Robust $H_\infty$ Control of Uncertain Switched Delay Systems Using Multiple Lyapunov Functions

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Abstract—This paper investigates the robust $H_\infty$ control problem for a class of uncertain switched delay systems that involve parameter uncertainties and unknown nonlinear disturbances. Based on the multiple Lyapunov functions method, a sufficient condition for the solvability of the robust $H_\infty$ control problem is derived by employing a hysteresis switching law and variable structure controllers. When the upper bounds of the nonlinear disturbances are unknown, an adaptive variable structure control strategy is developed. The use of adaptive technique is to adapt the unknown upper bounds of the nonlinear disturbances so that the objective of asymptotic stabilization with $H_\infty$-norm bound is achieved under the hysteresis switching law. A numerical example illustrates the effectiveness of the proposed design method.

I. INTRODUCTION

Switched systems constitute an important class of hybrid systems. A typical switched system consist of a family of subsystems and a switching law specifying which subsystem will be activated at each instant of time. The motivation for studying switched systems is from the fact that many practical systems are inherently multi-models in the sense that several dynamical subsystems are required to describe their behavior depending on various changing environmental factors. In the study of stability analysis for switched systems, the multiple Lyapunov functions method has been considered as an important analysis tool in [1-3]. Robust $H_\infty$ control and stabilization of uncertain switched linear systems were considered by utilizing a state-depend switching strategy with the multiple Lyapunov functions method in [4], [5] addressed the problem of stability and $L_2$-gain analysis for nonlinear switched systems via the multiple Lyapunov functions method. In this paper, the stability result has been generalized by defining more general weak multiple Lyapunov functions.

[6] designed a hysteresis switching law to avoid sliding motions that often occur in state-depended switching strategy. The value of the hysteresis switching signal is not determined by the current value of state alone, but depends also on the previous value of switching signal.

Time-delay commonly exists in various industrial systems, and its existence is frequently a source of instability. Switched systems with time-delay are one of the most useful models and have strong engineering background such as power systems [7] and networked control systems [8]. However, very few results on switched delay systems have been reported. Sufficient conditions of asymptotical stability were established for switched linear delay systems under arbitrary and constructed switching signals, respectively in [9], [10] investigated the problem of delay-dependent common Lyapunov functions for switched linear delay systems, which established the relationship between the delay-dependent common Lyapunov functions and the common Lyapunov functions for corresponding switched systems without delays. The stabilization problem of arbitrary switched linear systems with unknown time varying delays was considered in [11]. For uncertain linear discrete-time switched systems with state delays, sufficient conditions of robust stability and stabilizability in terms of matrix inequalities and Riccati-like inequalities were given in [12]. Stability of a class of switched delay systems was shown in [13] by using a common Lyapunov functional method. [14] studied stability and $L_2$-gain of switched delay systems based on the average time technique. However in above results, the value of switching signal only depends on state or time. There are no results for design of hysteresis switching law of uncertain switched delay systems in the current literature, which is indeed our motivation.

Another critical issue for switched systems is to enhance the robustness against system uncertainties and perturbations. Variable structure control with sliding mode or without sliding mode is an effective robust scheme for systems with uncertainties, which employs discontinuous control law to overcome the uncertainties and improve performance and stability.

In this paper, we consider the problem of robust $H_\infty$ control for a class of uncertain switched delay systems with parameter uncertainties and unknown nonlinear disturbances. Based on the multiple Lyapunov functions method, a sufficient condition for robust stability with $H_\infty$ disturbance attenuation level $\gamma$ is derived, and a hysteresis switching law is designed. For the case of known upper bounds of the nonlinear disturbances, variable structure controllers are developed such that the uncertain switched delay system is asymptotically stabilizable with $H_\infty$ disturbance attenuation.
level γ under the hysteresis switching law. For the case of unknown upper bounds of the nonlinear disturbances, adaptive variable structure controllers are developed so that the objective of asymptotic stabilization with disturbance attenuation level γ is achieved under the hysteresis switching law. A numerical example illustrates the effectiveness of the proposed design methods.

