2008 American Control Conference
Westin Seattle Hotel, Seattle, Washington, USA
June 11-13, 2008

Sampled-Data Piecewise Affine Slab Systems: A Time-Delay Approach

Behzad Samadi and Luis Rodrigues
Department of Mechanical and Industrial Engineering
Concordia University, Montreal, QC, H3G 1M8, Canada
email: {bsamadi, luisrod}@encs.concordia.ca

***Abstract*— This paper addresses stability analysis of sampled-data piecewise-affine (PWA) systems consisting of a continuous-time plant and a discrete-time emulation of a continuous-time state feedback controller. We consider the sampled-data system as a delayed system with a variable delay. The paper then presents conditions under which the trajectories of the sampled-data closed-loop system will converge to an attracting invariant set. It is also shown that when the sampling period converges to zero, the conditions derived in this paper reduce to sufficient conditions for the non-fragility of the stabilizing continuous-time PWA state feedback controller.**

I. INTRODUCTION

State and output feedback control of continuous-time PWA systems have received increasing interest over the last years [1]–[4]. However, none of these approaches would be applicable directly to controller synthesis for computer-controlled or sampled-data PWA systems. This is the scenario mostly encountered in applications given the flexibility of control implementation in a microprocessor. References [1]–[4] consider continuous-time processes controlled by continuous-time controllers while the implementation in a microprocessor requires emulation of a continuous-time controller as a discrete-time controller.

Although linear sampled-data systems are a well-studied matter [5], controller emulation for systems with possible discontinuities at the switching, such as sampled-data PWA systems, has not had many research contributions. In fact, only recently these systems have started to be addressed in the literature in references such as [6]–[12]. The approach by [9] established that, under certain conditions, the controllable subspaces of a continuous-time switched linear system and its discrete-time counterpart are the same. Canonical forms of switched linear systems based on controllability are presented in [10]. The approach described in [11] considers stability analysis of switched systems that can switch between a set of continuous-time plants and a set of discrete-time plants but does not handle sampled-data systems involving a cascade of a discrete-time system between a sample-and-hold and a continuous-time system. Furthermore, it does not address controller design. The work in [6]–[8] was probably the first where the term "sampled-data PWA systems" is used, although the systems described in this work do not posses the typical structure of a continuous-time plant being controlled by a discrete-time controller. The problem addressed in [6]–[8] is one where the controller is continuous-time and the switching events are the ones controlled by the system logic inside a computer. In other words, in these systems it is assumed that the designer has command over the switching times of the system. For this class of systems, reference [8] presents a probabilistic analysis of controllability. Reference [12] was the first to address the structure of a sampled-data PWA system whereby the system is continuous-time and the controller is emulated in discrete-time inside a computer. However, the sampling time must be constant in [12].

This paper addresses stability analysis of sampled-data PWA systems using a time delay approach, whereby the discrete-time PWA controller is seen as a continuous-time PWA controller with a delay that varies with time. Using a Lyapunov-Krasovskii functional, LMI conditions are derived to describe sufficient conditions for convergence of the sampled-data PWA system trajectories to an attracting invariant set. One of the advantages of the proposed method is that it can be applied to sampled-data PWA systems with variable sampling time. Furthermore, a very important and interesting property of the conditions derived in this paper is that when the sampling time converges to zero, they reduce to LMI conditions for the non-fragility of the continuous-time PWA controller. Therefore, to implement such a controller in discrete-time, it is required that the controller be robust to variations in its parameters.

The paper starts by a brief introduction to continuous-time PWA systems followed by stability analysis of sampled-data PWA systems when a continuous-time controller is emulated in discrete-time. A numerical example shows the performance of the proposed method, followed by the conclusions.

II. CONTINUOUS TIME PWA SLAB SYSTEMS

Consider the following continuous time PWA slab system

$$\dot{x}(t) = A_i x(t) + a_i + Bu(t),\ x(t) \in \mathcal{R}_i \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state and $u(t) \in \mathbb{R}^{n_u}$ is the control input. The initial state is $x(0) = x_0$. Slab regions $\mathcal{R}_i,\ i = 1, \ldots, M$ partition a slab subset of the state space $\mathcal{X} \subset \mathbb{R}^n$. Each region $\mathcal{R}_i$ is defined as

$$\mathcal{R}_i = \{x \mid \sigma_i < C_{\mathcal{R}} x < \sigma_{i+1}\}, \tag{2}$$

This work was supported by le Fonds Québécois de la Recherche sur la Nature et les Technologies (FQRNT) and the Natural Sciences and Engineering Research Council of Canada (NSERC).

