A New Method to Robust $H_\infty$ Control of Uncertain Switched Systems

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Abstract—This paper develops a new method to robust $H_\infty$ control problem for a class of uncertain switched systems by constructing single robust $H_\infty$ sliding surface. The method consists of two phases. One is to construct a single sliding surface which the reduced-order equivalent sliding motion is forced into, and to have the sliding motion robustly stabilized with $H_\infty$ disturbance attenuation level $\gamma$ under a hysteresis switching law to be designed; the other phase is to design variable structure controllers of the subsystems to thus drive the state of the switched system to reach the single sliding surface in finite time and remain on it thereafter. A numerical example is given to illustrate the effectiveness of the proposed method.

I. INTRODUCTION

Switched systems consist of a family of continuous-time or discrete-time systems and certain rules of logic specifying at each instant of time which subsystem is activated along the system trajectory, thus represent a rather important class of hybrid systems. As a result, switched systems and switching control have recently gained a great deal of attention [1-8] mainly because many real-world systems such as chemical processes and transportation systems can be modeled as switched systems under certain reasonable assumptions. In the literature, switched linear systems without uncertainties have been extensively investigated; for instance, see [3-5] and references therein. Since uncertainties are ubiquitous in system models due to the complexity of the system itself, exogenous disturbance and so on, from a practical point of view, the study of uncertain switched systems is relatively more important.

Among the existing results of switched systems with uncertainties, [6] considered quadratic stabilization of switched systems with norm-bounded time varying uncertainties. In [7], $L_2$ induced norm of switched systems with external disturbances was considered under the condition of large dwell time. Robust $H_\infty$ control and stabilization of uncertain switched linear systems were addressed in [8] based on multiple Lyapunov functions approach.

On the other hand, the sliding mode control (SMC) is one of most important methods in robust control domain, since it possesses various attractive features such as robustness, fast response, and good transient response [9, 10]. Over the years, there are many available papers [11-16] on SMC. Among the results concerning SMC, they are mainly partitioned into two ways, one is to develop SMC theory [11-13]; the other is to fuse SMC technique into other methods or other systems rather than the traditional ones, where there are already some exciting and significant results [14-16].

Along the latter, we will apply SMC to switched system. There are very few results focusing this interest except [17-20]. The authors of [17] proposed a SMC method to make a class of switched systems exponentially stable. [18] addressed the sliding mode control for planar switched systems under an arbitrary switching sequence. In [19], the sliding motion of switched systems without control input was analyzed and an approach was proposed to estimate the domain in which the sliding motion may occur. A variable structure controller with sliding mode sector for a hybrid system was presented which switches the hybrid system among subsystems to ensure its quadratic stability in [20]. As for tackling $H_\infty$ control problem with resort to SMC technique, to the best of our knowledge, there are almost no results in the current literature, which is indeed our motivation.

In this paper, we investigate and solve the robust $H_\infty$ sliding mode variable structure control problem for a class of uncertain switched linear systems. The outline of this paper is as follows. Section II presents the problem formulation and the necessary preliminaries. In Section III, the novel design is theoretically developed. In Section IV, the developed control design is applied to an illustrative example and numerical and simulation results are given to illustrate the effectiveness of the proposed design. Conclusion and references follow thereafter.

Throughout this paper, $\|\|$ denotes the Euclidean norm for a vector or the matrix induced norm for a matrix.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following uncertain switched system
\[ \dot{x}(t) = (A_x + \Delta A_x)x(t) + Bu_x + f_x(x(t)) + B_x\omega(t), \]
\[ z(t) = Cx(t), \]
where \( x(t) \in \mathbb{R}^n \) is the system state, \( \sigma(t): [0, \infty) \to \Xi = [1, 2, \ldots, l] \) is the piecewise constant switching signal that may depend on either time \( t \) or state, \( u_i \in \mathbb{R}^m \) is the control input of the \( i \)-th subsystem, \( z(t) \) is the controlled output, \( \omega(t) \in L_2([0, \infty)) \) is the external disturbance input, \( B_x \), \( B_x \), \( C \) and \( A_x \) are constant matrices with appropriate dimensions, \( \Delta A_x \) denote system parameter uncertainties, \( f_x(x(t)) \) represent nonlinearities of the system. The following assumptions are introduced.

**Assumption 1.** The parameter uncertainties can be represented and emulated as
\[ \Delta A_x = E\Sigma_i(t)F_i, i \in \Xi, \]
where \( E \) and \( F_i \) are known constant matrices with appropriate dimensions, \( \Sigma_i(t) \) are unknown matrices with Lebesgue measurable elements and satisfy \( \Sigma_i^T(t)\Sigma_i(t) \leq I \).

