Average Dwell-Time Method to $L_2$-Gain Analysis and Control Synthesis for Uncertain Switched Nonlinear Systems

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Abstract—This paper addresses the $L_2$-gain analysis and control synthesis problem for a class of switched nonlinear systems affected by both time-varying uncertainties and external disturbances. Firstly, the $L_2$-gain for the autonomous switched disturbed uncertain system is analyzed. Then, a switched state feedback control law is designed and the $L_2$-gain is analyzed for the disturbed uncertain non-autonomous switched system. Sufficient conditions for these two cases are obtained using average dwell-time method incorporated with piecewise Lyapunov functions. The corresponding closed-loop disturbed uncertain switched system and the disturbed uncertain autonomous switched system are globally exponentially stable with a weighted $L_2$-gain under the sufficient conditions. Both the piecewise Lyapunov functions and the average dwell-time based switching laws are constructed based on the structural characteristics of the uncertain switched system.

I. INTRODUCTION

Switched systems are consist of a family of continuous-time and/or discrete-time processes interacting with a logical or decision-making process. Analysis and synthesis of this kind of system have attracted lots of attention in recent years. Interests that focus on switched systems are mainly stability analysis [1-5], stabilization [6-8], controllability [9], observability [10], switching optimal control [11, 12], $H_\infty$ control [13-15], $L_2$-gain analysis [16-18] and so on. Stability is of great importance in the analysis of switched systems, and lots of researches are devoted to the study of this property. Among these researches the common Lyapunov function technique was introduce to check the uniformly stability or stabilizability of switched systems [1-3, 8]. But a common Lyapunov function may not exist or is too different to find. In this paper, the multiple Lyapunov functions technique [4, 6] and the average dwell-time technique [5] were generally proposed to analyze the stability property for the switched systems under some designed switching laws for the purpose of more flexibility in choosing a Lyapunov function.

On the other hand, switched systems with disturbances are commonly found in practice. Thus, the stability and $L_2$-gain analysis problem for the disturbed switched system becomes an interesting issue due to its value both in practical and in theoretical practice. But researches studying this problem are relatively few. [16] addressed the $L_2$-gain analysis and control synthesis problem with Linear matrices inequality method for a class of discrete-time disturbed uncertain switched linear systems. [17] investigated the disturbance attenuation problem for a class of disturbed autonomous switched linear system using the average dwell-time method, and a weighted $L_2$-gain is achieved. The $L_2$-gain analysis problem for a class of disturbed switched delay linear system was addressed in [18]. All the papers mentioned above are mainly about switched linear systems. But as we all know that nonlinear phenomenon exists in almost all the dynamics in practice, so analyzing the $L_2$-gain property for switched nonlinear disturbed systems deserves to be paid more attention.

In this paper, we investigate the $L_2$-gain analysis and control synthesis problem for a class of disturbed uncertain switched nonlinear systems using average dwell-time method incorporated with piecewise Lyapunov functions. The switched system under consideration is composed of a nonlinear part and a linear part. This research will address the $L_2$-gain analysis and control synthesis problem for this disturbed uncertain switched system in the case that both the linear and the nonlinear parts are stabilizable under some average dwell-time based switching laws. The average dwell-time for the switched system that the designed switching laws satisfied is designed recursively, and the relationship between the average dwell-time for the switched system and the average dwell-times for the linear and the nonlinear parts of the switched system is analyzed. Firstly, the $L_2$-gain is analyzed for the autonomous switched system. Then, the switched state feedback is synthesized for the non-autonomous switched system. Sufficient conditions are expressed in the form of Linear matrices inequalities under which both the autonomous switched system and the non-autonomous switched system are globally exponentially stable and have a weighted $L_2$-gain. Moreover, the state decay are calculated explicitly.