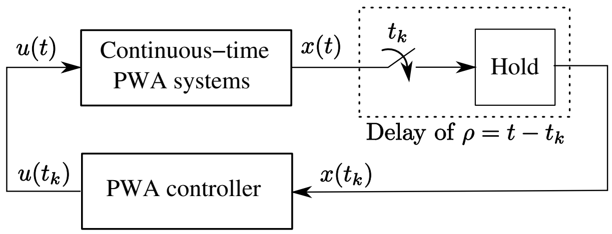

Fig. 1. Sampled-data PWA system

where $C_{\mathcal{R}} \in \mathbb{R}^{1\times n}$ and σ_i for $i = 1, \ldots, M+1$ are scalars such that

$$\sigma_1 < \sigma_2 < \ldots < \sigma_{M+1} \tag{3}$$

It is assumed that the vector field of the open loop PWA system (1) with $u(t) = 0$ is continuous across the boundaries of two or more regions and $a_i = 0$ for $i \in \mathcal{I}(0)$ where

$$\mathcal{I}(x) = \{i | x \in \overline{\mathcal{R}}_i\}. \tag{4}$$

and $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Note that if $x \in \mathcal{R}_i$, then $\mathcal{I}(x) = \{i\}$. Each slab region can be described by the following degenerate ellipsoid

$$\mathcal{R}_i = \{x \mid \|L_i x + l_i\| < 1\} \tag{5}$$

where $L_i = 2C_{\mathcal{R}}/(\sigma_{i+1}-\sigma_i)$ and $l_i = -(\sigma_{i+1}+\sigma_i)/(\sigma_{i+1}-\sigma_i)$.

III. STABILITY OF SAMPLED-DATA PWA SLAB SYSTEMS

Consider that a PWA controller of the form

$$u(t) = K_i x(t) + k_i,\ x(t) \in \mathcal{R}_i \tag{6}$$

has been designed for the PWA system (1) so that the closed-loop system is asymptotically stable. If the PWA controller (6) is implemented as a digital controller and is connected to the PWA system (1) through a sample-and-hold block (Figure. 1), the closed-loop system can be described by

$$\dot{x}(t) = A_i x(t) + a_i + B(K_j x(t_k) + k_j), \tag{7}$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$ where t_k for $k \in \mathbb{N}$ is the sampling time and $t_k \le t < t_{k+1}$. The closed-loop system (7) can be rewritten as

$$\dot{x}(t) = A_i x(t) + a_i + B(K_i x(t_k) + k_i) + Bw, \tag{8}$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$ where

$$w(t) = (K_j - K_i)x(t_k) + (k_j - k_i),\ x(t) \in \mathcal{R}_i,\ x(t_k) \in \mathcal{R}_j \tag{9}$$

The input $w(t)$ is a result of the fact that $x(t)$ and $x(t_k)$ are not necessarily in the same region.

Following [13], the time elapsed since the last sampling time will be denoted by

$$\rho := t - t_k,\ t_k \le t < t_{k+1} \tag{10}$$

and τ_M (τ_D) is defined as the maximum (minimum) interval between sampling times.

$$\tau_D \le t_{k+1} - t_k \le \tau_M, \forall k \in \mathbb{N} \tag{11}$$

We now consider a Lyapunov-Krasovskii functional of the form

$$V(x_s, \rho) := V_1(x) + V_2(x_s) + V_3(x_s, \rho) \tag{12}$$

where

$$x_s(t) \ := \ \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix},\ t_k \le t < t_{k+1}$$

$$\begin{aligned} V_1(x) &:= x^{\mathrm{T}} P x \\ V_2(x_s) &:= \int_{-\tau_M}^{0} \int_{t+r}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds dr \\ V_3(x_s, \rho) &:= (\tau_M - \rho)(x(t) - x(t_k))^{\mathrm{T}} X (x(t) - x(t_k)) \end{aligned}$$

and P, R and X are positive definite matrices. The Lyapunov-Krasovskii functional $V(x_s, \rho)$ is by its definition positive definite. At the sampling times, $V(x_s, \rho)$ does not increase because $V_3(x_s, \rho)$ is non-negative right before each sampling time and it becomes zero right after the sampling time [13]. It can be shown that $V(x_s, \rho)$ satisfies the following inequality [14]

$$\lambda_{\min}(P)\|x\|^2 \le V(x_s, \rho) \le \sigma_a \|x_s\|^2 + \sigma_b \tag{13}$$

where

$$\begin{aligned} \sigma_a &= \lambda_{\max}(P) + 2(\tau_M - \rho)\lambda_{\max}(X) + \frac{\tau_M^2}{2}\lambda_{\max}(\bar{R}) \\ \sigma_b &= \frac{\tau_M^2}{2}\lambda_{\max}(\bar{R}) \end{aligned}$$

$\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ are the minimum and maximum eigenvalues of a matrix, respectively, and

$$\bar{R} = \arg\max_{i,j} \lambda_{\max}(\bar{R}_{ij}) \tag{14}$$

$$\bar{R}_{ij} = \begin{bmatrix} A_i^{\mathrm{T}} \\ K_j^{\mathrm{T}} B^{\mathrm{T}} \\ a_i^{\mathrm{T}} + k_j^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} A_i & BK_j & a_i + Bk_j \end{bmatrix} \tag{15}$$

Theorem 1: For the sampled-data PWA system (8), assume there exist symmetric positive matrices P, R, X and matrices N_i for $i = 1, \ldots, M$ such that