In this paper, \(|\cdot|\) denotes the Euclidean norm for a vector or the matrix induced norm for a matrix; \(Z^+\) denotes the set of all nonnegative integers.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the uncertain switched delay system of the form

\[
x(t) = (A_i + \Delta A_i)x(t) + (A_{i\omega} + \Delta A_{i\omega})x(t-\tau) + B_iu_i(t) + f_i(x(t), t) + G_i\alpha(t),
\]

where \(x(t) \in \mathbb{R}^n\) is the state, \(u_i(t) \in \mathbb{R}^m\) is the control input of the \(i\)-th subsystem, \(r_i(t) \in \mathbb{R}^r\) denotes the disturbance input which belongs to \(L_2[0, \infty)\), \(z(t) \in \mathbb{R}^p\) is the controlled output, \(A_i, A_{i\omega}, B_i, G_i, C_i\) are constant matrices with appropriate dimensions, \(\tau\) denotes unknown constant time-delay which is bounded by the known constant \(\tau\), \(\alpha(t)\) is a differentiable vector-valued initial function on \([-\tau, 0]\), \(\Delta A_i, \Delta A_{i\omega}\) represent the system uncertainties, \(f_i(x, t)\) is an unknown nonlinear function. The following assumptions are introduced.

Assumption 1. The uncertainties can be represented and emulated as

\[
\Delta A_i = D_i \Sigma_i(t)E_i, \quad \Delta A_{i\omega} = B_i \Delta M_{i\omega}(t),
\]

where \(D_i\) and \(E_i\) are constant matrices with appropriate dimensions, \(\Sigma_i(t)\) are unknown matrices with Lebesgue measurable elements and satisfy \(\Sigma_i(t)\Sigma_i(t) \leq I\) , \(\Delta M_{i\omega}(t)\) are unknown but bounded as \(\|\Delta M_{i\omega}(t)\| \leq \alpha_i\), with known nonnegative constants \(\alpha_i\).

Assumption 2. There exist known nonnegative scalar-valued functions \(\phi_i(x, t), i \in \Xi\) such that \(|f_i(x, t)| \leq \phi_i(x, t)\).

A switching sequence is expressed by

\[
\Psi = \{x_0, (i_0, t_0), (i_1, t_1), \ldots, (i_j, t_j), \ldots, i_j \in \Xi, j \in \mathbb{Z}^+\}.
\]

In which \(t_0\) is the initial time, \(x_0\) is the initial state, \((i_k, t_k)\) means that the \(i_k-th\) subsystem is activated for \([t_k, t_{k+1})\).

**Lemma 1 ([15])**. Given real matrices \(R_1\) and \(R_2\) with appropriate dimensions and an unknown matrix \(\Sigma(t)\) with Lebesgue measurable elements such that \(\Sigma(t)\Sigma(t) \leq I\), then we have

\[
R_1\Sigma R_1^T + R_2\Sigma R_2^T \leq \beta R_1 R_1^T + \beta^{-1} R_2^T R_2,
\]

where \(\beta > 0\).

III. MAIN RESULTS

The objective in this paper is to design a switching law \(\sigma(t)\) and an associated controller \(u_\sigma\) such that the system (1) is stabilizable with an \(H_\infty\)-norm bound. To formulate the problem clearly, we give the following definition.

**Definition 1.** Given a constant \(\gamma > 0\), the uncertain switched delay system (1) is said to be robust stabilizable with \(H_\infty\) disturbance attenuation level \(\gamma\) if there exists a switching law \(\sigma(t)\) and an associated controller \(u_\sigma\) such that

i). the resulting closed-loop system of the system (1) with \(\omega(t) = 0\) is stable for all admissible uncertainties;

ii). with zero-initial condition \(\omega(\theta) = 0\), \(\theta \in [-\tau, 0]\), \(\|\omega(t)\|_2^\infty < \gamma\|\omega(t)\|_2^\infty\) holds for all admissible uncertainties and all nonzero \(\omega \in L_2[0, \infty)\).

