Stability Analysis of Discrete Fault Tolerant Control Systems with Parameter Uncertainties

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Abstract—The stochastic behavior of DTFTCS with norm bounded parameter uncertainties in noisy environments is studied. The uncertainties are assumed to be unknown but bounded. Second moment stability for the uncertain DTFTCS driven by a state feedback control law is to be investigated. Sufficient conditions that guarantee the second moment stability and achieve a minimum of $\delta$-level of disturbance rejection are to be derived. Conditions are to be formulated as a feasibility solution for a set of linear matrix inequalities (LMIs) which allow the utilization of linear optimization tools. The results are verified by means of a numerical example.

I. INTRODUCTION

Active Fault Tolerant Control Systems (AFTCSs) are highly sophisticated control functions designed to maintain high levels of system survivability for safety-critical systems. An AFTCS utilizes a fault detection and isolation (FDI) scheme and a control reconfiguration mechanism to respond to a faulty condition. Fault induced changes as detected by the FDI scheme are used to reconfigure the control law in real time basis. Control reconfiguration mechanisms can be broadly classified as: projection-based methods, or automatic on-line controller calculation methods. In projection-based methods the control gains for possible fault scenarios are computed off-line and only gain selection is completed on-line based on the decisions of the FDI scheme. On the other hand, on-line controller calculation techniques never assume any faulty scenario a priori. The on-line controller computation is more involved and put heavy constraints on response time, yet they are able to deal with unanticipated faults. A bibliographical review on the definition of FTCS, classes and major components of FTCS, and preview of some design methodologies can be found in [14]. Generally speaking, the dynamical behavior of AFTCS is governed by stochastic differential equations and modeled as a hybrid system combining the Euclidean space for system dynamics and the discrete space for fault-induced changes.

The research of hybrid systems evolved into two major classes: Jump Linear Systems (JLS) and Fault Tolerant Control Systems with Markovian Parameters (FTCSMP). The modelling of JLS defines a single finite state Markov chain called plant regime mode [3], [7] to represent the random variations in the coefficients of the system to be controlled. This modelling lead to an inherit assumption of perfect regime knowledge which can not be enforced in practical environments corrupted by all types of disturbances and noises. This limitation motivated the introduction of FTCSMP [13]. In FTCSMP two separate random processes with different state spaces are defined: one represents system component failures and the second represents the non-deterministic decisions of the FDI process. This unique modeling allows the consideration of practical implementation issues and physical limitations for a particular system to be controlled. were studied in [8],[9] and [10]. A comprehensive review of the stochastic stability and stabilization of continuous FTCSMP using Lyapunov function approach can be found in [11]. The citation also studied the stochastic behavior of FTCSMP in the presence of noise, detection errors, detection delays, parameter uncertainties and actuator saturation. Just lately, the analysis of stochastic stability and $H_{\infty}$ stabilization of continuous FTCSMP was revisited in [1] and [2] using convex programming framework. The results provided an LMI characterization of output feedback controllers that stochastically stabilize FTCSMP and ensures $H_{\infty}$ constraints. Integral Quadratic Constraints were defined for FTCSMP and a stabilizing controller was synthesized in [15] and [16], optimal $H_2$ performance was investigated in [17] and [18]. In [19] FTCSMP were modeled and analyzed using randomized algorithms. The vital issue of detection delays has been revisited in a more rigorous form in [20]. Just recently the class of discrete time fault tolerant control systems (DTFTCS) has drawn the attention of several researchers. The reason for the delay in studying and characterizing the stochastic behavior of DTFTCS was due to the complexity of the model and tools needed to complete the studies. [6] studied the stochastic stability and controller design for the nominal DTFTCS, [12] extended the results to include norm bounded parameter uncertainties. In both citations, a control law was synthesized by solving a set of Riccati-Like matrix inequalities. It was concluded that this DTFTCS model yield results that are more complex than the case of continuous FTCSMP.