II. PROBLEM STATEMENT AND PRELIMINARIES

In this paper, we study the $L_2$-gain analysis and control synthesis problem for the following uncertain switched nonlinear system:

\[
\begin{align*}
\dot{x}_1(t) &= \hat{A}_{1\sigma(t)}x_1(t) + A_{2\sigma(t)}x_2(t) + \hat{B}_{\sigma(t)}u_{\sigma(t)}(t) \\
&\quad + G_{\sigma(t)}w(t), \\
\dot{x}_2(t) &= f_{2\sigma(t)}(x_2(t)), \\
y(t) &= C_{\sigma(t)}x_1(t),
\end{align*}
\]
where \( x_1(t) \in \mathbb{R}^{n-d} \), \( x_2(t) \in \mathbb{R}^d \) are the states, \( u(t) \in \mathbb{R}^m \) is the control input, \( w(t) \in L^2[0, \infty) \) is the external disturbance input, and \( y(t) \in \mathbb{R}^p \) is the controlled output. \\
\( \sigma(t) : [0, \infty) \to I_N = \{1, \ldots, N\} \) is the switching signal, which is a piecewise constant function of time and will be determined later. \( \sigma(t) = i \) means that the ith subsystem is activated. \( A_{1i} = A_{1i} + \Delta A_{1i}, B_i = B_i + \Delta B_i(t), A_{2i}, B_i, G_i \) and \( C_i (i \in I_N) \) are constant matrices of appropriate dimensions which describe the nominal systems. \( f_{2i}(x_2(t)) \) are smooth vector fields with \( f_{2i}(0) = 0 \). \( \Delta A_{1i}(t) \) and \( \Delta B_i(t) \) are uncertain time-varying matrices denoting the uncertainties in the system matrices and having the following form

\[
\begin{align*}
\Delta A_{1i}(t), \Delta B_i(t) &= E_i, \Gamma(t)[F_{1i}, F_{2i}], \quad i \in I_N. \quad (2)
\end{align*}
\]

where \( E_i \in \mathbb{R}^{(n-d) \times i}, F_{1i} \in \mathbb{R}^{k \times (n-d)} \), and \( F_{2i} \in \mathbb{R}^{k \times m} \) are given constant matrices which characterize the structure of uncertainty, and \( F_{2i} \) is of full column rank. \( \Gamma \) is the norm-bounded time-varying uncertainty, i.e.,

\[
\Gamma = \Gamma(t) \in \{\Gamma(t) : \Gamma(t)^T \Gamma(t) = I, \Gamma(t) \in \mathbb{R}^{d \times k}, \text{the elements of } \Gamma(t) \text{ are Lebesgue measurable}\}. 
\]

There are several reasons for assuming that the system uncertainties have the structures given in (2), see [14] for details.

We are interested in \( L_2 \)-gain analysis and control synthesis of uncertain switched nonlinear systems under some average dwell-time based switching law. This analysis is to establish sufficient conditions that the switched system

\[
\begin{align*}
\dot{x}_1(t) &= \hat{A}_{1\sigma(t)}x_1(t) + A_{2\sigma(t)}x_2(t) + G_{\sigma(t)}w(t), \\
\dot{x}_2(t) &= f_{2\sigma(t)}(x_2(t)), \\
y(t) &= C_{\sigma(t)}x_1(t),
\end{align*}
\]

is globally exponentially stable with a weighted \( L_2 \)-gain (see Definition 3 below), whereas control synthesis is to design a switched state feedback control law

\[
u_{\sigma(t)} = K_{\sigma(t)}x_1(t),
\]

such that the corresponding closed-loop switched system (1) with \( w(t) \equiv 0 \) is globally exponentially stable for all admissible uncertainties.

Consider the switched system

\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}w(t), \\
y(t) &= C_{\sigma(t)}x(t),
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \), \( w(t), y(t), \sigma(t) \) are the same as stated in (1), \( A_i, B_i, C_i \ (1 \leq i \leq N) \) are known constant matrices.

Definition 3. System (6) is said to have a \( e^{-\lambda t} \)-weighted \( L_2 \)-gain over \( \sigma(t) \), from the disturbance input \( w(t) \) to the controlled output \( y(t) \), if the following inequality holds for \( \sigma(t) \) and some real-valued function \( \beta(t) \) with \( \beta(0) = 0 \)

\[
\int_0^\infty e^{-\lambda t} y(t)y(t)dt \leq \gamma^2 \int_0^\infty w^T(t)w(t)dt + \beta(x(0)),
\]

along the solution to (6), where \( w(t) \in L_2(0, +\infty) \), \( x(0) \neq 0 \) is the initial state.

The following lemmas will be used in the development of the main results.