For simplicity of presentation only the case for \(k = 2\) is considered in here. It is easy and straightforward to extend the result to the case for \(k > 2\).

First, we design a switching law \(\sigma(t)\) and an associated controller \(u_\sigma\) such that the system (1) with \(\omega(t) = 0\) is stabilizable via the multiple Lyapunov functions method.

**Lemma 2.** Suppose Assumptions 1 and 2 hold. The system (1) with \(\omega(t) = 0\) is stabilizable if there exist matrices \(P_i > 0\), \(Q_i < 0\), \(Q > 0\), scalars \(\alpha_i < 0\), \(\eta_i > 0\) \((i = 1, 2)\), \(\epsilon > 0\) such that the following matrix inequalities

\[
\begin{bmatrix}
\Theta_1 + Q + \alpha_1(P_1 - P_2 + \eta_1 Q_1) & P_1 A_{i1} \\
A_{i1}^T P_1 + \epsilon & -Q
\end{bmatrix} < 0
\]

and

\[
\begin{bmatrix}
\Theta_2 + Q + \alpha_2(P_2 - P_1 + \eta_2 Q_2) & P_2 A_{i2} \\
A_{i2}^T P_2 + \epsilon P_1 D_i^T P_i + \epsilon^{-1} D_i P_i D_i^T P_i & -Q
\end{bmatrix} < 0
\]

are satisfied with \(\Theta_i = A_i^T P_i + P_i A_i - P_i B_i R_i^T P_i + \epsilon^{-1} P_i D_i^T P_i + \epsilon E_i^T E_i\) \((i = 1, 2)\). The stabilizing variable structure controllers for subsystems are given by

\[
u_i(t) = -\frac{1}{2} B_i^T P_i x(t) - \hat{u}_i(t),
\]

where \(\hat{u}_i(t) = (\lambda_i \|x(t)\| + \phi_i(x, t) + \mu_i \text{sign}(s_i(t)))\), \(s_i(t) = B_i^T P_i x(t)\), \(\mu_i\) is positive constant.

**Proof.** We define regions

\[
\Phi_1 = \{x(t) | x^T(t) (P_1 - P_2 + \eta_1 Q_2) x(t) \leq 0\},
\]

\[
\Phi_2 = \{x(t) | x^T(t) (P_2 - P_1 + \eta_2 Q_2) x(t) \leq 0\}.
\]

Obviously \(\Phi_1 \cup \Phi_2 = R^n\).

