State-feedback Control Design for Continuous-time Piecewise Linear Systems: an LMI Approach

Da-Wei Ding and Guang-Hong Yang

Abstract—This paper studies the problem of state-feedback control design for a class of continuous-time piecewise linear systems. A new framework for the synthesis of piecewise linear control systems is given based on piecewise quadratic Lyapunov function and Reciprocal Projection Lemma. Under the new framework, Lyapunov variables and system dynamic matrices are separated by adding extra variables. And state-feedback control design for piecewise linear systems can be expressed in terms of linear matrix inequalities (LMIs). An example is given to illustrate the effectiveness of the proposed method.

Key words: Piecewise linear systems; piecewise quadratic Lyapunov function; linear matrix inequalities; Reciprocal Projection Lemma.

I. INTRODUCTION

Piecewise linear systems arise often in practical control systems when piecewise linear components such as deadzone, saturation, relays and hysteresis are encountered. In addition, many other classes of nonlinear systems can also be approximated by piecewise linear systems. In fact, piecewise linear systems are a broad modeling class in the sense that they have been shown to be equivalent to many other classes of systems, such as mixed logic dynamical systems [1] and extended linear complementary systems [2].

Since the author in [6] presented a pioneering work on the analysis of discrete-time piecewise linear systems in the early 1980s, numerous results [5]-[19] have been obtained on analysis and synthesis of piecewise linear systems. For example, the authors in [7], [8] presented results on stability and optimal performance analysis for piecewise linear systems based on a piecewise quadratic Lyapunov function. The author in [10] extended the stability analysis method [7] to discrete-time piecewise linear systems. Meanwhile, controller design for piecewise linear systems based on piecewise quadratic Lyapunov function arises, such as [11]-[16] for continuous-time systems and [17]-[19] for discrete-time systems.

By allowing the Lyapunov function to be piecewise quadratic and introducing S-procedure, piecewise quadratic Lyapunov function [7] is less conservative than quadratic Lyapunov function [4], [24] for stability analysis of piecewise linear systems. However, for state-feedback control design, it is hard to deduce an LMI condition to obtain the controller gain using piecewise quadratic Lyapunov function, due to parametrization of Lyapunov variables and existence of S-procedure. Usually, we can only obtain a nonconvex condition and achieve the controller gain by solving a set of bilinear matrix inequalities (BMIs) using kinds of methods [12], [14].

To solve this problem, a new framework for the synthesis of piecewise linear control systems is given based on piecewise quadratic Lyapunov function and Reciprocal Projection Lemma. Under the new framework, Lyapunov variables and system dynamic matrices are separated by adding extra variables. By a simple change of variables, state-feedback control design can be addressed in terms of LMIs, which is tractable with numerical software [24], [25]. Similar ideas have been presented for linear systems [22], [23].

The rest of the paper is organized as follows. Section 2 introduces the piecewise linear system model and piecewise quadratic stability, and presents the difficulty of state-feedback control design using piecewise quadratic Lyapunov function. Section 3 is the main result of this paper, which presents LMI conditions for state-feedback control design. Section 4 gives an example to illustrate the effectiveness of the proposed method. Finally, conclusions are given in Section 5.

Notations: We use standard notations throughout this note. $M^T$ is the transpose of the matrix $M$. $M > 0$ ($M < 0$) means that $M$ is positive definite (negative definite). The symbol $*$ will be used in some matrix expressions to induce a symmetric structure. For example, if $N$ is symmetric matrix, then

$$\begin{bmatrix} L & N \\ * & M \end{bmatrix} := \begin{bmatrix} L & N \\ NT & M \end{bmatrix}.$$

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider piecewise linear systems

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad \text{for } x(t) \in X_i \quad (1)$$

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where \( \{X_i\}_{i \in I} \subseteq \mathbb{R}^n \) is a partition of the state space into a number of closed (possibly unbounded) polyhedral cells with pairwise disjoint interior. The index set of the cells is denoted \( I = \{1, 2, \ldots, I\} \). Let \( x(t) \in \bigcup_{i \in I} X_i \) be a continuous piecewise \( C^1 \) function on the time interval \([t_0, t]\) such that the derivative \( \dot{x}(t) \) is defined, the equation \( \dot{x}(t) = A_i x(t) + B_i u(t) \) holds for all \( i \) with \( x(t) \in X_i \). \( x(t) \in \mathbb{R}^n \) is the state system variable, and \( u(t) \in \mathbb{R}^m \) the system input variable. Here, we assume there is no sliding mode, and all the local systems are controllable.