**Assumption 2.** There exist known nonnegative scalar-valued functions \( \phi_i(x,t), i \in \Xi \) such that \( \|f(x,t)\| \leq \phi_i(x,t) \) for all \( t \).

**Assumption 3.** There exists a known nonnegative constant \( \sigma \) such that \( \|\omega(t)\| \leq \sigma \) for all \( t \).

**Assumption 4.** The input matrix \( B_x \) has full rank \( m \) and \( m < n \).

**Remark 1.** Assumptions 1–4 are standard assumptions in the study of variable structure control.

We now recall the concept of asymptotic stability with \( H_\infty \) disturbance attenuation level \( \gamma \).

**Definition 1 ([21]).** Consider the following uncertain switched linear system
\[ \dot{x} = A_x x + B_x\omega, \]
\[ z = Cx. \]
For a given positive constant \( \gamma > 0 \), if there exists a switching law \( \sigma = \sigma(x) \) and a positive definite matrix \( P \), such that the inequality
\[ x^T (A^T_x P + PA_x + \gamma^2 PBB^T P + C^T C)x < 0 \]
holds, then system (2) is called asymptotically stable and satisfies \( H_\infty \) disturbance attenuation level \( \gamma \).

**Lemma 1 ([22]).** Given real matrices \( R_1 \) and \( R_2 \) with appropriate dimensions and an unknown matrix \( \Sigma(t) \) with Lebesgue measurable elements such that \( \Sigma^T(t)\Sigma(t) \leq I \), then we have
\[ R_1\Sigma R_2 + R_1^T\Sigma^T R_2^T \leq \beta R_1 R_1^T + \beta^{-1} R_2 R_2^T, \]
where \( \beta > 0 \).

Now, we introduce a convex combination of the system (1) without the matched uncertainties \( f_x(x,t) \) as follows
\[ x(t) = (\bar{A} + \Delta \bar{A})x(t) + Bu + B_x\omega(t), \]
\[ z(t) = Cx(t), \]
where \( \bar{A} = \sum_{i=1}^l \alpha_i A_i, \Delta \bar{A} = \sum_{i=1}^l \alpha_i \Delta A_i, \alpha_i \geq 0, \sum_{i=1}^l \alpha_i = 1. \)

**Lemma 2.** Given a constant \( \gamma > 0 \), if there exist matrix \( P > 0 \), state feedback gain \( K \), constant \( \lambda > 0 \) and scalars \( \alpha_i \) such that
\[ (\bar{A} - BK)^T P + P(\bar{A} - BK) + \gamma^2 PBB^T P + \gamma^{-2} PB_x B_x^T P \]
\[ + \frac{1}{\lambda^2} F^T F + C^T C < 0, \]
then the system (4) is robustly stabilized with \( H_\infty \) disturbance attenuation level \( \gamma \).

**Proof.** Let
\[ Q = (\bar{A} + \Delta \bar{A} - BK)^T P + P(\bar{A} + \Delta \bar{A} - BK) \]
\[ + \gamma^2 PB_x B_x^T P + C^T C \]
\[ = (\bar{A} - BK)^T P + P(\bar{A} - BK) + \gamma^2 PB_x B_x^T P \]
\[ + C^T C + \Delta \bar{A}^T P + \Delta \bar{A}P. \]

Using Lemma 1, one obtains
\[ \Delta \bar{A}^T P + \Delta \bar{A}P = (\sum_{i=1}^l \alpha_i A_i)^T P + P(\sum_{i=1}^l \alpha_i \Delta A_i) \]
\[ = [E(\sum_{i=1}^l \alpha_i \Sigma_i(t) F_i) F_i^T P + P(E(\sum_{i=1}^l \alpha_i \Sigma_i(t) F_i) F_i) \]
\[ \leq \dot{x}^2 PE^T P + \dot{x}^2 F^T F. \]

Hence, we have
\[ Q \leq (\bar{A} - BK)^T P + P(\bar{A} - BK) + P(\dot{x}^2 EE^T + \gamma^2 BB^T) P \]
\[ + \frac{1}{\lambda^2} F^T F + C^T C < 0, \]
which implies that the system (4) is robustly stabilized with \( H_\infty \) disturbance attenuation level \( \gamma \). This completes the proof.

**Remark 2.** The inequality (5) can be converted into a linear matrix inequality (LMI) by using Schur complement and the change of variable such that \( \bar{K} = KP^{-1} \). Hence, the feasible solutions can be globally found by the LMIs method [16].