Then, we design a hysteresis switching law by

\[
\sigma = \begin{cases}
1, & \text{if } (x(0) \in \Phi_1) \text{ or } (x(t) \in \Phi_1 \text{ and } \sigma(t^-) = 1) \\
& \text{or } (x(t) \notin \Phi_1 \text{ and } \sigma(t^-) = 2),
2, & \text{if } (x(0) \notin \Phi_1) \text{ or } (x(t) \in \Phi_2 \text{ and } \sigma(t^-) = 2) \\
& \text{or } (x(t) \notin \Phi_1 \text{ and } \sigma(t^-) = 1).
\end{cases}
\]
Choose the Lyapunov functional candidate
\[ V(t) = x^T(t)P_i x(t) + \int_{t_i}^{t} x^T(\theta)Q x(\theta) d\theta, \] (6)
where \( P_i (i=1,2) \), \( Q \) satisfy (2-a) and (2-b). Differentiating (6) with respect to \( t \), we obtain
\[ \dot{V}(t) = 2x^T(t)P_i \dot{x}(t) + x^T(t)Q x(t) - x^T(t-\tau)Q x(t-\tau). \] (7)
Using (3) in the systems (1) with \( \omega(t) = 0 \), we have
\[ \dot{x}(t) = (A_i - \frac{1}{2}B_i B_i^T P_i + \Delta A_i)x(t) + (A_d + \Delta A_d)x(t-\tau) - B_i \hat{u}_i(t) + B_i f_i(x,t). \] (8)
Let \( \zeta(t) = [x^T(t) \ x^T(t-\tau)]^T \). Substituting (8) into (7) and rearranging, we have
\[ \dot{V}(t) = \zeta^T(t) \Pi_i \zeta(t) + 2x^T(t)P_iB_i \Delta A_d x(t-\tau) - 2x^T(t)P_iB_i \hat{u}_i(t) + 2x^T(t)P_iB_i f_i(x,t), \] (9)
where
\[ \Pi_i = \begin{bmatrix} A_i^T P_i + P_i A_i - P_i B_i B_i^T P_i + Q + \Delta A_i^T P_i + P_i \Delta A_i + P_i A_d & P_i A_d \\ A_i^T P_i & -Q \end{bmatrix}. \]
It follows from Lemma 1 that
\[ \Delta A_i^T P_i + P_i \Delta A_i \leq \varepsilon^{-1} D_i D_i^T P_i + \varepsilon E_i^T E_i. \]
Applying the Razumikin theorem [16] we know that for any solution \( x(t-\tau) \) of (1) there exists a constant \( \lambda > 0 \) such that
\[ \|x(t-\tau)\| < \lambda \|x(t)\| \leq t < T. \] (10)
Thus, it holds that
\[ \dot{V}_i(t) < 0, [t_j, t_{j+1}). \]
Let \( \Phi_1 = \{x(t)| x^T(t)(P_i - P_2 + \eta_i Q_i) x(t) = 0\}, \)
\[ \Phi_2 = \{x(t)| x^T(t)(P_i - P_2 + \eta_i Q_i) x(t) = 0\}. \] (11)
Obviously, \( \Phi_1 \) is the boundary of \( \Phi_i (i=1,2) \).

According to the switching law (5), if \( \sigma(t) = 1 \) and \( x(t) \in \Phi_1 \), then the trajectory will remains in \( \Phi_1 \) until it hits the boundary \( \Phi_2 \). This means that switching only takes place on \( \Phi_1 \) or \( \Phi_2 \). In fact, \( x(t_j) \in \Phi_1 \) means that \( x^T(t_j)(P_i - P_2 + \eta_i Q_i) x(t_j) = 0 \) and \( x(t) \in \Phi_2 \). Thus,
\[ x^T(t_j)(P_2 - \eta_i Q_i) x(t_j) = x^T(t_j)(P_2 - \eta_i Q_i) x(t_j) > x^T(t_j)(P_2) x(t_j). \]
Then according to the switching law (5), at each switching time \( t_j \),
\[ V_{i,j}(t_j) < V_{i,j}(t_j) \]
is true. In view of the multiple Lyapunov functions method, the system (1) with \( \omega(t) = 0 \) is asymptotically stabilizable. This completes the proof.

**Remark 1.** The conventional state-depended switching rules appeared in many references [4, 17], which may result in sliding motions in switching surface. From the proof of Lemma 2, it is obviously that regions \( \Phi_1 \) and \( \Phi_2 \) overlap, thus we refer [6] to introduce hysteresis to avoid sliding motions by design a hysteresis switching law.

Next, based on the previous arguments, we consider stabilization with \( H_\alpha \) disturbance attenuation level \( \gamma \) of the uncertain switched delay system (1).

