Robust $H_{\infty}$ Performance Analysis for Continuous-Time Networked Control Systems

Xun-Lin Zhu and Guang-Hong Yang

Abstract—This paper studies the problems of robust $H_{\infty}$ performance analysis and controller design for continuous-time networked control systems (NCSs). A new type of Lyapunov functionals is exploited to derive sufficient conditions for guaranteeing the robust exponential stability and $H_{\infty}$ performance of the considered system, and robust $H_{\infty}$ controller design is presented. It is shown that the newly obtained result is less conservative than the existing corresponding ones. Meanwhile, by using a method of eliminating redundant variables, the computation complexity is reduced. Numerical examples are given to illustrate the effectiveness of the proposed methods.

I. INTRODUCTION

Networks have received increasing attention in recent years because of the popularity and advantages of using network cables in control systems. The network itself is a dynamic system and induces possible delays via network communication due to limited bandwidth. A realistic networked control system design should take the communication delays into account, since the delays are widely known to degrade the performance of the control system.

In the past decade, the control problem of networked systems with time-delays has received increasing attention. [1] studied the problem of packet dropout and transmission delays induced by communication network of NCSs in both continuous time and discrete time cases. By using the Lyapunov-Razumikhin function techniques, [2] obtained the delay-dependent condition on the stabilization of NCSs in terms of linear matrix inequalities (LMIs). The admissible upper bounds of data packet loss and delays can be computed by using the quasi-convex optimization algorithm. [3] discussed the design of robust $H_{\infty}$ controllers for uncertain NCSs with both the network-induced delay and data dropout. [4] was concerned with the controller design of NCSs.

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A new model of NCSs was provided under consideration of both the network-induced delay and the data packet dropout in the transmission, and a controller design method was proposed based on a delay-dependent approach. [5] studied the stabilization problem of NCSs where the main focus was the packet-loss issue. Two types of packet-loss processes were considered. One was the Markovian packet-loss process, the other was the Markovian packet-loss process.

In this paper, the problem of robust $H_{\infty}$ performance analysis for continuous-time NCSs is investigated. The objective is to seek for improved LMI-based conditions for ensuring larger delay bounds and better $H_{\infty}$ performance. A new type of Lyapunov functionals is proposed, and new delay-dependent criteria for the $H_{\infty}$ performance analysis are derived. Using these criteria, an upper bound of time-delay can be obtained such that the considered system is robustly exponentially stable with a prescribed $H_{\infty}$ performance bound. It is shown that the new result is less conservative than the existing corresponding ones. Meanwhile, by using a method of eliminating redundant variables, the computational complexity is also reduced.

The organization of this paper is as follows. Section 2 models an NCS with data packet dropout and transmission delays as a linear system with time-varying input delay. Section 3 presents $H_{\infty}$ performance analysis, controller design, and a method of eliminating redundant variables. It proves that the newly obtained stability condition is less conservative than some latest results. Three numerical examples are given to show the effectiveness of the criteria in Section 4, and finally conclusions are stated in Section 5.

II. SYSTEM DESCRIPTION

Throughout this paper, we assume that the sensor is clock-driven, the controller and actuator are event-driven and hold the latest data, $h$ is the length of sampling period. Single packet transmission is considered throughout this paper. The actuator and the sensor are connected through a communication network with finite bandwidth. Data packet dropout and disordering in an NCS are unavoidable because of limited bandwidth. An NCS with the possibility of dropping data packet and disordering can be described as in Figure 1. The model presented here is the same as that in [3]:

\[ \dot{x}(t) = Ax(t) + Bu(t) + B_d \omega(t), \]
\[ x(t) = \phi(t), \quad t \in [t_1 - h_2, t_1], \]
\[ z(t) = Cx(t) + Du(t), \]

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^p$ is the control input vector, $z(t) \in \mathbb{R}^q$ is controlled output, $\omega(t) \in L_2[t_1, \infty)$.
denotes the external perturbation, and \( t_1 \) denotes the instant the actuator receives the 1st control signal. \( A, B, B_0, C, D \) are constant matrices of appropriate dimensions; \( x_c \) is the delayed version of \( x, u \) is the delayed version of \( u_c, \) and \( u_c(t) = Kx_c(t) \). Here, \( K \) is the state feedback gain matrix. Denote the instant the actuator receives the 4th control signal as \( t_k \) and this control signal is based on the state of plant at the instant \( ikh \), thus \( \{t_1, t_2, t_3, \cdots \} \subseteq \mathbb{Z}^+ \), and

\[
\begin{align*}
    u(t^+) &= Kx(t - \tau_k), \quad t \in [ikh + \tau_k, k = 1, 2, \cdots],
\end{align*}
\]

where time-delay \( \tau_k \) denotes the time from the instant \( ikh \) when the sensor node samples sensor data from a plant to the instant \( t_k \) when the actuator receives the control signal, i.e., \( \tau_k = t_k - ikh \), and \( \tau_k = \tau_k^{a} + \tau_k^{c} \), where \( \tau_k^{a} \) is the sensor-to-controller time-delay of \( x(ikh) \), and \( \tau_k^{c} \) is the controller-to-actuator time-delay of \( u(ikh - \tau_k^{c}) \). Obviously, \( \cup_{k=1}^{\infty} [ikh + \tau_k, ikh + \tau_k + 1] = [t_1, \infty), t_1 \geq 0 \).

