Stochastic Stability for Discrete-Time Singular Systems with Markov Jump Parameters

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Abstract—In this paper, stochastic stability for discrete-time singular systems with Markov jump parameters is addressed. We present a set of coupled generalized Lyapunov equations (CGLEs) that serves as a necessary and sufficient condition for stochastic stability. A method for solving the obtained CGLEs is also presented, based on iterations of usual descriptor Lyapunov equations. An illustrative example is included.

I. INTRODUCTION

Very recently, singular systems with Markovian jumping parameters have received attention in the literature [15], [19], [20]. The interest in these systems is motivated by the fact that they are sufficiently specialised to yield strong results and, simultaneously, provide useful models for applications as it comprises both the singular (also called descriptor) systems and the systems with Markovian jumping parameters. Singular systems applications include aircraft modeling [18], chemical processes [9], circuit systems [13], [14], economic systems [11], large-scale interconnected systems [10], [17], mechanical engineering systems [6], power systems [16], and robotics [12]. Markov jump systems are becoming more and more popular due to their capability to describe dynamic behavior of systems with abrupt changes in their structures. For applications modeled by Markov jump systems, see [1], [4] and the references therein.

In this paper, a Lyapunov-type analysis of discrete-time singular linear systems with Markov jump parameters (SLSMJP) is developed in order to check stochastic stability. The set of coupled generalised Lyapunov equations (CGLEs) developed in this paper, and its respective solution, extend the results presented in [7] and [8] for conventional singular systems, and the results presented in [2] for conventional linear systems with Markov jump parameters (LSMJP). For computation solution of the proposed CGLE, a recursive algorithm is proposed. The algorithm has the nice property that, if a sequence generated by the recursion converges, then it converges to a solution of the CGLE, whenever it exists, disregarding stability or any other condition. For the interested reader in tests methods for solving coupled algebraic Riccati equations that are available in the literature for Markov jump linear systems, we can refer [3] and [5].

The paper is organized as follows. In Section II we present the notation. The problem statement and previous concepts are presented in Section III. The coupled generalized Lyapunov equation is addressed in Section IV. An algorithm for solving the CGLE is presented and studied in Section V. In Section VI, a numerical example is presented.

II. NOTATION

Let \( \mathbb{R}^n \) be the Euclidean linear space formed by all \( n \)-vectors. Let \( \mathbb{R}^{r,n} \) (respectively, \( \mathbb{R}^r \)) represent the normed linear space formed by all \( r \times n \) real matrices (respectively, \( r \times r \)) and \( \mathbb{R}^{r,+} \) the closed convex cone \( \{ U \in \mathbb{R}^r : U = U' \geq 0 \} \), (the open cone \( \{ U \in \mathbb{R}^r : U = U' > 0 \} \)), where \( U' \) denotes the transpose of \( U \); \( U \geq V \) \((U > V)\) signifies that \( U - V \in \mathbb{R}^{r,0} \) \((U - V \in \mathbb{R}^{r,+})\). For \( U \in \mathbb{R}^{r,0} \), \( \sigma(U) \) stands for the maximal eigenvalue of \( U \). Let \( \mathcal{M}^{r,n} \) denote the linear space formed by a number \( N \) of matrices such that \( \mathcal{M}^{r,n} = \{ U = (U_1, \ldots, U_N) : U_i \in \mathbb{R}^{r,n}, i = 1, \ldots, N \} \); also, \( \mathcal{M}^{r} \equiv \mathcal{M}^{r,r} \). We denote by \( \mathcal{M}^{r,0} \) \((\mathcal{M}^{r,+})\) the set \( \mathcal{M} \) when it is made up of \( U_i \in \mathbb{R}^{r,0} \) \((U_i \in \mathbb{R}^{r,+})\) for all \( i = 1, \ldots, N \). For \( U \in \mathcal{M}^{r,0} \), we define \( ||U|| = \max_{0 \leq i \leq N} \sigma(U_i) \), and we denote as \( U_{i,j}(j,l) \) the \( j,l \)-element of matrix \( U_i \).

We define the operators \( \mathcal{L} : \mathcal{M}^{r,0} \to \mathcal{M}^{r,0} \), \( \bar{\mathcal{L}} : \mathcal{M}^{r,0} \to \mathcal{M}^{r,0} \) as follows:

\[
\mathcal{L}(U) = \sum_{j=1}^{N} p_{ij} U_j;
\]

\[
\bar{\mathcal{L}}(U) = \sum_{j \neq i}^{N} p_{ij} U_j;
\]

for \( i = 1, \ldots, N \).