The problem of stabilizing an uncertain DTFTCS was tackled in [12]. However, it dealt with the case of DTFTCS operating in ideal noise-free environments and carried the analysis utilizing a less unified approach that lead to stability results in terms of nonlinear Riccati-like matrix equations that are difficult to solve and to test. It is the objective of this article to model and characterize the stochastic behavior of uncertain DTFTCS. Parameter uncertainties are assumed to be unknown yet bounded having norm bounded structure. Stability performance of uncertain DTFTCS actuated by a state feedback controller in noisy environments is to be studied. A unified approach of analysis is outlined and an
The DTFTCS (1) is assumed to satisfy both the growth and global uniform Lipschitz condition, the solution, \( x_k \), determines a family of unique discrete stochastic processes, one for each choice of the random variable \( x_0 \). The joint process \( \{x_k, \eta_k, \Psi_k; k \in \mathbb{N}\} \) is a discrete Markov process.

**Notations** The following notations are used in the paper, the notation \( M > N (\geq, \leq, >, \leq, 0 \) is used to denote that \( M - N \) is positive definite (positive semi-definite, negative definite, negative semi-definite) matrix. \( \lambda_{\text{min}}(\cdot), \lambda_{\text{max}}(\cdot) \) denote the minimum and the maximum eigenvalue, respectively. \( \mathbb{E}[\cdot] \) stands for the mathematical expectation. Also, \( A(\eta_k) = A_i, \Delta A(\eta_k) = \Delta A_i, B(\eta_k) = B_i, \Delta B(\eta_k) = \Delta B_i; C(\eta_k) = C_i, D(\eta_k) = D_i, E(\eta_k) = E_i, \varphi_x(\eta_k) = \varphi_x, \) and \( \varphi_y(\eta_k) = \varphi_y \), when \( \eta_k = i \in S \) and \( K(\Psi_k) = K_j \) when \( \Psi_k = j \in R \). To reduce repetition, a symmetric matrix \( \Lambda \) is written as

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{12}^T & a_{22}
\end{bmatrix} \triangleq 
\begin{bmatrix}
a_{11} & a_{12} \\
a_{12}^T & a_{22}
\end{bmatrix}
\]

**Plant Uncertainty Models**

The plant models commonly used in robust control system design and analysis are: state-variable models, transfer-function matrix models, and matrix-fraction models. Each plant model can have its own type of uncertainty representation. In general, system uncertainty is classified in a number of different ways. For example, uncertainty is characterized as parametric versus non-parametric, structured versus non-structured, etc. Norm bounded uncertainty is the most adopted form of structured parameter uncertainties in robust stability analysis [11]. In this form, the admissible parameter uncertainty is modelled as:

\[
\begin{align*}
\Delta A(\eta_k) &= H^a(\eta_k)F^a(\eta_k, k)M^a(\eta_k) \\
\Delta B(\eta_k) &= H^b(\eta_k)F^b(\eta_k, k)M^b(\eta_k)
\end{align*}
\]

where \( H^a(\eta_k), H^b(\eta_k) \in \mathbb{R}^{n \times n_k} \), and \( M^a(\eta_k), M^b(\eta_k) \in \mathbb{R}^{n_k \times n} \) are known constant matrices \( \forall k \geq 0 \). \( F^a(\eta_k, k) \) and \( F^b(\eta_k, k) \) are Lipschitz measurable unknown matrix functions satisfying the condition

\[
\begin{align*}
F^{aT}(\eta_k, k)F^a(\eta_k, k) &\leq I_{n_k}, \forall k \geq 0 \\
F^{bT}(\eta_k, k)F^b(\eta_k, k) &\leq I_{n_k}, \forall k \geq 0, \eta_k = i \in S
\end{align*}
\]