As pointed out in [3], under assumption:

\[
\begin{align*}
    (ikh + 1)h + \tau_{k+1} &\leq h_2, \quad k = 1, 2, \cdots, \\
    0 \leq h_1 &\leq \tau_{k}, \quad k = 1, 2, \cdots, \\
\end{align*}
\]

where \( h_1, h_2 \) are constants, then the system (1)-(4) can be rewritten as follows:

\[
\begin{align*}
    \dot{x}(t) &= A x(t) + B K x(t - d(t)) + B_0 \omega(t), \\
    x(t) &= \phi(t), \quad t \in [t_1 - h_2, t_1], \\
    z(t) &= C x(t) + D K x(t - d(t)), \\
    0 &\leq h_1 \leq d(t) \leq h_2,
\end{align*}
\]

where \( d(t) = t - ikh, \quad t \in [ikh + \tau_k, ikh + 1 + \tau_k + 1] \), which denotes the time-varying delay in the control signal. Obviously, \( d(t) \) is not always differentiable in the interval \([t_1, \infty)\].

In this paper, we analyze the robust \( H_{\infty} \) performance of the closed-loop system (7)-(10).

**Remark 1.** The above mentioned problem was also studied in [3]. In fact, many results of time-delay systems can be applied to deal with this problem, among them the result in [6] is one of the latest and it is listed as follows.

**Lemma 1.** [6] For given scalars \( h_1, h_2 \) \((h_2 > h_1 \geq 0)\) and a matrix \( K \), the linear system (7)-(8) with time-varying delay \( d(t) \) satisfying (10) and \( B_0 = 0 \) is asymptotically stable if there exist matrices \( P_i > 0, S_i > 0, Q_i \geq 0, Y_i, T_i, V_i \) \((i = 1, 2)\), such that the following LMI holds:

\[
\Lambda = \begin{bmatrix}
\Lambda_1 & \Lambda_2 & \Lambda_3 \\
\Lambda_1 & \Lambda_2 & \Lambda_3 \\
\end{bmatrix} < 0,
\]

where

\[
\begin{align*}
    \Lambda_1 &= \begin{bmatrix}
        \Lambda_{11} & \Lambda_{12} & V_1 & -T_1 & h_2 Y_1 & h_{12} T_1 \\
        * & \Lambda_{22} & V_2 & -T_2 & h_2 Y_2 & h_{12} T_2 \\
        * & * & -Q_1 & 0 & 0 & 0 \\
        * & * & * & -Q_2 & 0 & 0 \\
        * & * & * & * & -h_2 S_1 & 0 \\
        * & * & * & * & * & -h_2 \sum_{i=1}^{2} \tilde{S}_i \\
    \end{bmatrix}, \\
    \Lambda_{11} &= P_i A + A^T P_i + Q_i + Q_2 + Y_1 + Y_1^T, \\
    \Lambda_{12} &= P_i (B K) + Y_2 - Y_1 + T_1 - V_1, \\
    \Lambda_{22} &= T_2 + T_2^T - Y_2 - Y_2^T - V_2 - V_2^T, \\
    \Lambda_{55} &= -Q_2 - h_2 S_1 - h_{12} S_1 - h_{12} S_2, \\
    \Lambda_{55} &= -Q_2 - h_2 S_1 - h_{12} S_1 - h_{12} S_2,
\end{align*}
\]

III. MAIN RESULTS

In this section, a new type of Lyapunov-Krasovskii functionals is proposed to derive new delay-dependent robust exponential stability criteria for the system (7)-(10) with a prescribed \( H_{\infty} \) performance level. By eliminating redundant variables, a method for designing state feedback \( H_{\infty} \) controllers is presented.

**A. \( H_{\infty} \) performance analysis**

In the following, we will give new sufficient conditions which can ensure the exponential stability of NCS (7)-(10) with a prescribed \( H_{\infty} \) performance bound.