**Theorem 1.** Suppose Assumptions 1 and 2 hold. Given a constant \( \gamma > 0 \), the uncertain switched delay system (1) is stabilizable with \( H_\alpha \) disturbance attenuation level \( \gamma \), if there exist matrices \( P_i > 0, Q < 0, Q > 0 \) and scalars \( \alpha_i < 0, \eta_i > 0 (i=1,2) \), \( \varepsilon > 0 \) such that the following matrix inequalities
\[ W_i + Q + \alpha_i (P_i - P_2 + \eta_i Q_i) P_i A_{i1} A_i^T P_i < 0 \] (12-a)
and
\[ W_2 + Q + \alpha_2 (P_2 - P_2 + \eta_2 Q_2) P_2 A_{i2} < 0 \] (12-b)
are satisfied with \( W_i = A_i^T P_i + P_i A_i - P_i B_i B_i^T P_i + \Delta A_i^T P_i + P_i \Delta A_i + P_i A_d \) and scalars
\[ \alpha_i < 0, \eta_i > 0 (i=1,2), \varepsilon > 0 \] such that the following matrix inequalities
\[ \Delta A_i^T P_i + P_i \Delta A_i \leq \varepsilon^{-1} D_i D_i^T P_i + \varepsilon E_i^T E_i. \]

**Proof.** First, take the multiple Lyapunov functions \( V_i(t) \) (i=1,2) as (6), where \( P_i, Q \) satisfy (12-a) and (12-b). By Lemma 2, the system (1) is asymptotically stabilizable with \( \omega(t) = 0 \) for all admissible uncertainties.

Secondly, under zero initial condition, introduce the performance
\[ J = \int_0^T (z^T z - \gamma^2 \omega^T \omega) dt. \] (13)
Without loss of generality, we assume that the subsystem 1 is first activated, and the switching sequence is
\[ \Sigma \{t_0, t_1, t_2, t_3, \ldots\}. \] (14)
The subsystem 1 is activated on \([t_{2m}, t_{2m+1})\), the subsystem 2 is activated on \([t_{2m+1}, t_{2m+2})\), where \( m \in Z^+, t_0 = \). Then for \( \forall \omega \in L_1(0, \infty) \), we have
\[ J = \int_{t_0}^{t_1} (z^T z - \gamma^2 \omega^T \omega + \frac{d}{dt} V_i(x(t_i)) dt - V_i(x(t_i))) + V_i(x(t_0)) \]
\[ \quad + \int_{t_1}^{t_2} (z^T z - \gamma^2 \omega^T \omega + \frac{d}{dt} V_i(x(t))) dt - V_i(x(t))) + V_i(x(t_1)) \]
\[ \quad + \int_{t_2}^{t_3} (z^T z - \gamma^2 \omega^T \omega + \frac{d}{dt} V_i(x(t))) dt - V_i(x(t))) + V_i(x(t_2)) + \ldots \]
\[ \leq \int_{t_0}^{t_1} (z^T z - \gamma^2 \omega^T \omega + \frac{d}{dt} V_i(x(t)) dt + \int_{t_1}^{t_2} (z^T z - \gamma^2 \omega^T \omega + \frac{d}{dt} V_i(x(t)) dt + \ldots \]
\[ \quad + \int_{t_{m-1}}^{t_m} (z^T z - \gamma^2 \omega^T \omega + \frac{d}{dt} V_i(x(t)) dt + \ldots + \int_{t_{m+1}}^{t_{m+2}} (z^T z - \gamma^2 \omega^T \omega + \frac{d}{dt} V_i(x(t)) dt + \ldots, \] (15)
where \( V_i(t) (i=1,2) \), as (6) \( P_i, Q \) satisfy (12-a) and (12-b). Then the right hand side of (15) can be written as
The uncertain switched delay system (1) is

\[ \sum_{i=1}^{n} \int_{t_{i}}^{t_{i+1}} (z^T z - \gamma^2 \omega^2 \omega + \frac{d}{dt} V_i) dt + \sum_{i=1}^{n} \int_{t_{i}}^{t_{i+1}} (z^T z - \gamma^2 \omega^2 \omega + \frac{d}{dt} V_i) dt. \]

Thus, during any \([t_k, t_{k+1})\), \(k \in \mathbb{Z}^+\), it holds that

\[ z^T z - \gamma^2 \omega^2 \omega + \frac{d}{dt} V_i \leq \left[ \begin{array}{c} x(t) \\ x(t-\tau) \end{array} \right]^T \left[ \begin{array}{cc} W_i + Q & P_iA_i \\ A_i^T P_i & -Q \end{array} \right] \left[ \begin{array}{c} x(t) \\ x(t-\tau) \end{array} \right] - (\gamma^2 G^T_i P_i x(t) - \gamma \omega) x(t) < 0. \]