Remark 1: The operator \( \mathcal{L} \) is monotonically increasing in the positive semidefinite sense, i.e., for matrices \( U, V \in \mathcal{M}^{r,0} \),

\[
\mathcal{L}(V) \leq \mathcal{L}(U) \quad \text{whenever} \quad V \leq U.
\]

III. PROBLEM STATEMENT AND PREVIOUS CONCEPTS

We consider the discrete-time SLSMJP, described by
\[ \Phi : \left\{ \begin{aligned} E\theta(k+1)x(k+1) & = A\theta(k)x(k), \quad k = 0, 1, \ldots \\ x(0) & = x_0, \quad \theta(0) = \theta_0 \end{aligned} \right. \]

where \( x \in \mathbb{R}^n \) is the state variable, \( \theta \) is the state of an underlying discrete-time homogeneous Markov chain \( \Theta = \{\theta(k); k \geq 0\} \) having \( N = \{1, \ldots, N\} \) as state space and \( P = [p_{ij}] \), \( i, j = 1, \ldots, N \) as the transition matrix.

The system is defined in a fundamental probability space \((\Omega, \mathcal{F}, P)\). Matrices \( A_i, E_i \in \mathbb{R}^n \), \( i = 1, \ldots, N \), belong to \( N \) real constant matrices: \( A = (A_1, \ldots, A_N) \) and \( E = (E_1, \ldots, E_N) \) may be singular, with rank \( E_i \leq n \).

**Definition 1:** A matrix \( V \) is stable if and only if all its eigenvalues have magnitude less than 1.

The following notion is standard for stochastic systems, and have been considered in the literature of singular systems as well, see e.g. [19].

**Definition 2:** The discrete SLSMJP in (3) is said to be stochastically stable (SS) if there exist a scalar \( M(x_0, \theta_0) > 0 \) such that

\[ \lim_{N \to \infty} E \left\{ \sum_{k=0}^{N} ||x(k, x_0, \theta_0)||^2 \mid x_0, \theta_0 \right\} \leq M(x_0, \theta_0) \]

where \( x(k, x_0, \theta_0) \) denotes the solution to the System \( \Phi \) at time \( k \) under the initial conditions \( x_0 \) and \( \theta_0 \).

**IV. Coupled Generalized Lyapunov Equation**

Let \( x = \{x(k)\}_{k=0}^{\infty} \) be a solution to the System \( \Phi \).

Consider the following Lyapunov function candidate

\[ V(x(k), \theta(k)) := x'(k)E\theta(k)X\theta(k)E\theta(k)x(k) \geq 0 \]

where \( X_i \geq 0 \) for \( \theta(k) = i, \ i = 1, \ldots, N \). Then, for a set of matrices \( X_i = X_i' \geq 0, \ X_i \in \mathbb{R}^n, \ R_i \in \mathbb{R}^{(n-r) \times n}, \) and \( W \in \mathbb{R}^{n0} \) we define the following CGLEs as follows

\[ A_i' \left( \sum_{j=1}^{N} p_{ij} X_j \right) A_i + A_i' \left( \sum_{j=1}^{N} p_{ij} \bar{E}_j \right) R_i + R_i' \left( \sum_{j=1}^{N} p_{ij} \bar{E}_j \right) A_i - E_i' X_i E_i + W_i = 0. \]

for \( i = 1, \ldots, N, \) where \( E_i, A_i \in \mathbb{R}^n, \) and \( \bar{E}_i \in \mathbb{R}^{(n-r)} \) is a full column rank matrix such that \( E_i' \bar{E}_i = 0 \) and \( r = \text{rank}(E_i) \). With these assumptions, we are in position to state the following theorem

**Theorem 1:** The System \( \Phi \) is stochastically stable if and only if for \( W_i \in \mathbb{R}^{n0} \) there exist solutions \( E_i' X_i E_i \geq 0 \) and \( R_i \), \( i = 1, \ldots, N, \) to the CGLEs in (6).