- for all $i \in \mathcal{I}(0)$,

$$\Omega_i + \tau_M M_{1i} + \tau_M M_{2i} < 0 \tag{16}$$

$$\begin{bmatrix} \Omega_i + \tau_M M_{1i} & \tau_M \begin{bmatrix} N_i \\ 0 \end{bmatrix} \\ \tau_M \begin{bmatrix} N_i^{\mathrm{T}} & 0 \end{bmatrix} & -\tau_M R \end{bmatrix} < 0 \tag{17}$$

- for all $i \notin \mathcal{I}(0)$, $\bar{\Lambda}_i \succ 0$,

$$\overline{\Omega}_i + \tau_M \overline{M}_{1i} + \tau_M \overline{M}_{2i} < 0 \tag{18}$$

$$\begin{bmatrix} \overline{\Omega}_i + \tau_M \overline{M}_{1i} & \tau_M \begin{bmatrix} N_i \\ 0 \\ 0 \end{bmatrix} \\ \tau_M \begin{bmatrix} N_i^{\mathrm{T}} & 0 & 0 \end{bmatrix} & -\tau_M R \end{bmatrix} < 0 \tag{19}$$

where

$$\Omega_i = \begin{bmatrix} \Psi_i & \begin{bmatrix} P \\ 0 \end{bmatrix} B \\ B^{\mathrm{T}} \begin{bmatrix} P & 0 \end{bmatrix} & -\gamma I \end{bmatrix},$$

$$\begin{aligned} \Psi_i = & \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i \end{bmatrix} + \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} \\ & - \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix} \\ & - N_i \begin{bmatrix} I - I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N_i^{\mathrm{T}} + \eta I_{2n\times 2n}, \end{aligned}$$

$$M_{1i} = \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} A_i & BK_i & B \end{bmatrix},$$

$$\begin{aligned} M_{2i} = & \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & B \end{bmatrix} \\ & + \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix}, \end{aligned}$$

$$\overline{\Omega}_i = \begin{bmatrix} \overline{\Psi}_i & \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} B \\ B^{\mathrm{T}} \begin{bmatrix} P & 0 & 0 \end{bmatrix} & -\gamma I \end{bmatrix}, \tag{20}$$

$$\begin{aligned} \overline{\Psi}_i = & \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i & Bk_i + a_i \end{bmatrix} \\ & + \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ k_i^{\mathrm{T}} B^{\mathrm{T}} + a_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P & 0 & 0 \end{bmatrix} \\ & - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix} \\ & - \begin{bmatrix} N_i \\ 0 \end{bmatrix} \begin{bmatrix} I - I & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} \begin{bmatrix} N_i^{\mathrm{T}} & 0 \end{bmatrix} \\ & + \begin{bmatrix} \eta I_{2n\times 2n} & 0 \\ 0 & -\lambda_i \end{bmatrix} \\ & + \lambda_i \begin{bmatrix} L_i^{\mathrm{T}} \\ 0 \\ l_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} L_i & 0 & l_i \end{bmatrix} \end{aligned} \tag{21}$$

Let there be constants Δ_K and Δ_k such that

$$\|w\| \leq \Delta_K \|x(t_k)\| + \Delta_k \tag{22}$$

Define

$$\mu_\theta = \frac{\sqrt{\gamma}\Delta_k}{\sqrt{\theta\eta} - \sqrt{\gamma}\Delta_K} \tag{23}$$

and the region

$$\Phi_\theta = \{x_s | \; \|x_s\| \leq \mu_\theta\} \tag{24}$$

for any positive $\theta < 1$ that verifies

$$\Delta_K < \sqrt{\frac{\theta\eta}{\gamma}} \tag{25}$$

Then, all the trajectories of the system (8) in $\mathcal{X}$ converge to the following invariant set

$$\Omega = \{x_s | \; V(x_s, \rho) \leq \sigma_a \mu_\theta^2 + \sigma_b\} \tag{26}$$

□

Proof: The proof is divided in two parts.

1) At first, we will show that the inequalities (16), (17), (18) and (19) are sufficient conditions for the following inequality

$$\dot{V}(x_s, \rho) \leq -\eta x_s^{\mathrm{T}} x_s + \gamma w^{\mathrm{T}} w \tag{27}$$

for $t_k < t < t_{k+1}$. Along the trajectories of the system (8), one has

$$\dot{V}_1(x) = \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} \tag{28}$$

The term $V_2(x_s)$ can be written in the following form

$$V_2(x_s) = \int_{-\tau_M}^{0} g(t, r) dr \tag{29}$$

where

$$g(t, r) = \int_{t+r}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds \tag{30}$$

Thus, since $\dot{\rho} = 1$ for $t_k < t < t_{k+1}$,

$$\dot{V}_2(x_s) = \int_{-\tau_M}^{0} \frac{d}{dt} g(t, r) dr \tag{31}$$

The expression

$$\frac{d}{dt} g(t, r) = \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - \dot{x}^{\mathrm{T}}(t + r) R \dot{x}(t + r) \tag{32}$$

then yields

$$\dot{V}_2(x_s) = \tau_M \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - \int_{t-\tau_M}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds \tag{33}$$