In view of \(V_{\sigma(t_k)}(t_k) < V_{\sigma(t_k)}(t_k)\), we have

\[ J < 0 \]

for \(\forall \omega \in L_1[0, \infty)\) and all admissible uncertainties. That is \(\|z(t)\| < \|z_0(t)\|\). This completes the proof.

From practical point of view, the bounds \(\phi(x, t)\) of the disturbances \(f_i(x, t)\) are not easy to known. In order to handle this problem, we introduce the following assumption. Assumption 3. There exist unknown positive constants \(g_{i1}\) and \(g_{i2}\) \((i = 1, 2)\) such that

\[ \|f_i(x, t)\| \leq g_{i1} + g_{i2}\|x(t)\|. \]

We now have to deal with \(g_{i1}\) and \(g_{i2}\). To this end, we employ parameter estimates \(\tilde{g}_{i1}\) and \(\tilde{g}_{i2}\) to adapt the unknown constant parameters \(g_{i1}\) and \(g_{i2}\), respectively. The adaptation error of each parameter estimated is defined as \(\tilde{g}_{i2} = g_{i2} - g_{i2}\) and \(\tilde{g}_{i2} = g_{i2} - g_{i2}\).

Theorem 2. Suppose Assumptions 1 and 3 hold. Given a constant \(\gamma > 0\), the uncertain switched delay system (1) is stabilizable with \(H_\infty\) disturbance attenuation level \(\gamma\), if the matrix inequalities (12-a) and (12-b) are feasible. The switching law is as (5) and the stabilizing adaptive variable structure controller for subsystems are given by

\[ u_i(t) = -\frac{1}{2} B_i^T P_i x(t) - \widetilde{u}_i(t), \]

where \(\widetilde{u}_i(t) = (\lambda_2 \|x(t)\| + \tilde{g}_{i1} + \tilde{g}_{i2}\|x(t)\| + \mu) \mathrm{sign}(s_i(t)), s_i(t) = B_i^T P_i x(t) \). Let \(\lambda, \mu\) are positive constants, \(P_i\), \((i = 1, 2)\) \(Q\) satisfy (12-a) and (12-b). The parameter update laws of the \(i\)th subsystem are chosen as

\[ \tilde{g}_{i1} = \begin{cases} 0, & \text{if } \sigma = i, \\ \beta_i \|x(t)\| \sigma = i, \end{cases} \quad \tilde{g}_{i2} = \begin{cases} 0, & \text{if } \sigma = i, \\ \beta_i \|x(t)\| \sigma = i, \end{cases} \]

where \(\beta_i\) and \(\beta_i\) are positive constants specified by the designer.

Proof. We choose the Lyapunov functional candidate

\[ V_i(t) = x^T(t) P_i x(t) + \int_{t_{i-1}}^{t} x^T(t) \Theta \Omega \Theta x(t) dt + \beta_i^1 \sum_{j=1}^{n} \tilde{g}_{i1}^2 + \beta_i^2 \sum_{j=1}^{n} \tilde{g}_{i2}^2, \]

where \(P_i, Q\) satisfy (12-a) and (12-b).

In subsequent arguments, we shall first verify the stabilizability of the system (1) with \(\omega(t) = 0\).