The state cells \( X_i \) are polyhedrons, so we can construct matrices \( E_i \in \mathbb{R}^{l \times n} \), \( F_i \in \mathbb{R}^{(l+n) \times n} \) such that

\[
E_i x \geq 0 \quad x \in X_i, \quad i \in I \tag{2}
\]

\[
F_i x = F_j x \quad x \in X_i \cap X_j, \quad i, j \in I \tag{3}
\]

Here, (2),(3) describe state cells and cell boundaries respectively [8].

Quadratic stability (Proposition 3 in Appendix) of piecewise linear systems is attractive, since stability follows independently of cell partition and for a large class of switching schemes [7]. Meanwhile, state feedback control design based on quadratic Lyapunov is easy and can be addressed in terms of LMIs [11]. However, quadratic stability is very conservative for piecewise linear systems. To reduce the conservatism, the authors in [7] introduced a piecewise quadratic Lyapunov function method [7]. Meanwhile, due to existence of S-procedure, the usual conversion technique from BMIs into LMIs does not work in this case. This is the main problem when we design state feedback control for piecewise linear systems using piecewise quadratic Lyapunov function.

In this paper, to circumvent the nonlinearity problem, an alternative strategy is adopted to obtain an LMI condition for the state-feedback controller.

III. MAIN RESULTS

In this section, we provide the main contribution of the paper. An LMI-based state-feedback controller will be designed based on piecewise quadratic Lyapunov function and Reciprocal Projection Lemma (see Lemma 2). The following two lemmas are essential for later developments.

**Lemma 1** [21]: For definite matrix \( P \), the following inequality holds

\[
G^T P^{-1} G \geq G^T + G - P \tag{10}
\]

**Lemma 2** [23]: (Reciprocal Projection Lemma) Let \( \Psi \) be any given positive definite matrix. The following statements are equivalent:

\[
i) : \Psi + S + S^T < 0 \tag{11}
\]

\[
ii) : \text{the LMI problem} \quad \begin{bmatrix} \Psi + \Phi - (W + W^T) & S^T + W^T \\ S + W & -\Phi \end{bmatrix} < 0 \tag{12}
\]

is feasible with respect to \( W \).

Now, we give the main result of the paper.

**Theorem 1**: For symmetric matrix \( T \) and \( U_i, W_i \), such that \( U_i, W_i \) have nonnegative entries, and general matrix \( V_i, R_i \), if the following LMIs are satisfied

\[
\begin{bmatrix}
-(V_i + V_i^T) & V_i T A_i^T + R_i^T B_i^T + P_i & V_i^T \\
* & -P_i & 0 \\
* & * & E_i^T U_i E_i - P_i
\end{bmatrix} < 0
\]

Then the closed-loop piecewise linear system (1) becomes

\[
\dot{x}(t) = (A_i + B_i K_i) x(t), \quad \text{for } x(t) \in X_i \tag{7}
\]

Based on Proposition 1, if the following conditions are satisfied

\[
(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) + E_i^T U_i E_i < 0 \tag{8}
\]

\[
P_i - E_i^T W_i E_i > 0 \tag{9}
\]

the closed-loop system (7) is stable, in other words, the piecewise linear system (1) is stabilized by the controller (6). Note that (8) and (9) are LMIs. In general, the conversion from BMIs into LMIs is performed by multiplying the inequalities on the left and right by \( P_i^{-1} \). However, to make piecewise quadratic Lyapunov function continuous across the cell boundaries, \( P_i = F_i^T T F_i \) with \( F_i \in \mathbb{R}^{(l+n) \times n} \) so that it is difficult to obtain \( P_i^{-1} \) parameterized in terms of \( F_i \) and \( T^{-1} \). Meanwhile, due to existence of S-procedure, the usual conversion technique from BMIs into LMIs does not work in this case. This is the main problem when we design state feedback control for piecewise linear systems using piecewise quadratic Lyapunov function.

\[
\begin{bmatrix}
-(V_i + V_i^T) & V_i T A_i^T + R_i^T B_i^T + P_i & V_i^T \\
* & -P_i & 0 \\
* & * & E_i^T U_i E_i - P_i
\end{bmatrix} < 0
\]
$P_i - E_i^T W_i E_i > 0$ \hspace{1cm} (14)

with $P_i = F_i^T T_i F_i$, $i \in I$, then the closed-loop piecewise linear system (7) is globally stable, and the controller gain for each local subsystem is given by

\[ K_i = R_i V_i^{-1} \] \hspace{1cm} (15)

**Proof:** It follows from Proposition 1 that if the following conditions are satisfied

\[ A_{cl_i}^T P_i + P_i A_{cl_i} + E_i^T U_i E_i < 0 \] \hspace{1cm} (16)

with $A_{cl_i} = A_i + B_i K_i$, then the closed-loop piecewise linear systems (7) is globally stable.