Other forms of admissible parameter uncertainties can be alternatively used in the modelling of uncertain DTFTCS, yet it was shown that these forms can be studied in unified systematic approach [11]. Therefore and without loss of generality, admissible parameter uncertainties in this article will be assumed to be time varying with norm bounded structure satisfying

\[
\begin{align*}
[\Delta A(\eta_k) &\quad \Delta B(\eta_k)] = H(\eta_k)F(\eta_k, k)[M^a(\eta_k) \quad M^b(\eta_k)] \\
\text{with} \quad F^T(\eta_k, k)F(\eta_k, k) &\leq I_{n_k}, \forall k \geq 0, \eta_k = i \in S
\end{align*}
\]

Before concluding this section, the following Lemmas will be used in the proof of the main results.

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Lemma 1: (Shur Complement [4]) For appropriately dimensioned constant matrices $\phi = \phi^T$, $\omega = \omega^T$, and $\theta$, the linear matrix inequality

$$
\begin{bmatrix}
\phi & \theta^T \\
\theta & -\omega
\end{bmatrix} < 0
$$

is equivalent to $\phi + \theta^T \omega^{-1} \theta < 0$ and $\omega > 0$.

Lemma 2: (21I) For appropriately dimensioned real matrices $\phi, \psi, \theta$ and $E$, such that $\theta^T \psi \theta \leq I$, $\varepsilon \geq 0$, and $W - \varepsilon \psi^T \psi > 0$, the following inequality is satisfied

$$(\phi + \psi \theta \omega)^T W^{-1} (\phi + \psi \theta \omega) \leq \phi^T (W - \varepsilon \psi^T \psi)^{-1} \phi + \varepsilon \omega^T \omega$$

III. Stability Properties of Uncertain DTFTCS

Second moment stochastic stability for the uncertain DTFTCS will be studied. The uncertain DTFTCS without input and noise is first considered, then $H_\infty$ performance of the uncertain DTFTCS is investigated. The investigation of $H_\infty$ stabilization given constant control gains will be deferred to the next section.

A. Stability for Autonomous Uncertain DTFTCS

An autonomous uncertain DTFTCS is a noise-free input-free uncertain DTFTCS and is described as

$$
x_{k+1} = [A_i + \Delta A_i]x_k \\
y_k = C_i x_k, \quad z_k = D_i x_k
$$

Select a quadratic Lyapunov function as

$$V(x_k, \eta_k, \Psi_k) = x_k^T P_k (\eta_k = i, \Psi_k = j) x_k$$

Given $Q_{ij} > 0$, the one-step forward increment for uncertain DTFTCS (9) gives

$$(A_i + \Delta A_i)^T \tilde{P}_{ij}(A_i + \Delta A_i) - P_{ij} + Q_{ij} < 0$$

where

$$\tilde{P}_{ij} = \sum_{n=1}^{s} \alpha_{ij} \sum_{m=1}^{r} \gamma_{nm} P_{nm}$$

Define $\chi_{ij} = \chi_i^T P_{ij}^{-1}$, pre- and post- multiply (11) by $\chi_i$, we get

$$\chi_{ij} (A_i + \Delta A_i)^T \tilde{X}_{ij}^{-1}(A_i + \Delta A_i) \chi_i - \chi_{ij} + \chi_{ij} Q_{ij} \chi_i < 0$$

Define

$$Z = \text{diag}([\chi_{i1}, \chi_{i2}, ..., \chi_{ir}])_{j=1,2,...,s}$$

$$\ii_{ij} = \nabla_{ij}[Z]$$

where

$$\nabla_{ij}[:] = \begin{bmatrix}
\sqrt{\alpha_{11} \gamma_{11}}[:]
\vdots
\sqrt{\alpha_{1r} \gamma_{1r}}[:]
\sqrt{\alpha_{21} \gamma_{21}}[:]
\vdots
\sqrt{\alpha_{2r} \gamma_{2r}}[:]
\sqrt{\alpha_{31} \gamma_{31}}[:]
\vdots
\sqrt{\alpha_{3r} \gamma_{3r}}[:]
\end{bmatrix}$$