Lemma 1 [1]. Consider the nonlinear switched system

\[
\dot{x}(t) = f_i(x(t)), \quad i \in I_N = \{1, \ldots, N\}. \quad (8)
\]

Assume for each \( i \in I_N \) there exists a Lyapunov function \( V_i \) such that

\[
a_i \|x\|^2 \leq V_i(x) \leq b_i \|x\|^2,
\]

and

\[
\frac{\partial V_i(x)}{\partial x} f_i(x) \leq -c \|x\|^2.
\]

for some positive constants \( a_i, b_i, c \). Then, two positive constant \( \mu, \lambda \) can be found such that the switched nonlinear system (8) is globally exponentially stable for any switching signal that has the average dwell time property with \( \tau_a \geq \ln \mu/\lambda \).

Remark 1. It is not difficult to find from [1] that \( \mu = \sup_{i \in I_N} \frac{\lambda}{\beta_{ii}} \), \( p, q \in I_N \), \( \lambda \in (0, \lambda_0) \) and \( \lambda_0 = \min\{\frac{\mu}{\beta_{ii}} : i \in I_N\} \) in Lemma 1.

Lemma 2 [19]. Given a symmetric matrix \( G \), and any nonzero matrices \( M, N \) of appropriate dimensions. Then

\[
G + MTN + N^T GTM^T \leq 0
\]

for all \( \Gamma \) satisfying \( \Gamma^T \Gamma \leq I \) if and only if there exists a constant \( \varepsilon > 0 \) such that

\[
G + \varepsilon MM^T + \frac{1}{\varepsilon} N^T NT \leq 0.
\]

III. \( L_2 \)-Gain Analysis

This section gives the \( L_2 \)-gain analysis for the uncertain switched nonlinear system (3).

Theorem 1. Given any constant \( \gamma > 0 \), suppose that switched system (3) satisfies the following conditions

(i) if there exist constants \( \varepsilon_i > 0, \lambda_0 > 0, \mu_i \geq 1 \), such that the following inequalities

\[
A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i G_i G_i^T P_i + \varepsilon_i \gamma^2 F_i^T F_i + C_i^T C_i + \lambda_0 P_i + I < 0, \quad (11)
\]

where \( A_i(t) \in \mathbb{R}^n \), \( P(t) \in L^2[0, \infty) \) is the control input, and \( y(t) \in \mathbb{R}^p \) is the controlled output.
have positive definite solutions $P_i$. 
(ii) there exist proper, positive definite, and radially unbounded function $W_i(x(t))$ such that
\begin{align}
\frac{dW_i(x(t))}{dt} & \leq -\beta_i \|x(t)\|^2, \\
\alpha_i \|x(t)\|^2 & \leq \|W_i(x(t))\| \leq a_i \|x(t)\|^2.
\end{align}
for some constants $\beta_i > 0$, $a_i > 0$, $\alpha_i > 0$, $i = 1, \ldots, N$.

Then, switched system (3) is globally exponentially stable when $w(t) = 0$ and achieves a weighted $L_2$-gain which is less than or equal to $\gamma$ under arbitrary switching laws satisfying the average dwell time
\begin{equation}
\tau_a = \frac{\ln \hat{\mu}}{\lambda},
\end{equation}
where $\hat{\mu} = \max\{\mu, \frac{a_i}{\alpha_i} : i, j \in I_N\}$, $\lambda \in [0, \hat{\lambda}_0)$, $\hat{\lambda}_0 = \min\{\lambda_0, \frac{\beta_i}{\alpha_i} : i \in I_N\}$.

**Proof.** Define the following piecewise Lyapunov function candidate for switched system (3)
\begin{equation}
V(x_1, x_2) = x_i^T P_{\sigma(t)} x_1 + \omega P_{\sigma(t)} x_2.
\end{equation}
where $P_i$ are the solutions of (11) and (12).