The inequality (16) is equivalent to

\[ Q_i A_{cl_i}^T + A_{cl_i} Q_i + Q_i S_i Q_i < 0 \]

with $Q_i := P_i^{-1}$, $S_i := E_i^T U_i E_i$.

The use of Lemma 2 with $\Psi = Q_i S_i Q_i$ and $S = Q_i A_{cl_i}^T$, yields

\[ \left[ Q_i S_i Q_i + \Phi_i - (W_i + W_i^T) A_{cl_i} Q_i + W_i^T \right] < 0 \]

By the congruence transformation

\[ \begin{bmatrix} V_i & 0 \\ 0 & P_i \end{bmatrix}, \quad \text{with} \quad V_i := W_i^{-1}, \quad P_i = Q_i^{-1} \]

the inequality (18) becomes

\[ \left[ \Xi_i \quad V_i^T A_{cl_i} + P_i \right] < 0 \]

where

\[ \Xi_i = V_i^T (P_i^{-1} S_i P_i^{-1} + \Phi_i) V_i - (V_i^T + V_i) \]

By a Schur complement operation with the term $V_i^T (P_i^{-1} S_i P_i^{-1} + \Phi_i) V_i$, we obtains the following inequality

\[ \begin{bmatrix} -(V_i^T + V_i) & V_i^T A_{cl_i} + P_i & V_i^T \\ * & -P_i \Phi_i & 0 \\ * & * & -(P_i^{-1} S_i P_i^{-1} + \Phi_i)^{-1} \end{bmatrix} < 0 \]

The use of Lemma 1 with $\Phi_i = P_i^{-1}$ yields

\[ (P_i^{-1} S_i P_i^{-1} + \Phi_i)^{-1} = P_i (S_i + P_i) P_i \geq P_i - S_i \]

So, if the following inequality holds

\[ \begin{bmatrix} -(V_i + V_i^T) & V_i^T A_{cl_i} + P_i & V_i^T \\ * & -P_i & 0 \\ * & * & E_i^T U_i E_i - P_i \end{bmatrix} < 0 \]

the inequality (16) holds.

The dual of (20) in the transformation $A_{cl_i} \rightarrow A_{cl_i}^T$ is

\[ \begin{bmatrix} -(V_i + V_i^T) & V_i^T A_{cl_i} + P_i & V_i^T \\ * & -P_i & 0 \\ * & * & E_i^T U_i E_i - P_i \end{bmatrix} < 0 \]

Substituting $A_{cl_i} = A_i + B_i K_i$ and $R_i = K_i V_i$ into (21), we obtains (13).

So, if (13) holds, (16) holds. And (17) is the same as (14). Based on Proposition 1, the piecewise closed-loop linear system (7) is globally stable. In other words, the piecewise linear system (1) is stabilized by controller (6). And the control gain is given by $K_i = R_i V_i^{-1}$. Thus the proof is completed.

**Remark 1:** Conditions (13), (14) are LMIs with respect to $T_i U_i, W_i, V_i$ and $R_i$, which are well tractable with commercially available software.

**Remark 2:** Note that Proposition 1 doesn’t apply to piecewise affine systems that have multiple equilibria and therefore, the method in Theorem 1 doesn’t apply in this case. This is a limitation of the proposed method.

Based on Reciprocal Projection Lemma and piecewise quadratic stability without S-procedure (Proposition 2 in Appendix), we obtain the following corollary.

**Corollary 1:** For symmetric matrix $T_i$ and general matrix $V_i$ and $R_i$, if the following LMI is satisfied, with $P_i = F_i^T T_i F_i$ for $i \in I$,

\[ \begin{bmatrix} -(V_i + V_i^T) & V_i^T A_{cl_i}^T + R_i T_i B_i^T + P_i & V_i^T \\ * & -P_i & 0 \\ * & * & -P_i \end{bmatrix} < 0 \]

(22)

Then the state-feedback controller $u(t) = K_i x(t)$ stabilizes the piecewise linear system (7), and the controller gain for each local subsystem is given by

\[ K_i = R_i V_i^{-1} \]

\hspace{1cm} (23)

**Proof:** It follows from Proposition 2 that if the following two inequalities are satisfied

\[ A_{cl_i}^T P_i + P_i A_{cl_i} < 0 \] \hspace{1cm} (24)

\[ P_i = P_i^T > 0 \] \hspace{1cm} (25)

with $A_{cl_i} = A_i + B_i K_i$, then the closed-loop piecewise linear systems (7) is stable.