From (11) one has $\rho \leq \tau_M$ and considering the fact that R is positive definite, this leads to

$$\dot{V}_2(x_s) \leq \tau_M \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - \int_{t-\rho}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds \tag{34}$$

Since R is positive definite, for any matrix $N_i \in \mathbb{R}^{n\times 2n}$

$$\begin{bmatrix} \dot{x}^{\mathrm{T}}(s) & x_s^{\mathrm{T}}(t) N_i \end{bmatrix} \begin{bmatrix} R & -I \\ -I & R^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ N_i^{\mathrm{T}} x_s(t) \end{bmatrix} \geq 0 \tag{35}$$

and therefore

$$-\dot{x}(s)R\dot{x}(s) \leq x_s^{\mathrm{T}}(t)N_iR^{-1}N_i^{\mathrm{T}}x_s(t) - 2x_s^{\mathrm{T}}(t)N_i\dot{x}(s) \tag{36}$$

Integrating both sides from $t-\rho$ to t,

$$\begin{aligned} -\int_{t-\rho}^{t} \dot{x}(s)R\dot{x}(s)ds \quad \leq \quad & \rho x_s^{\mathrm{T}}(t)N_iR^{-1}N_i^{\mathrm{T}}x_s(t) \\ & -2x_s^{\mathrm{T}}(t)N_i \begin{bmatrix} I & -I \end{bmatrix} x_s(t) \end{aligned} \tag{37}$$

It follows from (34) and (37) that

$$\begin{aligned} \dot{V}_2(x_s) \quad \leq \quad & \tau_M\dot{x}^{\mathrm{T}}R\dot{x} + \rho x_s^{\mathrm{T}}N_iR^{-1}N_i^{\mathrm{T}}x_s \\ & -2x_s^{\mathrm{T}}N_i \begin{bmatrix} I & -I \end{bmatrix} x_s \end{aligned} \tag{38}$$

For $V_3(x_s, \rho)$, one can write

$$\begin{aligned} \dot{V}_3(x_s, \rho) \quad = \quad & -(x(t) - x(t_k))^{\mathrm{T}}X(x(t) - x(t_k)) \\ & +2(\tau_M - \rho)(x(t) - x(t_k))^{\mathrm{T}}X\dot{x}(t) \end{aligned} \tag{39}$$

From (28), (38) and (39), it follows that a sufficient condition for (27) is the following inequality

$$\begin{aligned} & \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} + \tau_M\dot{x}^{\mathrm{T}}R\dot{x} + \rho x_s^{\mathrm{T}}N_iR^{-1}N_i^{\mathrm{T}}x_s \\ & -2x_s^{\mathrm{T}}N_i \left[\begin{smallmatrix} I & -I \end{smallmatrix} \right] x_s - x_s^{\mathrm{T}} \left[\begin{smallmatrix} I \\ -I \end{smallmatrix} \right] X \left[\begin{smallmatrix} I & -I \end{smallmatrix} \right] x_s \\ & +2(\tau_M - \rho)x_s^{\mathrm{T}} \left[\begin{smallmatrix} I \\ -I \end{smallmatrix} \right] X\dot{x} - \eta x_s^{\mathrm{T}}x_s + \gamma w^{\mathrm{T}}w < 0 \end{aligned} \tag{40}$$

For $i \in \mathcal{I}(0)$,

$$\dot{x} = \begin{bmatrix} A_i & BK_i \end{bmatrix} x_s + Bw, \tag{41}$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$. Replacing $\dot{x}$ from (41) into (40) and considering N_i instead of N for region $\mathcal{R}_i$, yields

$$\begin{aligned} & x_s^{\mathrm{T}} \Big(\left[\begin{smallmatrix} P \\ 0 \end{smallmatrix} \right] \left[\begin{smallmatrix} A_i & BK_i \end{smallmatrix} \right] + \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{smallmatrix} \right] \left[\begin{smallmatrix} P & 0 \end{smallmatrix} \right] \\ & +\tau_M \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{smallmatrix} \right] R \left[\begin{smallmatrix} A_i & BK_i \end{smallmatrix} \right] + \rho N_iR^{-1}N_i^{\mathrm{T}} \\ & -N_i \left[\begin{smallmatrix} I & -I \end{smallmatrix} \right] - \left[\begin{smallmatrix} I \\ -I \end{smallmatrix} \right] N_i^{\mathrm{T}} \\ & - \left[\begin{smallmatrix} I \\ -I \end{smallmatrix} \right] X \left[\begin{smallmatrix} I & -I \end{smallmatrix} \right] \\ & +(\tau_M - \rho) \left[\begin{smallmatrix} I \\ -I \end{smallmatrix} \right] X \left[\begin{smallmatrix} A_i & BK_i \end{smallmatrix} \right] \\ & +(\tau_M - \rho) \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{smallmatrix} \right] X \left[\begin{smallmatrix} I & -I \end{smallmatrix} \right] + \eta I \Big) x_s \\ & +x_s^{\mathrm{T}} \left[\begin{smallmatrix} P \\ 0 \end{smallmatrix} \right] Bw + w^{\mathrm{T}}B^{\mathrm{T}} \left[\begin{smallmatrix} P & 0 \end{smallmatrix} \right] x_s \\ & +\tau_M x_s^{\mathrm{T}} \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{smallmatrix} \right] RBw + \tau_M w^{\mathrm{T}}B^{\mathrm{T}}R \left[\begin{smallmatrix} A_i & BK_i \end{smallmatrix} \right] x_s \\ & +\tau_M w^{\mathrm{T}}B^{\mathrm{T}}RBw + \gamma w^{\mathrm{T}}w < 0 \end{aligned} \tag{42}$$