Then, based on Lemma 2, when $\sigma(t) = i$, the time derivative of $V(x_1, x_2)$ along the trajectory of the switched system (3) is
\begin{align}
\dot{V} & = x_i^T (\dot{A}_i^T P_i + A_i^T P_i) x_1 + 2 x_i^T P_i A_2 x_2 + 2 x_i^T P_i G_i w \\
& \quad + k \frac{dW_i(x_2)}{dx_2} \dot{f}_2(x_2) \\
& \leq x_i^T (A_i^T P_i + A_i P_i) x_1 + 2 x_i^T P_i A_2 x_2 + 2 x_i^T P_i G_i w \\
& \quad + k \frac{dW_i(x_2)}{dx_2} \dot{f}_2(x_2) \\
& \leq x_i^T (A_i^T P_i + A_i P_i) x_1 + 2 x_i^T P_i E_i F_i x_1 \\
& \quad + 2 x_i^T P_i A_2 x_2 + 2 x_i^T P_i G_i w - k \beta_i \|x_2\|^2 \\
& \leq x_i^T (A_i^T P_i + A_i P_i) x_1 + \varepsilon_i^2 x_i^T E_i F_i x_1 \\
& \quad + 2 x_i^T P_i A_2 x_2 + 2 x_i^T P_i G_i w \\
& \quad - k \beta_i \|x_2\|^2 \\
& = x_i^T (A_i^T P_i + A_i P_i + \varepsilon_i^2 P_i E_i F_i P_i + \varepsilon_i^2 F_i^T F_i) x_1 \\
& \quad + 2 x_i^T P_i A_2 x_2 + 2 x_i^T P_i G_i w - k \beta_i \|x_2\|^2.
\end{align}

It is easy to see that there exist constants $l_i > 0$, $q_i > 0$, $i \in I_N$ such that
\begin{align}
\|A_2 x_2\| & \leq l_i \|x_2\|, \\
\|x_i^T P_i\| & \leq q_i \|x_1\|.
\end{align}

Let $l = \max\{l_i q_i : i \in I_N\}$, $b = \min\{\frac{\beta_i}{\alpha_i} : i \in I_N\}$, from (11) and (14), we can obtain
\begin{align}
\dot{V} + y^T y & \leq x_i^T (A_i^T P_i + A_i P_i + \varepsilon_i^2 P_i E_i F_i P_i + \varepsilon_i^2 F_i^T F_i) x_1 \\
& \quad + 2 l \|x_1\| \|x_2\| - k \beta_i \|x_2\|^2 \\
& \quad + 2 x_i^T P_i A_2 x_2 + 2 x_i^T P_i G_i w - k \beta_i \|x_2\|^2 \\
& \leq x_i^T (A_i^T P_i + A_i P_i + \varepsilon_i^2 P_i E_i F_i P_i + \varepsilon_i^2 F_i^T F_i) x_1 \\
& \quad + 2 l \|x_1\| \|x_2\| - k \beta_i \|x_2\|^2 \\
& \quad + 2 x_i^T P_i A_2 x_2 + 2 x_i^T P_i G_i w - k \beta_i \|x_2\|^2.
\end{align}

Hence, the globally exponential stability of system (3) when $w(t) = 0$ follows.
Integrating both sides of (17), and from (19), we can get
\[ V(t) \leq V(t_{N\sigma(0,t)}) e^{-\hat{\lambda}_0(t-t_{N\sigma(0,t)})} - \int_{t_{N\sigma(0,t)}}^{t} e^{-\hat{\lambda}_0(t-\tau)} \cdot y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau) d\tau \]
\[ \leq \mu V(t_{N\sigma(0,t)}) e^{-\hat{\lambda}_0(t-t_{N\sigma(0,t)})} - \int_{t_{N\sigma(0,t)}}^{t} e^{-\hat{\lambda}_0(t-\tau)} \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)] d\tau - \ldots \]
\[ = e^{-\hat{\lambda}_0(t-t_{N\sigma(0,t)})} [y^T(t_{N\sigma(0,t)})y(t_{N\sigma(0,t)}) - \int_{t_{N\sigma(0,t)}}^{t} e^{-\hat{\lambda}_0(t-\tau)} \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)] d\tau]. \]