This will lead to the identification

$$\chi_{ij} (A_i + \Delta A_i)^T \tilde{X}_{ij}^{-1}(A_i + \Delta A_i) \chi_i = \chi_i [\ii_{ij}^T A_i + \ii_{ij}^T H_i F_i M_i^a]^T Z^{-1} [\ii_{ij}^T A_i + \ii_{ij}^T H_i F_i M_i^a] \chi_i$$

For the particular parameter uncertainties and the results of Lemma 2, Inequality (13) can be equivalently written as

$$\chi_{ij} A_i^T \ii_{ij} (Z - \varepsilon_{ij} H_i^T H_i^T \ii_{ij})^{-1} \ii_{ij} \chi_{ij} + \varepsilon_{ij}^{-1} \chi_{ij} M_i^a T M_i^a \chi_{ij} \chi_{ij} + \chi_{ij} Q_{ij} \chi_{ij} < 0$$

Applying Shur complement to the term $\chi_{ij} A_i^T \ii_{ij} (Z - \varepsilon_{ij} H_i^T H_i^T \ii_{ij})^{-1} \ii_{ij} A_i \chi_{ij}$, we get

$$\begin{bmatrix}
\Omega_{11} \\
\ast \\
\ast
\end{bmatrix} \begin{bmatrix}
\chi_{ij} A_i^T \ii_{ij} \\
\ast \\
\ast
\end{bmatrix} \begin{bmatrix}
\chi_{ij} A_i^T \ii_{ij} \\
\ast \\
\ast
\end{bmatrix} \begin{bmatrix}
\Omega_{11} \\
\ast \\
\ast
\end{bmatrix} \begin{bmatrix}
\chi_{ij} A_i^T \ii_{ij} \\
\ast \\
\ast
\end{bmatrix}$$

where $\Omega_{11} = -\chi_{ij} + \chi_{ij} Q_{ij} \chi_{ij} + \varepsilon_{ij}^{-1} \chi_{ij} M_i^a T M_i^a \chi_{ij}$.

Applying Shur complement to the term $\chi_{ij} M_i^a T \varepsilon_{ij} M_i^a \chi_{ij}$, we get

$$\begin{bmatrix}
-\chi_{ij} + \chi_{ij} Q_{ij} \chi_{ij} & \chi_{ij} A_i^T \ii_{ij} \\
\ast & (\varepsilon_{ij} H_i^T H_i^T \ii_{ij} - Z)
\end{bmatrix} \begin{bmatrix}
\chi_{ij} M_i^a T \\
\ast
\end{bmatrix} < 0$$

The above result is formally stated in Theorem 1.

Theorem 1: A sufficient condition for the second moment stability for the autonomous uncertain DTFTCS (9) is the existence of $\chi_{ij} = \chi_i^T > 0$ and scalars $\varepsilon_{ij} > 0$, for some preselected $Q_{ij} > 0$, satisfying the matrix inequality (19).

B. $H_\infty$ Performance of Uncertain DTFTCS

A noisy input-free uncertain DTFTCS is described as

$$x_{k+1} = [A_i + \Delta A_i]x_k + \varphi_x w_k \\
y_k = C_i x_k + \varphi_y w_k \\
z_k = D_i x_k$$

For a quadratic Lyapunov function, (10), the one-step increment for the noisy input-free uncertain DTFTCS (20) gives

$$x_k^T (A_i + \Delta A_i)^T \tilde{P}_{ij}(A_i + \Delta A_i) x_k + x_k^T [-P_{ij} + Q_{ij}] x_k$$