Since (42) is affine in ρ, if it holds for $\rho = 0$ and $\rho = \tau_M$ then it is satisfied for any $\rho \in [0, \tau_M]$. For $\rho = 0$, the inequality (42) can be written as (16). Using Schur complement for $\rho = \tau_M$, the inequality (42) can be converted to (17).

For $i \notin \mathcal{I}(0)$,

$$\dot{x} = \begin{bmatrix} A_i & BK_i & a_i + Bk_i \end{bmatrix} \bar{x}_s + Bw, \; x \in \mathcal{R}_i \tag{43}$$

where

$$\bar{x}_s = \begin{bmatrix} x_s \\ 1 \end{bmatrix} \tag{44}$$

It follows from (5) that

$$1 - (L_ix + l_i)^{\mathrm{T}}(L_ix + l_i) > 0, \; x \in \mathcal{R}_i \tag{45}$$

Using (43) and (45), a sufficient condition for (42) when $x \in \mathcal{R}_i$ with $i \notin \mathcal{I}(0)$ can be written as

$$\begin{aligned} & \bar{x}_s^{\mathrm{T}} \Big(\left[\begin{smallmatrix} P \\ 0 \\ 0 \end{smallmatrix} \right] \left[\begin{smallmatrix} A_i & BK_i & a_i+Bk_i \end{smallmatrix} \right] \\ & + \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \\ a_i^{\mathrm{T}}+k_i^{\mathrm{T}}B^{\mathrm{T}} \end{smallmatrix} \right] \left[\begin{smallmatrix} P & 0 & 0 \end{smallmatrix} \right] \\ & +\tau_M \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \\ a_i^{\mathrm{T}}+k_i^{\mathrm{T}}B^{\mathrm{T}} \end{smallmatrix} \right] R \left[\begin{smallmatrix} A_i & BK_i & a_i+Bk_i \end{smallmatrix} \right] \\ & +\rho \left[\begin{smallmatrix} N_i \\ 0 \end{smallmatrix} \right] R^{-1} \left[\begin{smallmatrix} N_i^{\mathrm{T}} & 0 \end{smallmatrix} \right] \\ & - \left[\begin{smallmatrix} N_i \\ 0 \end{smallmatrix} \right] \left[\begin{smallmatrix} I & -I & 0 \end{smallmatrix} \right] - \left[\begin{smallmatrix} I \\ -I \\ 0 \end{smallmatrix} \right] \left[\begin{smallmatrix} N_i^{\mathrm{T}} & 0 \end{smallmatrix} \right] \\ & - \left[\begin{smallmatrix} I \\ -I \\ 0 \end{smallmatrix} \right] X \left[\begin{smallmatrix} I & -I & 0 \end{smallmatrix} \right] \\ & +(\tau_M - \rho) \left[\begin{smallmatrix} I \\ -I \\ 0 \end{smallmatrix} \right] X \left[\begin{smallmatrix} A_i & BK_i & a_i+Bk_i \end{smallmatrix} \right] \\ & +(\tau_M - \rho) \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \\ a_i^{\mathrm{T}}+k_i^{\mathrm{T}}B^{\mathrm{T}} \end{smallmatrix} \right] X \left[\begin{smallmatrix} I & -I & 0 \end{smallmatrix} \right] \\ & +\eta \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right] \Big) \bar{x}_s + \bar{x}_s^{\mathrm{T}} \left[\begin{smallmatrix} P \\ 0 \\ 0 \end{smallmatrix} \right] Bw \\ & +\tau_M \bar{x}_s^{\mathrm{T}} \left[\begin{smallmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}}B^{\mathrm{T}} \\ k_i^{\mathrm{T}}B^{\mathrm{T}}+a_i^{\mathrm{T}} \end{smallmatrix} \right] RBw \\ & +\tau_M w^{\mathrm{T}}B^{\mathrm{T}}R \left[\begin{smallmatrix} A_i \\ BK_i \\ Bk_i+a_i \end{smallmatrix} \right] \bar{x}_s \\ & +w^{\mathrm{T}}B^{\mathrm{T}} \left[\begin{smallmatrix} P & 0 & 0 \end{smallmatrix} \right] \bar{x}_s \\ & +\tau_M w^{\mathrm{T}}B^{\mathrm{T}}RBw \\ & +\gamma w^{\mathrm{T}}w - \lambda_i + \lambda_i \left[\begin{smallmatrix} L_i^{\mathrm{T}} \\ 0 \\ l_i^{\mathrm{T}} \end{smallmatrix} \right] \left[\begin{smallmatrix} L_i & 0 & l_i \end{smallmatrix} \right] < 0 \end{aligned} \tag{46}$$