Multiplying both sides of the above inequality by \( e^{-N\sigma(t)\ln(\mu)} \), leads to
\[ e^{-N\sigma(t)\ln(\mu)} V(t) \leq e^{-\hat{\lambda}_0(t)N\sigma(t)} V(0) - \int_{0}^{t} e^{-\hat{\lambda}_0(t-\tau)N\sigma(t)\ln(\mu)} \cdot [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)] d\tau. \]
Thus, the following inequality follows from (20)
\[ \int_{0}^{t} e^{-\hat{\lambda}_0(t-\tau)\gamma^2} y^T(\tau)y(\tau) d\tau \leq e^{-\hat{\lambda}_0(t)N\sigma(t)} V(0) + \int_{0}^{t} e^{-\hat{\lambda}_0(t-\tau)\gamma^2} w^T(\tau)w(\tau) d\tau. \]
Integrating both sides of the foregoing inequality from \( t = 0 \) to \( \infty \) and rearranging the double-integral area, we obtain
\[ \int_{0}^{\infty} e^{-\hat{\lambda}_0 t} y^T(\tau)y(\tau) d\tau \leq \gamma^2 \int_{0}^{\infty} w^T(\tau)w(\tau) d\tau + V(0). \]

From Definition 3, we know that system (3) has an weighted \( L_2 \)-gain.

**Remark 2.** Applying Shur complement formula, the first matrix inequality of condition (i) can be easily transformed into the LIMs form. The second inequality of condition (i) is trivial, as long as we let \( \mu = \sup_{i,j \in I_N} \frac{\lambda_{\max}(F_i^T)}{\lambda_{\min}(P_i)}. \)

**Remark 3.** When \( x_2(t) = 0 \), it is easy to verify that the \( x_1 \)-subsystem of system (3) is exponentially stable with an weighted \( L_2 \)-gain under arbitrary switching laws which satisfy the average dwell-time \( \tau_1 \geq \tau^*_{a_1} = \frac{\ln \mu_i}{\hat{\lambda}_1} \), \( \lambda_1 \in [0, \lambda_0) \), \( \mu_1 \) is equivalent to \( \mu \) in (12). And from Lemma 1, we know that the \( x_2 \)-subsystem of system (3) is exponentially stable under arbitrary switching laws which satisfy the average dwell-time \( \tau_2 \geq \tau^*_{a_2} = \frac{\ln \mu_2}{\hat{\lambda}_2} \), \( \mu_2 = \max\{ \frac{\mu_{2i}}{\lambda_{2i}} : i \in I_N \}, \lambda_2 \in [0, \lambda_0) \), and \( \hat{\lambda}_0 = \min\{ \frac{\mu_{2i}}{\lambda_{2i}} : i \in I_N \}. \)

(15), we know that the average dwell-time for the whole cascade switched system satisfies \( \tau_a \geq \tau^*_{a} \geq \max\{ \tau^*_{a_1}, \tau^*_{a_2} \}. \) Consequently, the two parts of the switched system are all globally exponentially stable under arbitrary switching laws that satisfy the average dwell-time \( \tau_a \). Based on the cascade system theory introduced in [20], we know that the whole switched system is exponentially stable under arbitrary switching laws that satisfy the average dwell time \( \tau_a \).

**IV. CONTROL SYNTHESIS**

In this section, we design an switched state feedback controller, such that the corresponding closed-loop system (1) is globally exponentially stable with an weighted \( L_2 \)-gain under some switching laws satisfying an average dwell-time. By Theorem 1 this problem reduces to finding \( u_{\sigma(i)} = K_{\sigma(i)}x_1 \), such that
\[
\dot{x}_1(t) = \left( \dot{A}_{\sigma(i)} + \dot{B}_{\sigma(i)}K_{\sigma(i)} \right)x_1(t) + A_{2\sigma(i)}x_2(t) + B_{\sigma(i)}u_{\sigma(i)} \]
\[
\dot{x}_2(t) = f_{2\sigma(i)}(x_2(t)) + y(t) = C_{\sigma(i)}x_1(t), \]  
(28)
is globally exponentially stable with an weighted \( L_2 \)-gain.

**Theorem 2** Given any constant \( \gamma > 0 \), suppose that switched system (1) satisfies the following conditions
(i) if there exist constants \( \varepsilon_1 > 0, \lambda_0 > 0, \mu > 1 \), such that the following inequalities
\[
A_{1i}^T P_i + P_i A_{1i} + \varepsilon_1^{-2} P_i \dot{E}_i^T E_i P_i + \gamma^2 P_i G_i^T G_i P_i + \varepsilon_1^{-2} F_i^T F_i + C_i^T C_i + \lambda_0 P_i + I - (\varepsilon_1^{-1} P_i B_i + \varepsilon_1^{-1} F_i F_i)^{-1}(\varepsilon_1^{-1} P_i B_i + \varepsilon_1^{-1} F_i F_i)^T \leq 0, \]
(29)
\[
P_i \leq \mu P_j, \quad i, j = 1, \ldots, N. \]
(30)