Define $\chi_{ij} = \chi_i^T P_{ij}^{-1}$, and using the identification (14) gives

$$\chi_{ij} (A_i + \Delta A_i)^T \tilde{X}_{ij}^{-1}(A_i + \Delta A_i) = (A_i + \Delta A_i)^T Z^{-1} \ii_{ij} (A_i + \Delta A_i) = [\ii_{ij}^T A_i + \ii_{ij}^T H_i F_i M_i^a]^T Z^{-1} [\ii_{ij}^T A_i + \ii_{ij}^T H_i F_i M_i^a]$$

The results of Lemma 2 give

$$x_k^T (A_i + \Delta A_i)^T \tilde{X}_{ij}^{-1}(A_i + \Delta A_i) x_k + x_k^T [-\chi_{ij}^{-1} + Q_{ij}] x_k$$

The noisy input-free uncertain DTFTCS (20) has a $\delta$-level disturbance rejection, if the following is satisfied

$$z_k^T z_k - \delta^2 w_k^T w_k < 0$$

The one-step increment for the controlled output gives

$$x_k^T D_i^T D x_k - w_k^T (\delta^2 T) w_k \leq 0$$
Combining (23) and (24) by $y_k^T = [x_k^T \ w_k^T]$ gives

$$y_k^T \Theta y_k \leq 0 \quad \text{(25)}$$

where

$$\Theta_{11} = A_i^T \epsilon_{ij}(Z - \xi_{ij}^T H_i^T \epsilon_{ij}^T)^{-1} \xi_{ij}^T A_i + \epsilon_{ij}^T M^a_i T^a_i D_i^T \epsilon_{ij}^{-1} + Q_{ij}$$

$$\Theta_{12} = \epsilon_{ij}^T \chi_{ij} \chi_{ij}^{-1} (A_i + \Delta A_i) = \Theta_{21}$$

$$\Theta_{22} = -\delta^2 \mathcal{I} + \epsilon_{ij}^T \chi_{ij} \chi_{ij}^{-1} \epsilon_{ij}$$

$$\text{(26)}$$

Applying Shur complement to the term $\chi_{ij} A_i^T \epsilon_{ij}(Z - \xi_{ij}^T H_i^T \epsilon_{ij}^T)^{-1} \xi_{ij}^T A_i \chi_{ij}$, Inequality (27) is feasible if the following inequality is feasible

$$\begin{bmatrix} \Lambda_{11} & 0 \\ -\delta^2 \mathcal{I} + \omega^T \epsilon_{ij} \chi_{ij}^{-1} \omega_i \\ 0 & \epsilon_{ij}^T \chi_{ij} \chi_{ij}^{-1} \omega_i \end{bmatrix} < 0$$

where

$$\Lambda_{11} = \chi_{ij} A_i^T \epsilon_{ij}(Z - \xi_{ij}^T H_i^T \epsilon_{ij}^T)^{-1} \xi_{ij}^T A_i \chi_{ij} + \chi_{ij} D_i^T \epsilon_{ij} \chi_{ij} + \chi_{ij} Q_{ij} \chi_{ij} - \chi_{ij}.$$  

$$\text{(28)}$$

with $\Lambda_{11} = -\chi_{ij} + \chi_{ij} Q_{ij} \chi_{ij}$. Define

$$J_N = \mathbb{E} \left\{ \sum_{k=0}^{N} z_k^T z_k + V(x_{k+1}, \eta_{k+1}, \Psi_{k+1}) - V(x_k, \eta_k, \Psi_k) \right\}$$

$$\text{(30)}$$

Dykin’s formula leads to

$$\mathbb{E} \left\{ V(x_N, \eta_N, \Psi_N) - V(x_0, \eta_0, \Psi_0) \right\}$$

$$= \mathbb{E} \left\{ \sum_{k=0}^{N-1} V(x_{k+1}, \eta_{k+1}, \Psi_{k+1}) - V(x_k, \eta_k, \Psi_k) \right\}$$