Based on Lemma 2, (24) is equivalent to

\[ Q_i A_{cl_i}^T + A_{cl_i} Q_i < 0 \]

with $Q_i := P_i^{-1}$.

The use of Lemma 2 with $\Psi = 0$ and $S = Q_i A_{cl_i}^T$, yields

\[ \left[ \Phi_i - (W_i + W_i^T) A_{cl_i} Q_i + W_i^T \right] < 0 \]

By the congruence transformation

\[ \begin{bmatrix} V_i & 0 \\ 0 & P_i \end{bmatrix}, \quad \text{with} \quad V_i := W_i^{-1}, \quad P_i = Q_i^{-1} \]
the inequality above becomes

\[
V_i^T \Phi_i V_i - (V_i^T + V_i) V_i^T A_{cl_i} + P_i \ 
\begin{bmatrix}
0 \\
* \\
* 
\end{bmatrix}
\Phi_i P_i
\]
< 0

By a Schur complement operation with the term \(V_i^T \Phi_i V_i\), we obtain the following inequality

\[
-\begin{bmatrix}
(V_i^T + V_i) \\
* \\
* 
\end{bmatrix}
\begin{bmatrix}
V_i^T A_{cl_i} + P_i \\
* \\
* 
\end{bmatrix}
\begin{bmatrix}
V_i^T \\
* \\
* 
\end{bmatrix}
\Phi_i P_i
\]
< 0

Let \(\Phi_i = P_i^{-1}\), then the inequality above becomes

\[
-\begin{bmatrix}
(V_i + V_i^T) \\
* \\
* 
\end{bmatrix}
\begin{bmatrix}
V_i^T A_{cl_i} + P_i \\
* \\
* 
\end{bmatrix}
\begin{bmatrix}
V_i^T \\
* \\
* 
\end{bmatrix}
\Phi_i P_i
\]
< 0

The dual of (26) in the transformation \(A_{cl_i} \rightarrow A_{cl_i}^T\) is

\[
-\begin{bmatrix}
(V_i + V_i^T) \\
* \\
* 
\end{bmatrix}
\begin{bmatrix}
V_i^T A_{cl_i}^T + P_i \\
* \\
* 
\end{bmatrix}
\begin{bmatrix}
V_i^T \\
* \\
* 
\end{bmatrix}
\Phi_i P_i
\]
< 0

Substituting \(A_{cl_i} = A_i + B_i K_i\) and \(R_i = K_i V_i\) into (27) obtains (22). Thus (22) is equivalent to (24) and (25), and the proof is completed.

Remark 3: On the one hand, Theorem 1 is less conservative than Corollary 1, since Proposition 1 is less conservative than Proposition 2. And it is easy to see that Theorem 1 reduces to Corollary 1 without S-procedure “\(E_i^T U_i E_i\)” and “\(E_i^T W_i E_i\)” in (13) and (14). On the other hand, both Theorem 1 and Corollary 1 are less conservative than quadratic stabilization [11].

IV. NUMERICAL EXAMPLES

This example is given to illustrate that Theorem 1 is less conservative than quadratic stabilization [11] for piecewise linear systems.

Consider a continuous-time piecewise linear system

\[
\dot{x}(t) = \begin{cases}
A_1 x(t) + B_1 u(t), & x_1 - x_2 \geq 0, x_1 + x_2 \geq 0 \\
A_2 x(t) + B_2 u(t), & x_1 - x_2 < 0, x_1 + x_2 > 0 \\
A_3 x(t) + B_3 u(t), & x_1 - x_2 \leq 0, x_1 + x_2 \leq 0 \\
A_4 x(t) + B_4 u(t), & x_1 - x_2 > 0, x_1 + x_2 < 0
\end{cases}
\]