**Proof:** (Necessity). Consider the sequence \( q_T(k, x, i), k \geq 0, \) defined by

\[ q_T(k, x, i) := E \left\{ \sum_{t=k}^{k+T} x'(t)W\theta(t)x(t) \mid x(k) = x, \theta(k) = i \right\} \]

We can show that \( X_i(T), i = 1, \ldots, N, T \geq 0, \) satisfying

\[ x(k) E_i' X_i(T) E_i x(k) = q_T(k, x, i), \]

for any \( x \in \mathbb{R}^n, i = 1, \ldots, N, \) and \( k \geq 0. \) Since \( \Phi \) is stochastically stable, we can evaluate

\[ \lim_{T \to \infty} x'(T)E_i' X_i(T) E_i x(T) \leq x'(0)E_i' X_i(0) E_i x(0), \]

for any \( x \) and \( i = 1, \ldots, N, \) hence

\[ X_i = \lim_{T \to \infty} X_i(T) \in \mathbb{R}^n. \]

From (7) and (9) yield

\[ \lim_{T \to \infty} x'(T)E_i' X_i(T) E_i x(T) \leq x'(0)E_i' X_i(0) E_i x(0), \]

and

\[ q_{T-1}(1, x(1), \theta(1)) \]

By subtracting (12) from (11), we have that

\[ x'(1)E_{\theta(1)}(T-1)E_{\theta(1)}x(1) + x'(1)E_{\theta(0)}E_{\theta(1)}R_{\theta(0)}x(0) + x'(0)R_{\theta(0)}E_{\theta(1)}x(1) - x'(0)E_{\theta(0)}X_{\theta(0)}(T)E_{\theta(1)}x(1) \]

\[ = -E \left\{ x'(0)W_{\theta(0)}x(0) \mid x(0), \theta(0) \right\} \]
and taking the expectation in (13) yields
\[
\mathbb{E}\left\{ x''(0) \left[ A'_{\theta(0)} X_{\theta(1)} (T - 1) A_{\theta(0)} + A'_{\theta(0)} \bar{E}_{\theta(0)} R_{\theta(0)} + R'_{\theta(0)} \bar{E}_{\theta(1)} A_{\theta(0)} - E'_{\theta(0)} X_{\theta(0)} (T - 1) A_{\theta(0)} \right] x(0) \mid x(0), \theta(0) = i \right\} = -x''(0) W_i x(0),
\]
or equivalently,
\[
x''(0) \left[ A' \sum_{j=1}^{N} (p_{ij} X_j (T - 1)) A_i + A' \sum_{j=1}^{N} (p_{ij} E_j) R_i + R' \sum_{j=1}^{N} (p_{ij} E_j') A_i - E' X_i (T - 1) E_i + W_i = 0
\]
\]

Finally, we take the limit with \( T \to \infty \) in (15), leads to
\[
A' \sum_{j=1}^{N} (p_{ij} X_j (T - 1)) A_i + A' \sum_{j=1}^{N} (p_{ij} E_j) R_i + R' \sum_{j=1}^{N} (p_{ij} E_j') A_i - E' X_i E_i + W_i = 0
\]
\]
when \( W_i > 0 \) for any \( i = 1, \ldots, N \),
\[
A' \sum_{j=1}^{N} (p_{ij} X_j) A_i + A' \sum_{j=1}^{N} (p_{ij} E_j) R_i + R' \sum_{j=1}^{N} (p_{ij} E_j') A_i - E' X_i E_i < 0
\]

(Sufficiency). For \( x_0 = 0 \), we can consider \( x(k) = 0, k = 0, 1, \ldots \) and so, (4) holds trivially. Therefore, we consider \( x_0 \neq 0 \) in what follows. We can rewrite (6) equivalently as
\[
A' \sum_{j=1}^{N} (p_{ij} X_j) A_i + A' \sum_{j=1}^{N} (p_{ij} E_j) R_i + R' \sum_{j=1}^{N} (p_{ij} E_j') A_i - E' X_i E_i + W_i = 0
\]