Note that

\[ \tilde{g}_{i1} = \tilde{g}_{i1}, \tilde{g}_{i2} = \tilde{g}_{i2}. \]

Differentiating (20) with respect to \(t\), we obtain

\[ \dot{V}_i(t) = 2x^T(t) P_i x(t) + x^T(t) \Omega x(t) - x^T(t - \tau) \Omega x(t - \tau) + 2\beta_i^1 \sum_{j=1}^{n} \tilde{g}_{i1} \dot{\tilde{g}}_{i1} + 2\beta_i^2 \sum_{j=1}^{n} \tilde{g}_{i2} \dot{\tilde{g}}_{i2}. \]

Substituting (18) into the system (1), gives

\[ \dot{x}(t) = (A_i - \frac{1}{2} B B_i^T P_i x(t) + \Delta A_i) x(t) + (A_i + \Delta A_i) x(t - \tau) - \dot{B}_{i} \tilde{u}_i(t) + B_i f_i(x, t), \]

where \(\tilde{u}_i(t) = x(t) \in [x(t) - x(t - \tau)]^T, \Pi_i\) are defined as (9).

Thus, it holds that

\[ \dot{V}_i(t) < 0, \quad [t_j, t_{j+1}) \]

Note that \(\sum_{j=1}^{n} \tilde{g}_{i1}^2 + \sum_{j=1}^{n} \tilde{g}_{i2}^2 \) are continuous all time.

Similar to the proof of Lemma 1, each switching time \(t_j\), we have

\[ V_i(t_{j}) < V_{i} (t_{j}). \]

In view of multiple Lyapunov functions method, the system (1) with \(\omega(t) = 0\) is asymptotically stabilizable.

Secondly, let

\[ J = \int_0^\infty (z^T z - \gamma^2 \omega^2 \omega) dt \]

Without loss of generality, we assume that the subsystem 1 is first activated and the switching sequence is expressed as (14). Then, similar to the proof of Theorem 1, we have

\[ J \leq \int_{t_{i}}^{t_{i+1}} (z^T z - \gamma^2 \omega^2 \omega + \frac{d}{dt} V_i) dt + \int_{t_{i}}^{t_{i+1}} (z^T z - \gamma^2 \omega^2 \omega + \frac{d}{dt} V_i) dt \]

where \(V_i(t) \) \((i = 1, 2)\) as (19).

Similar to the proof of Theorem 1, in view of \(V_{\sigma(t_k)}(t_k) < V_{\sigma(t_k)}(t_k)\), we have

\[ J < 0 \]

for \(\forall \omega \in L_1[0, \infty)\) and all admissible uncertainties. That is \(\|z(t)\| < \|z_0(t)\|\). This completes the proof.

Remark 2. From practical point of view, the chattering phenomenon may occur around \(\|z(t)\| = 0\). Therefore, \(\|z(t)\|\) cannot be precisely guaranteed equal to zero. However, according to (19), \(\tilde{g}_{i1}\) and \(\tilde{g}_{i2}\) will be increasing all the time as \(\|z(t)\| \neq 0\). Motivated by [18], to overcome this problem, we may apply the following modified parameter update laws.
\[
\dot{\xi}_i = \begin{cases} 0, & \text{if } \sigma \neq i, \\
0, & \text{if } \|x(t)\| < d, \sigma = i,
\end{cases}
\]
\[
\dot{\hat{g}}_{i1} = \begin{cases} 0, & \text{if } \|x(t)\| < d, \sigma = i, \\
\beta_1 \|x(t)\| \text{sgn}(x(t)) & \text{if } \sigma = i.
\end{cases}
\]

**Remark 3.** The use of sign function \( \text{sign}(x) \) may cause chattering effect. This may be undesirable in practical engineering systems. To overcome this drawback, the function \( \tanh(x) \) can be used to replace the function \( \text{sign}(x) \) [19].