The system matrices are given by:

\[
A_1 = \begin{bmatrix}
0.8143 & -1.5415 \\
-0.0601 & 0.6594
\end{bmatrix},
A_2 = \begin{bmatrix}
-0.5385 & 0.2671 \\
-1.4398 & 1.2920
\end{bmatrix},
A_3 = \begin{bmatrix}
0.6958 & 1.8465 \\
1.9978 & -1.7646
\end{bmatrix},
A_4 = \begin{bmatrix}
-0.5588 & -0.9529 \\
0.1941 & 0.3894
\end{bmatrix},
B_1 = \begin{bmatrix}
-1.8029 \\
0.2842
\end{bmatrix},
B_2 = \begin{bmatrix}
0.8034 \\
1.8492
\end{bmatrix},
B_3 = \begin{bmatrix}
1.0021 \\
0.9000
\end{bmatrix},
B_4 = \begin{bmatrix}
-0.2725 \\
0.5371
\end{bmatrix}.
\]

The matrices describing the cells of the state are given as follows:

\[
E_1 = -E_3 = \begin{bmatrix}
-1 & 1 \\
-1 & -1
\end{bmatrix},
E_2 = -E_4 = \begin{bmatrix}
-1 & 1 \\
1 & 1
\end{bmatrix},
F_1 = \begin{bmatrix}
E_1 \\
I
\end{bmatrix}, F_2 = \begin{bmatrix}
E_2 \\
I
\end{bmatrix},
F_3 = \begin{bmatrix}
E_3 \\
I
\end{bmatrix}, F_4 = \begin{bmatrix}
E_4 \\
I
\end{bmatrix}.
\]

Simulation shows that the open-loop piecewise linear system is unstable. Here, we design a state-feedback controller \(u(t) = K_i x(t), i = 1, \ldots, 4\), to stabilize the unstable piecewise linear system. Quadratic stabilization [11] has no solution. However, using Theorem 1, the following solutions have been found:

\[
T = \begin{bmatrix}
0.5507 & 0.1176 & 0.0300 & -0.1761 \\
0.1176 & 0.5682 & 0.0679 & 0.2618 \\
0.0300 & 0.0679 & 0.2084 & 0.0088 \\
-0.1761 & 0.2618 & 0.0088 & 0.3512
\end{bmatrix},
P_1 = \begin{bmatrix}
1.3668 & -0.0973 \\
-0.0973 & 0.3592
\end{bmatrix},
P_2 = \begin{bmatrix}
1.1679 & 0.5620 \\
0.5620 & 1.8766
\end{bmatrix},
P_3 = \begin{bmatrix}
1.7582 & 0.1498 \\
0.1498 & 2.1107
\end{bmatrix},
P_4 = \begin{bmatrix}
1.0163 & -0.5095 \\
-0.5095 & 1.5341
\end{bmatrix},
K_1 = \begin{bmatrix}
1.3836 & -1.2081 \\
0.6109 & -1.9962
\end{bmatrix},
K_2 = \begin{bmatrix}
-2.8732 & -1.5435 \\
1.8866 & -4.2790
\end{bmatrix}.
\]

Let the initial state system be \(x_0 = \begin{bmatrix} 1 & 0.9 \end{bmatrix}\). The response of the closed-loop control system is shown in Fig.1. From the simulation result, we know that the piecewise linear system has been stabilized. This example shows that Theorem 1 is less conservative than quadratic stabilization.

V. CONCLUSIONS

State-feedback control design for continuous-time piecewise linear systems using piecewise quadratic Lyapunov function is usually nonconvex. To solve this problem, a new framework for the synthesis of piecewise linear control systems has been given based on piecewise quadratic Lyapunov function and Reciprocal Projection Lemma. Under
the new framework, Lyapunov variables and system dynamic matrices are separated by adding extra variables. And state-feedback control design for continuous-time piecewise linear systems can be expressed in terms of LMIs. An example has been given to illustrate the effectiveness of the proposed method.

VI. APPENDIX

**Proposition 2:** (Piecewise quadratic stability without S-procedure) For symmetric matrix \( T \), if \( P_i = F_i^T T F_i \) satisfy

\[
\begin{cases}
A_i^T P_i + P_i A_i < 0 \\
P_i > 0
\end{cases}
\]  

(29)

Then \( x(t) \) tends to zero exponentially for every continuous piecewise \( C^1 \) trajectory in \( \bigcup_{i \in I} X_i \) satisfying (1) with \( u \equiv 0 \) for \( t \geq 0 \).

**Proposition 3:** (Quadratic stability) If there exists a matrix \( P = P^T > 0 \) such that \( A_i^T P + P A_i < 0 \), \( i \in I \), then \( x(t) \) tends to zero exponentially for every continuous piecewise \( C^1 \) trajectory in \( \bigcup_{i \in I} X_i \) satisfying (1) with \( u \equiv 0 \) for \( t \geq 0 \).

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