where we set stochastic Lyapunov function (5), we can evaluate \( W_i = W_i' > 0 \) conveniently. For
\[
\mathbb{E}\left\{ V \left( x(k+1), \theta(k+1) \right) \mid x(k) = x, \theta(k) = i \right\}
\]
\[
= \mathbb{E}\left\{ x''(k+1) E_{\theta(k+1)} X_{\theta(k+1)} E_{\theta(k+1)} x(k+1) + x' R_{\theta(k+1)} E_{\theta(k+1)} E_{\theta(k+1)} x(k+1) \mid x(k) = x, \theta(k) = i \right\}
\]
\[
= \mathbb{E}\left\{ x' A' \sum_{j=1}^{N} (p_{ij} X_j) A_i + x' A' \sum_{j=1}^{N} (p_{ij} E_j) R_i + x' R' \sum_{j=1}^{N} (p_{ij} E_j') A_i \mid x(k) = x, \theta(k) = i \right\}
\]
\[
\mathbb{E}\left\{ V \left( x(k+1), \theta(k+1) \right) \mid x(k) = x, \theta(k) = i \right\}
\]
\[
= x' A' \sum_{j=1}^{N} (p_{ij} X_j) A_i + x' A' \sum_{j=1}^{N} (p_{ij} E_j) R_i + x' R' \sum_{j=1}^{N} (p_{ij} E_j') A_i \mid x(k) = x, \theta(k) = i \right\}
\]
\]
\[
\Delta V(x(k) = x, \theta(k) = i)
\]
\[
= \mathbb{E}\left\{ V \left( x(k+1), \theta(k+1) \right) \mid x(k) = x, \theta(k) = i \right\}
\]
\[
- V(x(k) = x, \theta(k) = i)
\]
then,
\[
\Delta V(x(k) = x, \theta(k) = i)
\]
\[
= x' A' \sum_{j=1}^{N} (p_{ij} X_j) A_i + x' A' \sum_{j=1}^{N} (p_{ij} E_j) R_i + x' R' \sum_{j=1}^{N} (p_{ij} E_j') A_i \mid x(k) = x, \theta(k) = i \right\}
\]
\[
- V(x(k) = x, \theta(k) = i)
\]

We now split the proof in two parts. First, assume \( x \) and \( i \) are such that \( x'E' X_i E_i x > 0 \), and consequently \( V(\cdot) \neq 0 \). Defining the scalar
\[
\alpha := 1 - \min_{i=1, \ldots, N} \left\{ \frac{\lambda_{\min}(W_i)}{\lambda_{\max}(E_{\theta(k)} X_{\theta(k)} E_{\theta(k)})} \right\}
\]

where we set stochastic Lyapunov function (5), we can evaluate \( W_i = W_i' > 0 \) conveniently. For
we have that
\[
\mathbb{E} \left\{ V(\theta(k+1), x(k+1)) \mid x(k), \theta(k) \right\} - V(x(k), \theta(k)) \\
= - \left( \sum_{i=1}^{N} x_i E_i E_i x(k) \right)
\]}
\[
\leq \alpha
\]
(23)

Note that, when \( W_i \) and \( X_i \) are definite positive matrices, then \( 0 < \alpha < 1 \), from (23) we have that
\[
\mathbb{E} \left\{ V(x(k+1), \theta(k+1)) \mid x(k), \theta(k) \right\} - V(x(k), \theta(k)) \\
\leq \alpha V(x(k), \theta(k))
\]
(24)

and from (24) we conclude that
\[
0 < \alpha < 1, \text{ for } x E_i X_i E_i x = 0, \text{ we have that } V(\cdot) = 0 \text{ and (25) holds. Employing basic properties of the operator } \mathbb{E} \{ \cdot \}, \text{ from (25) we obtain with } k = 0
\]
\[
\mathbb{E} \left\{ V(x(1), \theta(1)) \mid x(0), \theta(0) \right\} \leq \alpha V(x(0), \theta(0))
\]
and with \( k = 1 \)
\[
\mathbb{E} \left\{ V(x(2), \theta(2)) \mid x(0), \theta(0) \right\} \\
= \mathbb{E} \left\{ V(x(2), \theta(2)) \mid x(0), \theta(0) \right\} + \sum_{i=1}^{N} x_i E_i X_i E_i x(k) \leq \alpha \mathbb{E} \left\{ V(x(0), \theta(0)) \right\}
\]
(25)
yielding
\[
\mathbb{E} \left\{ V(x(2), \theta(2)) \mid x(0), \theta(0) \right\} \leq \alpha^2 V(x(0), \theta(0))
\]
(26)

Similarly as above, for a general \( k \geq 0 \), we have
\[
\mathbb{E} \left\{ V(x(k), \theta(k)) \mid x(0), \theta(0) \right\} \leq \alpha^k V(x(0), \theta(0))
\]
(27)