To get a regular form of the system (1), we define a nonsingular matrix \( T \) and an associated vector \( \xi \) as follows
\[ T = \begin{bmatrix} \bar{B}^T \\ B^T \end{bmatrix}, \]
and
\[ \dot{\xi}(t) = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = T x(t) = \begin{bmatrix} \bar{B}^T \\ B^T \end{bmatrix} x(t), \]
with \( \xi_1 \in \mathbb{R}^{n-m}, \xi_2 \in \mathbb{R}^m \), where \( \bar{B} \) is an orthogonal complement of the matrix \( B_x \). We can easily show
By means of the state transformation $\dot{z}(t) = T x(t)$, the system (1) is transformed into the following regular form

$$\dot{\bar{z}} = (\bar{A}_o + \Delta \bar{A}_e) \bar{z} + \bar{B}(u_o + f_o(x,t)) + \dot{B}_1\omega(t),$$

$$z(t) = C \bar{z}(t),$$

where $\bar{A}_o = T A_o T^{-1}$, $\Delta \bar{A}_e = T \Delta A_e T^{-1}$, $\bar{B} = T B$, $\dot{B}_1 = T B_1$, $\bar{C} = C T^{-1}$. The system (9) is equivalent to the following form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{o11} & \bar{A}_{o12} \\ \bar{A}_{o21} & \bar{A}_{o22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 & \bar{B}_1 \end{bmatrix} \begin{bmatrix} u_o + f_o(x,t) \\ \omega(t) \end{bmatrix} + \bar{B}_1 \begin{bmatrix} B_1 \end{bmatrix} \omega(t),$$

$$z(t) = C \begin{bmatrix} \bar{B} (B^T B)^{-1} + (B^T B)^{-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

where

$$\bar{A}_{o11} = \bar{B}^T A_o \bar{B} (B^T B)^{-1} + B^T E \Sigma_e(t) F \bar{B} (B^T B)^{-1},$$

$$\bar{A}_{o12} = \bar{B}^T A_o B (B^T B)^{-1} + B^T E \Sigma_e(t) F B (B^T B)^{-1},$$

$$\bar{A}_{o21} = B^T \bar{A}_o \bar{B} (B^T B)^{-1} + B^T E \Sigma_e(t) F B^T \bar{B}(B^T B)^{-1},$$

$$\bar{A}_{o22} = B^T \bar{A}_o B (B^T B)^{-1} + B^T E \Sigma_e(t) F B^2 (B^T B)^{-1}.$$
By multiplying (17) with $[T_{1m}, -T_{12} P_{2}^{-1}]$ and $[I_{m}, -T_{12} P_{2}^{-1}]^T$ from left and right, respectively, we have

$$
(\bar{A}_i - \bar{A}_{12} P_{2}^{-1} \bar{P}_{1}) \bar{P}_{r} + \bar{P}_{r}(\bar{A}_{11} - \bar{A}_{12} P_{2}^{-1} \bar{P}_{1})
+ \bar{P}_{r} \bar{B}_{i} (\lambda^2 EE^T + \gamma^{-1} B_{i} B_{i}^T) \bar{P}_{r} + [\bar{B}(\bar{B}^T B_{i}^{-1}) - B(\bar{B}^T B_{i}^{-1})] M^T \left( \frac{1}{\lambda^2} F^T F + C^T C \right)
$$

$$
= [\bar{B}(\bar{B}^T B_{i}^{-1}) - B(\bar{B}^T B_{i}^{-1}) \bar{P}_{2}^{-1} \bar{P}_{1}] < 0,
$$

where $\bar{P}_r = \bar{P}_{11} - \bar{P}_{12} P_{2}^{-1} \bar{P}_{12}$, $\bar{P}_r > 0$ since $\bar{P}_r > 0$. Therefore, by setting $M = [(\bar{B}^T B_{i}^{-1}) B_{i} P(B B_{i}^{-1})]^{-1} (\bar{B}^T B_{i}^{-1} B_{i} P \bar{B}) \bar{B}^T B_{i}^{-1} = \bar{P}_{2}^{-1} \bar{P}_{12}$, (18) becomes

$$
(\bar{A}_{11} - \bar{A}_{12} M)^T \bar{P}_r + \bar{P}_r(\bar{A}_{11} - \bar{A}_{12} M) + \bar{P}_r \bar{B}_{i} (\lambda^2 EE^T + \gamma^{-1} B_{i} B_{i}^T) \bar{P}_r + [\bar{B}(\bar{B}^T B_{i}^{-1}) - B(\bar{B}^T B_{i}^{-1})] M^T \left( \frac{1}{\lambda^2} F^T F + C^T C \right) (\bar{B}^T B_{i}^{-1}) - B(\bar{B}^T B_{i}^{-1}) M < 0.
$$