where $\lambda_i < 0$. Inequality (18) is equivalent to (46) for $\rho = 0$ and inequality (19) is equivalent to (46) for $\rho = \tau_M$. Since (46) is affine in ρ, inequalities (18) and (19) imply that (46) is satisfied for any $\rho \in [0, \tau_M]$.
In conclusion, (27) is satisfied for $t_k < t < t_{k+1}$, $k = 0, 1, 2, \ldots$, any $x(t) \in \mathcal{R}_i$, $i = 1, 2, \ldots, M$ and any $x(t_k) \in \mathcal{R}_j$.

2) In the second part of the proof, we show that Ω is an attracting invariant set. The inequality (27) can be written as

$$\dot{V}(x_s, \rho) \leq -(1-\theta)\eta x_s^{\mathrm{T}}x_s - \theta\eta x_s^{\mathrm{T}}x_s + \gamma w^{\mathrm{T}}w \tag{47}$$

for $0 < \theta < 1$. Therefore if

$$\theta\eta x_s^{\mathrm{T}}x_s > \gamma w^{\mathrm{T}}w \tag{48}$$

then

$$\dot{V}(x_s, \rho) \leq -(1-\theta)\eta x_s^{\mathrm{T}}x_s \tag{49}$$

It follows from (22) that the following inequality is a sufficient condition for (48)

$$\sqrt{\theta\eta}\|x_s\| > \sqrt{\gamma}(\Delta_K\|x_s\| + \Delta_k) \tag{50}$$

Therefore if (25) holds and $\|x_s\| > \mu_\theta$, then

$$\dot{V}(x_s,\rho) \leq -(1-\theta)\eta\mu_\theta^2, \text{ for } t_k < t < t_{k+1} \tag{51}$$

For any $x_s \notin \Omega$,

$$V(x_s,\rho) > \sigma_a\mu_\theta^2 + \sigma_b \tag{52}$$

It follows from (13) that $\|x_s\| > \mu_\theta$. Therefore, using (51), $\dot{V}(x_s,\rho) \leq -(1-\theta)\eta\mu_\theta^2$ between the sampling times. As it was mentioned earlier, $V(x_s,\rho)$ decreases at each sampling time. Therefore there is a finite time t^θ such that $x_s(t^\theta) \in \Phi_\theta$ and therefore $V(x_s(t^\theta),\rho) \leq \sigma_a\mu_\theta^2 + \sigma_b$, which means that $x_s(t^\theta) \in \Omega$. Therefore, Ω is an attracting invariant set.

■

Remark 1: The upper bound for $\|w\|$ defined in (22) can be obtained as

$$\begin{aligned} \Delta_K &= \max_{i,j=1,\ldots,M} \|K_i - K_j\| \\ \Delta_k &= \max_{i,j=1,\ldots,M} \|k_i - k_j\| \end{aligned} \tag{53}$$

Note that for the case where $K_i = K_j$ and $k_i = k_j$, $\Delta_K = \Delta_k = 0$ and (25) is automatically satisfied. In this case $w = 0$ and $\mu_\theta = 0$.

Remark 2: For $\tau_M \to 0$ and

$$N_i = \begin{bmatrix} -PBK_i + I \\ -I \end{bmatrix}, \; X = (\beta - 2)I \tag{54}$$

where $\beta > \max(\eta, 2)$ and

$$\eta_c = \eta + \frac{\eta\beta}{\beta - \eta} \tag{55}$$

the inequalities (16), (17), (18) and (19) are reduced to the following inequalities for all $i \in \mathcal{I}(0)$

$$\begin{bmatrix} (A_i + BK_i)^{\mathrm{T}}P + P(A_i + BK_i) + \eta_c I & PB \\ B^{\mathrm{T}}P & -\gamma I \end{bmatrix} < 0 \tag{56}$$

and for $i \notin \mathcal{I}(0)$

$$\begin{bmatrix} \begin{pmatrix} (A_i + BK_i)^{\mathrm{T}}P \\ +P(A_i + BK_i) \\ +\lambda_i L_i^{\mathrm{T}} L_i + \eta_c I \end{pmatrix} & \begin{pmatrix} PBk_i + Pa_i \\ +\lambda_i L_i^{\mathrm{T}} l_i \end{pmatrix} & PB \\ \begin{pmatrix} a_i^{\mathrm{T}}P + k_i^{\mathrm{T}}B^{\mathrm{T}}P \\ +\lambda_i l_i^{\mathrm{T}} L_i \end{pmatrix} & \lambda_i(-1 + l_i^{\mathrm{T}} l_i) & 0 \\ B^{\mathrm{T}}P & 0 & -\gamma I \end{bmatrix} < 0 \tag{57}$$