have positive definite solutions \( P_i \),
(ii) there exist proper, positive definite, and radially unbounded functions \( W_i(x_2(t)) \) such that
\[
\frac{dW_i(x_2)}{dx_2} f_{2i}(x_2(t)) \leq -\beta_i \|x_2\|^2, \]
(31)
\[
\|a_{1i}\|_2 \leq \|W_i(x_2)\| \leq a_{2i} \|x_2\|^2. \]
(32)

for some constants \( \beta_i > 0, a_{1i} > 0, a_{2i} > 0, i = 1, \ldots, N. \)

Then, switched system (1) is globally exponentially stable with an weighted \( L_2 \)-gain which is less than or equal to \( \gamma \) when \( w(t) = 0 \) with the switched state feedback
\[
u_i = -(F_{2i}^T F_{2i})^{-1}(\varepsilon_1^{-2} B_i^T P_i + F_{2i}^T F_{2i})x_1(t). \]
(33)
under arbitrary switching laws satisfying the average dwell time
\[ \tau_a \geq \tau_a^* = \frac{\ln \hat{\mu}}{\lambda}, \] (34)
where \( \hat{\mu} = \max \{\mu, \frac{a_{ij}}{a_{ii}} : i, j \in I_N\} \), \( \lambda \in [0, \hat{\lambda}_0) \), \( \hat{\lambda}_0 = \min \{\lambda_0, \frac{a_{ii}}{a_{ii}} : i \in I_N\} \).

**Proof.** For switched system (1), define the following piecewise Lyapunov function
\[ V(x) = x_t^T P_{\sigma(t)} x_1 + \hat{k} W_{\sigma(t)} (x_2), \] (35)
where \( P_i \) are the solutions of (29) and (30).

Then, based on Lemma 2, from (33), when the \( i \)th switched subsystem is activated, we can get
\[ \dot{V} = x_t^T (A_i^T P_i + A_i P_i) x_1 + 2 x_t^T P_i \Delta A_i x_1 + 2 x_t^T P_i B_i u_i + 2 x_t^T P_i A_2 x_2 + k \frac{dW_1}{dx_2} f_2 (x_2) + 2 x_t^T P_i G_i w \]
\[ \leq x_t^T (A_i^T P_i + A_i P_i) x_1 + 2 x_t^T P_i E_i \Gamma (F_{i1} x_1 + F_{i2} u_i) + 2 x_t^T P_i B_i u_i + 2 x_t^T P_i A_2 x_2 - k \hat{\beta}_{i} ||x_2||^2 + 2 x_t^T P_i G_i w \]
\[ \leq x_t^T \left( A_i^T P_i + A_i P_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i F_{i1}^T F_{i1} \right) x_1 + 2 x_t^T P_i A_2 x_2 - k \hat{\beta}_{i} ||x_2||^2 + \frac{dW_1}{dx_2} f_2 (x_2) + 2 x_t^T P_i G_i w \]
It is easy to know that there exist constants \( m_i > 0, n_i > 0, i \in I_N \), such that
\[ ||x_t^T P_i|| \leq m_i ||x_1||, \quad ||A_i x_1|| \leq n_i ||x_2||. \]

Let \( \bar{V} = -\lambda_0 x_t^T P_i x_1 - \hat{k} b W(x_2) + \hat{k} b W(x_2) - x_t^T x_1 + 2 p ||x_1|| ||x_2|| - \hat{k} \beta_i ||x_2||^2 \]
\[ \leq -\lambda_0 V + k a_{2b} ||x_2||^2 - x_t^T x_1 + 2 p ||x_1|| ||x_2|| - \hat{k} \beta_i ||x_2||^2 \]
\[ \leq -\hat{\lambda}_0 V - \hat{k} (\beta_i - k a_{2b} b - p^2) ||x_2||^2 \]
where \( b = \min \{\frac{a_{2i}}{a_{ii}} : i \in I_N\} \), \( \hat{\lambda}_0 = \min \{\lambda_0, \frac{a_{ii}}{a_{ii}} : i \in I_N\} \).