hence,
\[
\mathbb{E} \left\{ \sum_{k=0}^{T} V(x(k), \theta(k)) \mid x(0), \theta(0) \right\} \\
= \mathbb{E} \left\{ V(x(0), \theta(0)) + \ldots \right. \\
+ V(x(T), \theta(T)) \mid x(0), \theta(0) \right\} \\
= \mathbb{E} \left\{ \sum_{k=0}^{T} V(x(k), \theta(k)) \mid x(0), \theta(0) \right\} \leq \alpha^k V(x(0), \theta(0))
\]
(28)
and we evaluate
\[
\lim_{T \to \infty} \left( \mathbb{E} \left\{ \sum_{k=0}^{T} V(x(k), \theta(k)) \mid x(0), \theta(0) \right\} \right) \\
\leq \lim_{T \to \infty} \left( \frac{1 - \alpha^T}{1 - \alpha} \right) V(x(0), \theta(0)) \\
= \left( \frac{1 - \alpha}{1 - \alpha} \right) V(x(0), \theta(0)) \\
= \left( \frac{1 - \alpha}{1 - \alpha} \right) x'(0) E_{\theta(0)} X_{\theta(0)} E_{\theta(0)} x(0).
\]
(29)

Remember (7), (8), and that by hypotheses \( W_i = W'_{i} > 0 \), \( i = 1, \ldots, N \). Then, we have that
\[
x(k) E'_i X_i (T) E_i x(k) \\
= \mathbb{E} \left\{ \sum_{t=k}^{k+T} x'(t) W_{\theta(t)} x(t) \mid x, i \right\}
\]
then, we can evaluate
\[
\lim_{T \to \infty} \mathbb{E} \left\{ \sum_{k=0}^{T} x'(k) W_{\theta(k)} x(k) \mid x(0), \theta(0) \right\} \\
\leq \lim_{T \to \infty} \mathbb{E} \left\{ \sum_{k=0}^{T} x'(k) E'_i X_i E_i x(k) \mid x(0), \theta(0) \right\} \\
\leq \lim_{T \to \infty} \mathbb{E} \left\{ \sum_{k=0}^{T} V(x(k), \theta(k)) \mid x(0), \theta(0) \right\} \\
= \lim_{T \to \infty} \mathbb{E} \left\{ \sum_{k=0}^{T} x'(k) E'_{\theta(0)} X_{\theta(0)} E_{\theta(0)} x(0) \right\} \\
:= M(x(0), \theta(0))
\]
(30)

Remark 2: It is a straightforward task to check that, when \( N = 1 \), the CGLE given in (6) reduces to the GLE of standard singular linear systems, see for instance [8]. Note that, in this situation, \( P = 1 \), and we can rewrite (6) as
\[
A'_i X_i A_i + A'_i E_i R_i + R'_i E'_i A_i \\
= E'_i X_i E_i + W_1 = 0.
\]

Remark 3: The following set of coupled generalized Lyapunov inequality (CGLI) has been considered in [19].
\[
E'_i X_i E_i \geq 0,
\]
\[
A'_i \left( \sum_{j=1}^{N} p_{ij} X_j \right) A_i - E'_i X_i E_i < 0.
\]

From the proposed CGLE in this paper, we can further associate for stochastic stability characterization, an alternative CGLI described as following
\[
A'_i \left( \sum_{j=1}^{N} p_{ij} X_j \right) A_i + A'_i \left( \sum_{j=1}^{N} p_{ij} E_j \right) R_i \\
+ R'_i \left( \sum_{j=1}^{N} p_{ij} E'_j \right) A_i - E'_i X_i E_i < 0.
\]
(31)

V. AN ALGORITHM FOR SOLVING THE CGLE

In this section we consider a method for solving the following CGLE. The CGLE consists of a set of CGLs with unknowns \( X_i \in \mathbb{R}^n \), \( i = 1, \ldots, N \) and associated matrices \( A_1, \ldots, A_N \) and similarly \( E, E, R \) and \( W \), as in (6).
Note that modifying the CGLEs, adding and subtracting $\kappa_ip_{ii}A'_iX_iA_i$, we obtain
\[
\kappa_ip_{ii}A'_iX_iA_i - E'_iX_iE_i + W_i \\
+ A'_i \left( (1 - \kappa_i) X_i + \sum_{j=1,j\neq i}^N p_{ij}X_j \right) A_i \\
+ A'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right) R_i + R'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right)' A_i = 0.
\]

The method for solving the CGLE generates a sequence $X(k) \in \mathcal{M}^{n_0}$, $k \geq 0$, as follows:

1. Set $X(0) \in \mathcal{M}^{n_0}$ and $W \in \mathcal{M}^{n_0}$ as $X_i(0) = 0$ and $W_i = I$, $i \geq 1$. For each $i \in N$, set $\kappa_i$ as the largest scalar such that the matrix $(\kappa_ip_{ii})^{1/2}A_i$ is stable, see Definition 1; if $\kappa_i > 1$ then set $\kappa_i = 1$.