Conditions (56) and (57) are sufficient conditions for input to state stability of the continuous-time PWA system (1) with the following condition satisfied for $V(x) = x^{\mathrm{T}}Px$

$$\dot{V}(x) < -\eta_c x^{\mathrm{T}}x + \gamma w^{\mathrm{T}}w \tag{58}$$

TABLE I

PARAMETERS OF THE HELICOPTER MODEL

Parameter	Value	Unit
I_{yy}	0.0283	kgm^2
m_{heli}	0.9941	kg
l_{cgx}	0.0134	m
l_{cgz}	0.0289	m
F_{kM}	0.0003	Nm
F_{vM}	0.0041	Nm/rad/s
g	9.81	m/s^2

This result establishes that the continuous-time PWA controller should satisfy a very important property: *non-fragility*. In other words, if there exists an error w in the implementation of the continuous-time PWA controller (6) as shown in the following

$$u(t) = K_i x(t) + k_i + w(t) \tag{59}$$

and the norm of w is bounded, the norm of the state vector $x(t)$ remains bounded.

IV. NUMERICAL EXAMPLE

Example 1: A state space model was built for an experimental two degrees of freedom helicopter in [15]. In this example, a simplified version of the pitch model of the experimental helicopter is considered. This model is described by the following equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{I_{yy}}(-m_{heli}l_{cgx}g\cos(x_1) - m_{heli}l_{cgz}g\sin(x_1) \\ &\quad - F_{kM}\,\mathrm{sgn}(x_2) - F_{vM}x_2 + u) \end{aligned} \tag{60}$$

where the values of the parameters are shown in Table 1.

The PWA approximation of the following nonlinear function in (60)

$$f(x_1) = -m_{heli}l_{cgx}g\cos(x_1) - m_{heli}l_{cgz}g\sin(x_1) \tag{61}$$

is then computed based on a uniform grid in x_1. The resulting approximation is shown in Figure 2. A PWA model is obtained by replacing $f(x_1)$ by $\hat{f}(x_1)$ in (60). The following PWA controller is then designed to stabilize the PWA system at the origin ($x_1 = x_2 = 0$).

$$\begin{aligned}
u &= -0.2919x_1 - 0.1092x_2 - 0.6313, && \text{for } x \in \mathcal{R}_1 \\
u &= 0.0900x_1 - 0.1092x_2 + 0.0887, && \text{for } x \in \mathcal{R}_2 \\
u &= 0.1579x_1 - 0.1092x_2 + 0.1314, && \text{for } x \in \mathcal{R}_3 \\
u &= -0.1961x_1 - 0.1092x_2 + 0.3538, && \text{for } x \in \mathcal{R}_4 \\
u &= -0.4475x_1 - 0.1092x_2 + 0.8278, && \text{for } x \in \mathcal{R}_5 \\
u &= -0.2919x_1 - 0.1092x_2 - 0.6319, && \text{for } x \in \mathcal{R}_6 \\
u &= 0.0900x_1 - 0.1092x_2 + 0.0881, && \text{for } x \in \mathcal{R}_7 \\
u &= 0.1579x_1 - 0.1092x_2 + 0.1308, && \text{for } x \in \mathcal{R}_8 \\
u &= -0.1961x_1 - 0.1092x_2 + 0.3532, && \text{for } x \in \mathcal{R}_9 \\
u &= -0.4475x_1 - 0.1092x_2 + 0.8272, && \text{for } x \in \mathcal{R}_{10}
\end{aligned}$$

Using Theorem 1, a sampling time for discrete time implementation of the proposed PWA controller can be computed

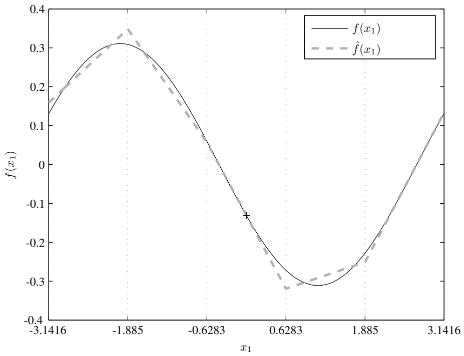

Fig. 2. PWA function - Helicopter model

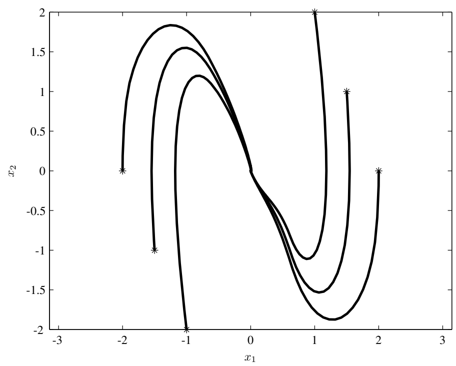

Fig. 3. Trajectories of the nonlinear Helicopter model - continuous time PWA controller