**IV. EXAMPLES**

In this section, we present a numerical example to demonstrate the effectiveness of proposed design method. Consider the following uncertain switched delay system

\[
\dot{x}(t) = (A_+ + D_+ x(t)) + (A_- + D_- x(t - \tau)) + B_g (u_x + f_x(x,t)) + G_0 \theta(t),
\]

where \( i \in \{1,2\} \), \( t \in [-\tau,0] \), \( z(t) = C_+ x(t) \), where \( \sigma(t) \in \Xi = \{1,2\} \), \( \tau \leq 1 \),

\[
A_i = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Sigma_i = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}
\]

\[
G_i = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0.5 \end{bmatrix}
\]

\( \Delta A_i = D_i \Sigma_i E_i, \quad \Delta A_2 = D_2 \Sigma_2 E_2 \),

where \( E_i = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \),

\( D_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix} \), \( \Sigma_1 = \begin{bmatrix} 0.5 \nu_{11} & 0.5 \nu_{12} \end{bmatrix} \), \( \nu_{11}, \nu_{12} \in [-1,1] \),

\( D_2 = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & 0.1 \end{bmatrix} \), \( \Sigma_2 = \begin{bmatrix} 0.5 \nu_{21} & 0.5 \nu_{22} \end{bmatrix} \), \( \nu_{21}, \nu_{22} \in [-1,1] \),

\( \Delta A_{i1} = \begin{bmatrix} 2 \nu_1 \\ 0.5 \nu_1 \end{bmatrix}, \quad \nu_1 \in [-1,1] \),

\( \Delta A_{i2} = \begin{bmatrix} 2 \nu_2 \\ 0.5 \nu_2 \end{bmatrix}, \quad \nu_2 \in [-1,1] \),

the unknown nonlinear functions \( f_1 = 0.5 x_1 \cos(x_2) - 0.5 \sin(x_1) + x_2 \),

\( f_2 = 1.5 x_1 x_2 + 1.5 \sin(x_1) + x_2 \).

We adopt adaptive control to estimate the upper bounds of the disturbances. It is easy to verify that the conditions of Theorem 2 are satisfied. Let \( \varepsilon = 0.1 \), \( \alpha_1 = \alpha_2 = -5 \), \( \eta_1 = \eta_2 = 0.1 \). The disturbance attenuation level is given by \( \gamma = 1/\sqrt{2} \). Then solving (12-a), (12-b), leads to

\[
P_1 = \begin{bmatrix} 0.1183 & 0.1126 \\ 0.1126 & 0.3687 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.1415 & 0.0313 \\ 0.0313 & 0.263 \end{bmatrix}
\]

\[
Q_1 = \begin{bmatrix} 0.5483 & 0.4009 \\ 0.4009 & 0.6249 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -0.0424 & 0.0363 \\ -0.0363 & 0.1016 \end{bmatrix}
\]

Let \( \mu = 2 \). According to (18), the adaptive variable structure controllers are given by

\[
\begin{align*}
u_1 &= -0.0873 x_1 - 0.1485 x_2 - (6 \|x(t)\| \\
+ \hat{g}_{11} + \hat{g}_{12} \|x(t)\| + 2) \text{sign}(s_1), \\
u_2 &= -0.0707 x_1 - 0.0157 x_2 - (8 \|x(t)\| \\
+ \hat{g}_{21} + \hat{g}_{22} \|x(t)\| + 2) \text{sign}(s_2).
\end{align*}
\]

The parameter update laws are designed as (28), with \( d = 0.04 \), \( \beta_1 = \beta_2 = 0.1 \).

By (5), we design the following switching law with initial state \( x_0 = [2,-2]^T \)

\[
\sigma(0) = 1,
\]

\[
\sigma = \begin{cases} 1, & \text{if } (x(t) \in \Phi_1 \text{ and } \sigma(t) = 1) \text{or}(x(t) \not\in \Phi_2 \text{ and } \sigma(t) = 1), \\
2, & \text{if } (x(t) \in \Phi_2 \text{ and } \sigma(t) = 2) \text{or}(x(t) \not\in \Phi_1 \text{ and } \sigma(t) = 2). \end{cases}
\]

The simulation results are depicted in Fig. 1-Fig. 4.
developed to stabilize the uncertain switched delay system with $H_\infty$ disturbance attenuation level $\gamma$ under the hysteresis switching law for the cases of known and unknown upper bounds of the nonlinear disturbances respectively.

REFERENCES