Let $k = 1$.

2. For each $i \in N$, solve\(^1\) the following equality in the variable $R_i(k)$,
\[
0 = R'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right) A_i + A'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right) R_i \\
+ \kappa_ip_{ii}A'_iX_i(k-1)A_i - E'_iX_i(k-1)E_i + 0
\]

We define the operator \(\hat{\mathcal{L}} : \mathcal{M}^{n_0} \to \mathcal{M}^{n_0}\) as follows:
where
\[
\hat{\mathcal{L}}_i(X) = (1 - \kappa_i)p_{ii}X_i + \tilde{X}_i(X),
\]
for $i = 1, \ldots, N$.

3. For each $i \in N$, solve the following generalized algebraic Lyapunov equation in the variable $X(k)$,
\[
\kappa_ip_{ii}A'_iX_i(k)A_i - E'_iX_i(k)E_i + \hat{Q}_i = 0
\]
where
\[
\hat{Q}_i = A'_i\hat{\mathcal{L}}_i(X(k-1))A_i + W_i \\
+ A'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right) R_i + R'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right)' A_i.
\]

4. If $X(k)$ satisfies the stopping criterion (e.g. $\|X(k) - X(k-1)\| \leq \epsilon$ for some pre-specified $\epsilon$), then stop, else set $k = k + 1$ and go to Step 2.

**Theorem 2:** If $X(k)$ converges to some $X_\infty \in \mathcal{M}^{n_0}$ as $k \to \infty$ then $X_\infty$ is a solution to the CGLE.

**Proof:** Assuming that $X(k)$ converges to some $X_\infty$ as $k \to \infty$, employing (33) we have that
\[
\kappa_ip_{ii}A'_iX_\infty(k)A_i - E'_iX_\infty(k)E_i + \hat{Q}_i = 0
\]

where
\[
\hat{Q}_i = A'_i\hat{\mathcal{L}}_i(X_\infty)A_i + W_i \\
+ A'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right) R_i + R'_i \left( \sum_{j=1}^N p_{ij}\bar{E}_j \right)' A_i
\]
that satisfies the CGLE in (6).

VI. NUMERICAL EXAMPLE

**Example 1:** Consider the System $\Phi$ for the following numerical example, with $N = 2$,
\[
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
\]
\[
A_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
\[
W_1 = W_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{E}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{E}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

For $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}$, in $k = 10$ steps $R$ converges to
\[
R_1 = \begin{bmatrix} 0 & 0.6235 & -5.6111 \\ 0 & -90 & 0 \end{bmatrix}
\]
and $R_2 = [-90 0 -10]$, and $X$ converges to semidefinite matrices
\[
X_1 = \begin{bmatrix} 1.1942 & 0 & 0 \\ 0 & 1.0225 & 0 \\ 0 & 0 & 1.1111 \end{bmatrix}
\]
and
\[
X_2 = \begin{bmatrix} 183.4164 & 0 & 0 \\ 0 & 0.7978 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
then for this transition matrix of Markov chain, the system is stochastically stable.

For $P = \begin{bmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{bmatrix}$, in $k = 13$ steps and $R$ converges to
\[
R_1 = \begin{bmatrix} 0 & 454.5 & -50.5 \\ 0 & -0.1235 & 0 \end{bmatrix}
\]
and $R_2 = [-0.1235 0 -1.1111]$, and $X$ converges to
\[
X_1 = \begin{bmatrix} 1.1942 & 0 & 0 \\ 0 & 82.8200 & 0 \\ 0 & 0 & 10 \end{bmatrix}
\]
and
\[
X_2 = \begin{bmatrix} 21.4411 & 0 & 0 \\ 0 & -8.0911 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
then, for this second transition matrix of Markov chain, the system is not stochastically stable.

\(^1\)Note that (32) forms a set of equations that are linear in the variables $R_{i,j}$, which can be easily solved by Gaussian elimination.
VII. CONCLUSION

In this paper, a new characterization of stochastic stability for discrete-time SLSMJP was proposed. The set of coupled generalized Lyapunov equations (CGLEs), developed in this paper, reduces to the usual generalized Lyapunov equation for descriptor systems when the Markov chain is degenerated to the case \( N = 1 \) \[8\], and to a standard coupled Lyapunov equation when there is no singularity, i.e., \( E \) is a positive definite matrix \[4\]. We further derived a recursive algorithm for solving the CGLEs, and so, the issue of practical implementation of the proposed stability test was also dealt with.

REFERENCES