Let \( \hat{k} \geq \frac{b^2}{\beta_i - k a_{2b} b - p^2} \) we have
\[ \dot{V} + y_t^T y - \gamma^2 w_t^T w \leq -\hat{\lambda}_0 V. \] (36)

The remainder of the proof for exponential stability when \( w(t) = 0 \) and the weighted \( L_2 \)-gain analysis for the closed-loop system (1), i.e., for switched system (28), is the same as that of Theorem 1.

V. EXAMPLE

Consider the switched system (3) with \( I_N = \{1, 2\} \), \( n = d = 2, d = 2 \) and
\[ A_{11} = \begin{bmatrix} -4 & 0 \\ 2 & 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, C \]
\[ B_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, E_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, F_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, F_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, G_1 = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, F_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, F_{22} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, G_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, F_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \gamma(t) = \begin{bmatrix} \sin t & 0 \\ 0 & \cos t \end{bmatrix}. \]

For \( \gamma = 1 \), let \( \varepsilon_1 = \varepsilon_2 = 1 \). Solving (29), gives
\[ P_1 = \begin{bmatrix} 2.6179 & 0.3342 \\ 0.3342 & 2.3329 \end{bmatrix}, P_2 = \begin{bmatrix} 2.9598 & 0.1577 \\ 0.1577 & 2.0921 \end{bmatrix} \]
It is easy to verify that \( P_1 \) and \( P_2 \) are positive definite matrices, which indicate that condition (i) in Theorem 2 is satisfied. Choosing
\[ W_1 = 1.5 x_3^2 + 0.8 x_3 x_4 + 1.5 x_4^2, \quad W_2 = x_3^2 + 2 x_4^2 \]
We have \( a_{11} = 1.1, a_{21} = 1.9, a_{12} = 1, a_{22} = 2 \), \( W_1 \leq -1.6(x_3^2 + x_4^2), W_2 \leq -4(x_3^2 + x_4^2). \) This implies that condition (ii) is satisfied. Using Theorem 2, we design the average dwell-time and switched state feedback. Let \( \mu = 1.4175 \), \( \lambda_0 = 0.8, \lambda = 0.7, \) we can get \( \hat{\mu} = 1.9 \), \( \tau_0^* = 0.9 \), \( \lambda_0 = 0.8 \), \( \lambda = 0.7 \). Design the switching law as
\[ \sigma(t) = \begin{cases} 1, & k = 0, 2, 4, \ldots, \\ 2, & k = 1, 3, 5, \ldots, \\ t_k = k. \end{cases} \] (37)
and the switched state feedback is given as:
\[
\begin{align*}
    u_i &= \begin{cases} 
        -1.6432x_1 - 3.5000x_2, & i = 1, \\
        -21.8392x_1 - 0.6308x_2, & i = 2.
    \end{cases}
\end{align*}
\]
A simple calculation shows that the average dwell time for the linear switched part of the system is \(\tau_{a1} \geq \tau_{a1}^* = \frac{\ln \mu}{\lambda} = 0.5\), and the average dwell time for the nonlinear switched part is \(\tau_{a2} \geq \tau_{a2}^* = 0.7\). Thus, \(\tau_a \geq \tau_a^* \geq \max\{\tau_{a1}, \tau_{a2}\}\) is obvious. Let \(x(0) = (-2, 1.5, 2.5, -2.5)^T\).

\[\text{Fig. 1. The state response of the switched system}\]

\[\text{Fig. 2. The switching signal for the switched system}\]

Fig. 1 and Fig. 2 are the state response and the switching signal of the whole switched system separately, which indicate the feasibility of our results.

VI. CONCLUSIONS

In this paper, we have studied the \(L_2\)-gain analysis and control synthesis problem for a class of uncertain switched nonlinear cascade systems with external disturbances input. Sufficient conditions for both the weighted \(L_2\)-gain analysis and the control synthesis have been expressed in the form of linear matrix inequalities. The disturbed uncertain autonomous switched system and the disturbed uncertain non-autonomous switched system with the designed switched state feedback are globally exponentially stable and achieved a weighted \(L_2\)-gain under arbitrary designed switching laws that satisfy some average dwell-time. Moreover, the average dwell-time and the state decay have been calculated explicitly.

REFERENCES