Further, denoting

$$
Q_i = (\bar{B}^T B_{i}^{-1} \bar{B}(\bar{B}^T B_{i}^{-1}) - \bar{B}_i) A(\bar{B}(B^T B_{i}^{-1}) M)^T \bar{P}_r
+ \bar{P}_r(\bar{B}^T B_{i}^{-1} \bar{B}_i A(\bar{B}(B^T B_{i}^{-1}) M) + \bar{P}_r \bar{B}_{i} (\lambda^2 EE^T + \gamma^{-1} B_{i} B_{i}^T) \bar{P}_r + [\bar{B}(\bar{B}^T B_{i}^{-1}) - B(\bar{B}^T B_{i}^{-1})] M^T \left( \frac{1}{\lambda^2} F^T F + C^T C \right) (\bar{B}^T B_{i}^{-1}) - B(\bar{B}^T B_{i}^{-1}) M,
$$

and substituting $\bar{A}_{11} = \bar{B}^T A \bar{B}(\bar{B}^T B_{i}^{-1} - \bar{B}_i) A(\bar{B}(B^T B_{i}^{-1}) M)^T \bar{P}_r$ and $\bar{A}_{12} = \bar{B}^T \bar{B}(\bar{B}^T B_{i}^{-1} - \bar{B}_i) A(\bar{B}(B^T B_{i}^{-1}) M)$

$$
\tilde{A} = \sum_{i=1}^{I} \alpha_i \tilde{A}_i,
$$

and $\tilde{A} = \sum_{i=1}^{I} \alpha_i \tilde{A}_i$, into inequality (19) gives

$$
\alpha_i Q_i + \alpha_1 Q_1 + \ldots + \alpha_{I} Q_{I} < 0.
$$

We define the regions

$$
\Omega_i = \{ \xi | \xi \in \Omega, \xi \in \Omega \},
$$

Obviously,

$$
\bigcup_{\xi \in \Omega} = R^{(m+1)} \setminus \{0\}.
$$

The hysteresis switching law for the sliding motion (12) is designed as follows

$$
\sigma(0) = \min \{ \Omega | \xi(0) \in \Omega \},
$$

for $t > 0$,

$$
\sigma(t) = \begin{cases}
1, & \text{if } \xi(t) \in \Omega, \text{ and } \sigma(t) = 1, \\
\min \{ \Omega | \xi(t) \in \Omega \}, & \text{if } \xi(t) \notin \Omega, \text{ and } \sigma(t) = i.
\end{cases}
$$

By virtue of Definition 1, we conclude that sliding motion (12) is robust stabilized with $H_{\infty}$ disturbance attenuation level $\gamma$ under the switching law (22). The proof is thus completed.

In the end, we give the following result.

**Theorem 2.** Assume that the conditions of Theorem 1 are satisfied and the sliding surface of system (1) is given by (13). Then under the control laws

$$
u_i = -(SB)^{-1} S A_i x - (SB)^{-1} \left[ SE \| F \| x + | SB \| \phi_i (x, t) + \sigma \| SB \| + \mu \text{sign}(\zeta), i \in \Xi, \right]
$$

the state of the system (1) can enter finite time and subsequently remains on the sliding surface, where $\mu$ is a positive scalar to adjust the convergent rate.

**Proof.** The derivative of the sliding function $\zeta(t) = Sx(t)$ along the trajectory of the system (1) is

$$
\dot{\zeta}(t) = S \dot{x}(t) = (A + \Delta A) x(t) + SBu + SBf + SB\phi(t).
$$

With regard to Assumptions 1~3, substituting the control laws (23) into (24) yields $\dot{\zeta}(t) \dot{\zeta}(t) \leq -\mu \| \zeta(t) \|^2$, which implies that the state of the system (1) reaches the sliding surface (13) in finite time and thereafter remains on it. This completes the proof.

**Remark 5.** The single sliding surface is reached in finite time according to the reaching rate of sliding surface $\mu$. The $\zeta(t) \dot{\zeta}(t) \leq -\mu \| \zeta(t) \|$ implies the decay rate of the sliding surface is no less than $e^{-\mu}$. 