$$\text{(31)}$$

Assuming zero initial conditions, we get

$$\mathbb{E} \left\{ V(x_N, \eta_N, \Psi_N) \right\} = \mathbb{E} \left\{ \sum_{k=0}^{N-1} V(x_{k+1}, \eta_{k+1}, \Psi_{k+1}) - V(x_k, \eta_k, \Psi_k) \right\}$$

$$\text{(32)}$$

Equations (30) and (32) give

$$\mathbb{E} \left\{ \sum_{k=0}^{N-1} y_k^T \Theta y_k \right\} < 0$$

Hence, the dissipativity condition

$$\| z_k^T z_k \| < \delta^2 \| w_k^T w_k \|^2$$

is satisfied and the uncertain DTFTCS (20) is second moment stable as stated in the following Theorem

**Theorem 2:** The noisy input-free uncertain DTFTCS (20) is second moment stable with $\delta$-disturbance attenuation for any noise disturbance $w_k \in \mathcal{L}_2$, if for some preselected $Q_{ij} > 0$, there exist symmetric $\chi_{ij} > 0$ and some scalars $\epsilon_{ij} > 0$ that satisfy matrix inequality (29).

**IV. STOCHASTIC STABILIZATION OF UNCERTAIN DTFTCS**

This section derives conditions that test the stability of the uncertain DTFTCS given a linear state feedback controller with constant gains $K_j \forall j \in R$. The study considers both noise-free and noisy environments.

**A. Uncertain DTFTCS in Noise-Free Environments**

An uncertain DTFTCS in noise-free environments driven by a state feedback controller $u_k = -K_j x_k$ can be described as

$$x_{k+1} = [\dot{A}_{ij} + \Delta A_{ij}] x_k$$

$$y_k = C_i x_k, \quad z_k = D_i x_k$$

$$\text{(34)}$$

Equation (8) combines admissible parameter uncertainties for the closed-loop system together as follows

$$\Delta A(\eta_k) = H(\eta_k)F(\eta_k) [M^a(\eta_k) - M^b(\eta_k) K(\Psi_k)]$$

$$\text{(36)}$$

**Theorem 3:** Given state feedback controller with constant gains, $K_j$, the noise-free uncertain DTFTCS (34) is second moment stable, if for some preselected $Q_{ij} > 0$, there exist $\chi_{ij} > 0$ and some scalars $\epsilon_{ij} > 0$ satisfying the following matrix inequality

$$\chi_{ij} Q_{ij} \chi_{ij} - \chi_{ij} \epsilon_{ij} A_{ij} \epsilon_{ij}^T (A_{ij}^T H_i^T \chi_{ij} - Z)$$

$$\begin{bmatrix} \chi_{ij} A_i^T \epsilon_{ij} & 0 & 0 \\ -\delta^2 \mathcal{I} + \epsilon_{ij}^T \chi_{ij} \epsilon_{ij}^{-1} \epsilon_{ij}^T \chi_{ij} \epsilon_{ij} \epsilon_{ij}^T \chi_{ij}^{-1} \omega_i \\ 0 & \epsilon_{ij}^T \chi_{ij} \chi_{ij}^{-1} \omega_i & \epsilon_{ij}^T \chi_{ij} \chi_{ij}^{-1} \omega_i \end{bmatrix} < 0$$

**Proof:** A careful look reveals that the noise-free uncertain DTFTCS (34) has similar structure to the autonomous uncertain DTFTCS (9) with the augmented system matrices $\hat{A}$, $\Delta \hat{A}$, and $\hat{D}$ replace the system open loop system matrices $A$, $\Delta A$, and $D$ respectively. As a result, the proof is omitted.
B. Uncertain DTFTCS in Noisy Environments

An uncertain DTFTCS in noisy environment with state feedback controller $u_k = -K_j x_k$ is described as

$$x_{k+1} = [\hat{A}_{ij} + \Delta \hat{A}_{ij}] x_k + \varphi_x, w_k$$
$$y_k = C_i x_k + \varphi_y, w_k$$
$$z_k = D_{ij} x_k$$

(37)

where $\hat{A}_{ij}$, $\Delta \hat{A}_{ij}$ and $D_{ij}$ were defined in (35) and the admissible parameter uncertainties were combined in (36).