so that the closed loop sampled data system converges to a bounded invariant set. In this example, we consider η and γ as optimization parameters. However, to provide a larger upper bound on Δ_K, we require that $\eta > \gamma$ and $\gamma > 1$. Now, solving an optimization problem to maximize τ_M subject to the constraints of Theorem 1 and $\eta > \gamma > 1$, one has

$$\tau_M^\star = 0.1465,\ \eta = 4.2403,\ \gamma = 4.2403$$

$$P = \begin{bmatrix} 30.4829 & 2.4706 \\ 2.4706 & 4.4771 \end{bmatrix},$$
$$R = \begin{bmatrix} 44.9622 & 9.0745 \\ 9.0745 & 3.1994 \end{bmatrix},$$
$$X = \begin{bmatrix} 499.9799 & 11.6429 \\ 11.6429 & 24.1825 \end{bmatrix}$$

Figure 3 shows the trajectories of the nonlinear model (60) in feedback connection with the continuous time PWA controller. The trajectories of a sampled data PWA controller with a sampling time of 0.1465 second is shown in Figure 4.

V. Conclusion

This paper has presented stability results for closed-loop sampled-data PWA and linear systems under state feedback. Sampled-data PWA and linear systems were considered as delay systems with variable delay. The result for PWA

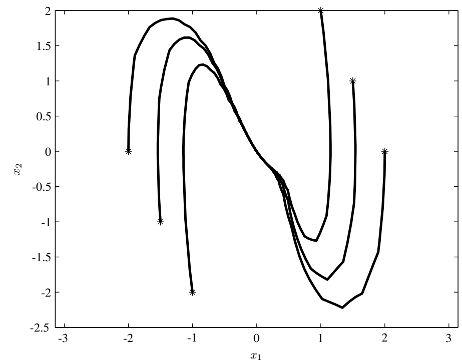

Fig. 4. Trajectories of the nonlinear Helicopter model - sampled data PWA controller

systems is equivalent to non-fragility of the continuous-time PWA controller when the sampling time converges to zero.

References

[1] A. Hassibi and S. Boyd, "Quadratic stabilization and control of piecewise-linear systems," in *Proc. American Control Conference*, Philadelphia, PA, Jun. 1998, pp. 3659–3664.

[2] M. Johansson, *Piecewise Linear Control Systems*. Berlin: Springer, 2003.

[3] L. Rodrigues and J. P. How, "Observer-based control of piecewise-affine systems," *International Journal of Control*, vol. 76, pp. 459–477, Mar. 2003.

[4] L. Rodrigues and S. Boyd, "Piecewise-affine state feedback for piecewise-affine slab systems using convex optimization," *Systems and Control Letters*, vol. 54, pp. 835–853, 2005.

[5] T. Chen and B. Francis, *Optimal Sampled Data Control Systems*. Springer-Verlag, 1995.

[6] J. Imura, "Optimal continuous-time control of sampled-data piecewise affine systems," in *Proc. American Control Conference*, 2003, pp. 5317–5322.

[7] ——, "Optimal control of sampled-data piecewise affine systems and its application to CPU processing control," in *Proc. of the 42nd IEEE Conf. on Decision and Control*, 2003, pp. 161–166.

[8] S. Azuma and J. Imura, "Probabilistic controllability analysis of sampled-data/discrete-time piecewise-affine systems," in *Proc. American Control Conference*, 2004, pp. 2528–2533.

[9] Z. Sun and S. S. Ge, "Sampling and control of switched linear systems," in *Proc. of the 41st IEEE Conference on Decision and Control*, 2002, pp. 4413–4418.

[10] Z. Sun, "Canonical froms of switched linear control systems," in *Proc. American Control Conference*, 2004, pp. 5182–5187.

[11] A. M. G. Zhai, H. Lin, and K. Yasuda, "Stability analysis for switched systems with continuous-time and discrete-time subsystems," in *Proc. American Control Conference*, 2004, pp. 4555–4560.

[12] L. Rodrigues, "Stability of sampled-data piecewise-affine systems under state feedback," *Automatica*, vol. 43, pp. 1249–1256, 2007.

[13] P. Naghshtabrizi, J. P. Hespanha, and A. R. Teel, "On the robust stability and stabilization of sampled-data systems: A hybrid system approach," in *45th IEEE Conference on Decision and Control*, San Diego, CA, USA, December 2006, pp. 4873–4878.

[14] B. Samadi, "Stability analysis and controller synthesis for a class of piecewise smooth systems," Ph.D. dissertation, Concordia University, 2008. [Online]. Available: http://users.encs.concordia.ca/~bsamadi/BSamadiThesis.pdf

[15] R. Endreß and L. Rodrigues, "Modeling and control of two degrees of freedom quanser helicopter," Concordia University, Tech. Rep., Jan. 2008.