**IV. AN ILLUSTRATIVE EXAMPLE**

In this section, we present a numerical example to demonstrate the effectiveness of the proposed design method. Consider the following uncertain switched linear system

$$
\dot{x}(t) = (A_s + \Delta A_s) x(t) + B(u_0 + f_0) + B_0 \phi(t),
$$

$$
z(t) = Cx(t),
$$

where $\sigma(t) \in \Xi = \{1, 2\}$,

$$
A_1 = \begin{bmatrix} -3 & -0.5 & 1 \\ 1 & -0.5 & 1 \\ 0 & 1 & -2 \end{bmatrix},
A_2 = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -2 \end{bmatrix},
B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
$$

$$
B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
C = 0, \text{ the parameter uncertainties } \Delta A_i = E,
$$

$$
\times \Sigma_1(t) F \text{ where } E = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
F = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\Sigma_1 = \eta_i \in [-1, 1], \text{ and }
$$

$$
f_1 = f_2 = 0.
$$

We choose the convex combination coefficients $\alpha_1 = \alpha_2 = 0.5$ and the constant $\lambda = \frac{1}{\sqrt{2}}$. The disturbance attenuation level is given by $\gamma = \frac{1}{\sqrt{2}}$.

Taking the matrix $K = B^T P$, by solving the inequality (5), one can obtain the following solution.
Then we obtain the matrix
\[ P = \begin{bmatrix} 9.2006 & 7.8894 & 3.7381 \\ 7.8894 & 8.1363 & 3.6193 \\ 3.7381 & 3.6193 & 3.2078 \end{bmatrix}. \]

The single robust \( H_\infty \) sliding function is given as follows
\[ \zeta = Sx = [-0.1591, -0.3458, 1.0771]^T x. \] (26)

According to (23) the subsystem control laws are given by
\[
\begin{align*}
    u_1 &= -0.1054x_1 - 1.0637x_2 + 2.1273x_3 \\
    &- 0.8(0.1866\|x_2 + x_1\| + 1)\text{sign}(\zeta), \\
    u_2 &= -0.4358x_1 + 0.1493x_2 + 1.5741x_3 \\
    &- 0.8(0.1866\|x_2 + x_1\| + 1)\text{sign}(\zeta).
\end{align*}
\] (27)

The simulation results for system state responses of the two subsystems alone with the initial state vector
\[ x_0 = [1, 2, -1]^T \] are shown in Fig. 1 and Fig. 2, respectively. We can easily see that both subsystems are unstable.

It is easy to verify that the conditions of Theorem 1 and 2 are satisfied.

The hysteresis switching law is
\[
\sigma(t) = \begin{cases} 
1, & \text{if } (x(t) \notin \Omega_i) \lor (x(t) \in \Omega_i \land \sigma(t^-) = 1) \\
2, & \text{if } (x(t) \notin \Omega_i) \lor (x(t) \in \Omega_i \land \sigma(t^-) = 2) \\
0, & \text{otherwise},
\end{cases}
\] (28)

where
\[
\Omega_i = \{x(t)|x^T(t) \begin{bmatrix} -32.5006, -13.8465, -6.9232 \\
-13.8465, -0.5939, -0.2969 \\
-6.9232, -0.2969, -0.1485 \end{bmatrix} x(t) < 0\},
\]
\[
\Omega_2 = \{x(t)|x^T(t) \begin{bmatrix} 23.2917, 9.9988, 4.9994 \\
9.9988, -2.3998, -1.1999 \\
4.9994, -1.1999, -0.5999 \end{bmatrix} x(t) < 0\}.
\]

The simulation results are depicted in Fig. 3-Fig. 6.
The simulation results for the system states in the closed-loop and with the same initial state vector $x_0 = [1, 2, -1]^T$ are shown in Fig. 3. It is clearly seen that the closed-loop system of the switched system (25) with the designed controller (27) and the switching law (28) is asymptotically stable. Fig. 4 is the input signal of the switched system (25). The trajectory of the sliding function (26) is shown in Fig. 5. The switching signal is given by Fig. 6.

V. CONCLUSION

This paper has developed the new method to robust $H_{\infty}$ control problem for a class of uncertain switched systems by constructing single robust $H_{\infty}$ sliding surface. The sufficient condition for the existence of the single robust $H_{\infty}$ sliding surface has been derived in terms of Riccati inequality associated with the convex combination of the switched system. The switching law has been constructed such that the $n-m$ dimensional sliding motion is robustly stabilized with $H_{\infty}$ disturbance attenuation level $\gamma$. Variable structure controllers have been designed to drive the state of the switched system to reach the single robust $H_{\infty}$ sliding surface in finite time.

REFERENCES