The one-step increment for the uncertain DTFTCS (37) gives

$$x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$= x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$+ x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) \hat{P}_{ij} (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$+ w_k^T x_k \varphi_x, w_k - x^T_k P_{ij} x_k + x^T_k Q_{ij} x_k < 0$$

(38)

Equivalently,

$$x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$= x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$+ x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) \hat{P}_{ij} (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$+ w_k^T x_k \varphi_x, w_k - x^T_k P_{ij} x_k + x^T_k Q_{ij} x_k < 0$$

(39)

Using the identification in (14), the results of Lemma 2 and the combined parameter uncertainties in (36) gives

$$x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$= x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$+ x^T_k (\hat{A}_{ij} + \Delta \hat{A}_{ij}) \hat{P}_{ij} (\hat{A}_{ij} + \Delta \hat{A}_{ij}) x_k$$
$$+ w_k^T x_k \varphi_x, w_k - x^T_k P_{ij} x_k + x^T_k Q_{ij} x_k < 0$$

(40)

The uncertain DTFTCS (37) have a δ-level disturbance rejection if

$$x^T_k D_{ij} x_k - w^T_k (\delta^2 \mathcal{I}) x_k < 0$$

(41)

Combining (40) and (41) using $y_k = [x^T_k \ w^T_k]$, pre- and post-multiply by $\text{diag}\{x_{ij}, \mathcal{I}\}$ and applying Shur complement to the term $\chi_{ij} \hat{A}_{ij} x_k (Z - \varepsilon_{ij} H_i H_i^T \hat{I}_{ij}^{-1})^{-1} \hat{I}_{ij} x_k$, and the term $\mathcal{W}_{ij}^T Z^{-1} \mathcal{W}_{ij}$, we get

$$\begin{bmatrix}
\Pi_{11} & 0 & 0 & \varepsilon_{ij} H_i H_i^T \hat{I}_{ij}^{-1} - Z \\
0 & -\delta^2 \mathcal{I} & 0 & 0 \\
0 & 0 & (\varepsilon_{ij} H_i H_i^T \hat{I}_{ij}^{-1} - Z) & 0 \\
0 & 0 & 0 & -\mathcal{I}
\end{bmatrix} < 0$$

(42)

where $\Pi_{11} = \varepsilon_{ij}^{-1} \chi_{ij} (M^a_i - M^b K_i)^T (M^a_i - M^b K_i) \chi_{ij} + \chi_{ij} \hat{D}_{ij} D_{ij} \chi_{ij} - \chi_{ij} + \chi_{ij} \hat{D}_{ij} \hat{D}_{ij} \chi_{ij}$. Further application of Shur complement to the term $\chi_{ij} (M^a_i - M^b K_i)^T \varepsilon_{ij}^{-1} (M^a_i - M^b K_i) \chi_{ij}$, we get

$$\begin{bmatrix}
\nabla_{11} & 0 & 0 & 0 \\
0 & -\delta^2 \mathcal{I} & 0 & 0 \\
0 & 0 & (\varepsilon_{ij} H_i H_i^T \hat{I}_{ij}^{-1} - Z) & 0 \\
0 & 0 & 0 & -\mathcal{I}
\end{bmatrix} < 0$$

(43)

where $\nabla_{11} = \chi_{ij} Q_{ij} x_k - \chi_{ij} + \chi_{ij} \hat{D}_{ij} D_{ij} \chi_{ij}$. Finally, Shur complement to the term $\chi_{ij} D_{ij} D_{ij} \chi_{ij}$, lead to

$$\begin{bmatrix}
\chi_{ij} Q_{ij} x_k - \chi_{ij} & 0 & \chi_{ij} \hat{A}_{ij} \hat{I}_{ij} \\
0 & -\delta^2 \mathcal{I} & 0 \\
0 & 0 & (\varepsilon_{ij} H_i H_i^T \hat{I}_{ij} - Z)
\end{bmatrix} < 0$$

