu) = r_1w_1\cos(w_2) \), with \( r_1 \) a nonzero unknown constant. The exosystem with output (3) admits a generalized immersion in observer form
\[ \Phi_o(t, \sigma) = \begin{pmatrix} -\alpha_3(t, \sigma) & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ -\alpha_0(t, \sigma) & 0 & 0 & 0 \end{pmatrix}, \]
with coefficients \( \alpha_3(t, \sigma) = 0, \alpha_2(t, \sigma) = (5 + w_1^2(0)) + (-w_2^2(0) + w_2^2(0))\sin^2(t) + (w_1(0)w_2(0))\sin(2t) \)
\[ \alpha_1(t, \sigma) = 3(w_1(0)w_2(0) - 6(w_1(0)w_2(0))\sin^2(t) \]
\[ + 1.5((-w_1^2(0) + w_2^2(0))\sin(2t) \]
\[ \alpha_0(t, \sigma) = (4 + w_1^2(0) + 3w_2^2(0)) - 2(-w_1^2(0) + w_2^2(0))\sin^2(t) - 2(w_1(0)w_2(0))\sin(2t). \]

The proposed solution applies to the given system with \( \theta = (w_1^2(0) - w_2^2(0) + w_2^2(0) - 2w_1(0)w_2(0))w_1^2(0) \).