Theorem 4: Given a state feedback controller with constant gains, $K_j$, the uncertain DTFTCS (37) is second moment stable with $\delta$-disturbance attenuation for any noise disturbance $w_k \in l_2$, if for some preselected $R_{ij} > 0$, there exist $\chi_{ij} = \tilde{\chi}_{ij} > 0$ and scalars $\varepsilon_{ij} > 0$ satisfying the following linear matrix inequality

$$\begin{bmatrix}
-\chi_{ij} & 0 & \chi_{ij} \hat{A}_{ij} \hat{I}_{ij} \\
0 & -\delta^2 \mathcal{I} & 0 \\
0 & 0 & (\varepsilon_{ij} H_i H_i^T \hat{I}_{ij} - Z)
\end{bmatrix} < 0$$

Proof: The same argument as those to get the matrix inequality (44) with the parametrization, $Q_{ij} = R_{ij}^{-1}$, is used to avoid the nonlinearity introduced by the slack term $\chi_{ij} \hat{D}_{ij} \hat{D}_{ij}$. The proof is omitted.

V. NUMERICAL EXAMPLE

A second-order DTFTCS subject to single actuator failure has the following system parameters

$$A_1 = \begin{bmatrix} -1.0 & 0.0 \\ 0.2 & 1.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.0 & -2.0 \\ 1.0 & 1.0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2.0 & 0.5 \\ 0.0 & 1.0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0.0 & 0.2 \end{bmatrix}, \quad M^a = \begin{bmatrix} 0.1 & 0.0 \end{bmatrix}, \quad M^b = \begin{bmatrix} 0.0 & -0.2 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}, \quad M^a = \begin{bmatrix} 0.3 & 0.0 \end{bmatrix}, \quad M^b = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.0 & 1.0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.0 & 0.5 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.2 & 0.6 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.5 & 0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.0 & -0.2 \end{bmatrix},$$

For such system, the actuator failure, $\eta_k$, has a state space $S = \{1, 2\}$ and the FDI process, $\Psi_k$, has $R = \{1, 2\}$. The failure rates representing these two modes of operation are $\alpha_{mn} = \begin{bmatrix} 0.70 & 0.30 \\ 0.60 & 0.40 \end{bmatrix}$.
The following gains were selected and the existence of feasible solutions, \( \chi_{ij}, \forall i, j = 1, 2 \), of the LMI in Theorem 4 is a sufficient condition for the stability of the system under study when driven by this preselected state feedback gains.

\[
K_1 = \begin{bmatrix} -0.20 & 1.00 \\ 1.00 & 1.20 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -2.20 & -2.20 \\ -0.20 & -2.23 \end{bmatrix},
\]

A sample path simulation for the state trajectories and the controlled output are shown in Figure 1 and 2, respectively.

Fig. 1: Single sample path simulation: State Trajectory

Fig. 2: Single sample path simulation: Controlled Output

VI. CONCLUSIONS

The stochastic behavior of DTFTCS with parameter uncertainties in noisy environments has been studied. The uncertainties are assumed to be unknown time varying with norm bounded structure. Second moment stability of the proposed uncertain DTFTCS has been investigated. Sufficient conditions that guarantee the second moment stability and achieve a minimum of \( \delta \)-level of disturbance rejection were derived and formulated as feasibility solutions for some LMIs. This framework allows the utilization of linear optimization tools. The major results were illustrated by a numerical example. The current results provide a test criteria given a state feedback controller, future work is to develop a methodology to synthesis a stabilizing reconfigurable state feedback fault tolerant controller.

REFERENCES