**Lemma 2.** For given scalars \( h_1, h_2 \) \((h_2 > h_1 \geq 0)\), \( \gamma > 0 \), and a matrix \( K \), the system (7)-(10) is exponentially stable with an \( H_{\infty} \) norm bound \( \gamma \) if there exist \( n \times n \) matrices \( P_{ij}, P_i = P_i^T, i, j = 1, 2, 3 \), \( P_1 > 0, Q_i \geq 0, Z_i \geq 0, N_i, S_i \geq 0 \) \((i = 1, 2)\), and matrices \( M_i \) \((i = 1, 2, \cdots, 9)\), such that

\[
\Gamma < 0,
\]

and

\[
P = \begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
* & P_{22} & P_{23} \\
* & * & P_{33} \\
\end{bmatrix} \geq 0, \quad H_i = \begin{bmatrix}
Z_i \\
S_i \\
\end{bmatrix} \geq 0,
\]

where

\[
\Gamma = \begin{bmatrix}
\Gamma_1 & \Gamma_2 & -M_1 \delta - \delta^T M_1^T, \\
\Gamma_2 & \Gamma_3 & -P_{12} - P_{13} \\
* & \Gamma_3 & 0 \\
\end{bmatrix}, \\
\Gamma_1 &= \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} & \Gamma_{13} & P_{13} & -P_{12} - P_{13} \\
\Gamma_{22} & 0 & 0 & 0 \\
* & \Gamma_{33} & h_{12} S_1 + h_{12} S_2 & h_{12} S_1 + h_{12} S_2 & h_{12} S_1 + h_{12} S_2 \\
* & * & -Q_1 - h_{12} S_2 & 0 & 0 \\
* & * & * & \Gamma_{55} & \Gamma_{55} \\
\end{bmatrix},
\]

\[
\Gamma_{11} = Q_1 + Q_2 + h_2 Z_1 + h_1 Z_2 + P_{12} + P_{13} + C T C - h_{12} S_1, \\
\Gamma_{12} = h_2 N_i + h_1 N_i + P_{11}, \\
\Gamma_{13} = C T D K + h_{12} S_1, \\
\Gamma_{22} = h_2 S_1 + h_{12} S_2, \\
\Gamma_{33} = (D K)^T (D K) - h_{12} S_1 - h_{12} S_1 - h_{12} S_2 - h_{12} S_2,
\]

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\[
\Gamma_2 = \begin{bmatrix}
P_{22} - h_{12}^{-1} N_{12}^T & P_{22} + P_{23} & P_{23} & 0 \\
K_7 & \frac{1}{h_{12}^{-1}} N_{12}^T & \frac{1}{h_{12}^{-1}} N_{12}^T & 0 \\
& \Gamma_73 & 0 & 0 \\
& & \Gamma_75 & 0
\end{bmatrix},
\]
\[
\Gamma_{73} = -h_{12}^{-1} (N_{11}^T + N_{12}^T),
\]
\[
\Gamma_{75} = -P_{22} - P_{23} - P_{33} - h_{12}^{-1} (N_{11}^T + N_{12}^T),
\]
\[
\delta = \begin{bmatrix}
A & -I & BK & 0 & 0 & 0 & 0 & B_w
\end{bmatrix},
\]
\[
h_{12} = h_2 - h_1.
\]

Proof: Construct a Lyapunov-Krasovskii functional as
\[
V(t) = \begin{bmatrix}
\int_{t-h_1}^{t} x(s) ds \\
\int_{t-h_2}^{t} x(s) ds
\end{bmatrix}^T P \begin{bmatrix}
\int_{t-h_1}^{t} x(s) ds \\
\int_{t-h_2}^{t} x(s) ds
\end{bmatrix} + \frac{1}{2} \sum_{i=1}^{2} \int_{t-h_i}^{t} \int_{t-h_\beta}^{t} Q x(s) x(s)^T ds \\
+ \int_{t-h_2}^{t} \int_{t-h_\beta}^{t} H_1^T \begin{bmatrix}
\int_{t-h_1}^{t} x(s) ds \\
\int_{t-h_2}^{t} x(s) ds
\end{bmatrix} d\beta \\
+ \int_{t-h_2}^{t} \int_{t-h_\beta}^{t} H_2^T \begin{bmatrix}
\int_{t-h_1}^{t} x(s) ds \\
\int_{t-h_2}^{t} x(s) ds
\end{bmatrix} d\beta,
\]
where \( P \geq 0, H_i \geq 0 \) (\( i = 1, 2 \)) are defined in (13) and \( Q_i \geq 0 \) (\( i = 1, 2, 3 \)).

Using the Cauchy-Schwarz inequality [7], and denoting \( \iota = [x^T(t), x^T(t - d(t))]^T \), \( x_f(t - d(t)) = (x^T(t - d(t)), x^T(t - d(t)))^T \), \( (\int_{t-d(t)}^{t} x(s) ds)^T (\int_{t-d(t)}^{t} x(s) ds)^T \omega^T(t) \), we can get
\[
\dot{V}(t) + \zeta^T(t) \zeta(t) - \gamma^2 \omega^T(t) \omega(t) \leq \zeta^T \Gamma \zeta,
\]
which implies that \( ||\dot{z}(t)||_2 \leq \gamma ||\omega(t)||_2 \) under zero initial condition.

Similar to [3], we can prove the exponential stability of system (7)-(10).

Thus, the proof is completed.

Remark 2. In Lemma 2, a sufficient condition of exponential stability for the system (7)-(10) with an \( H_m \) norm bound \( \gamma \) is given in terms of solutions to a set of LMIs. Different sufficient conditions are also given in [3] and [4]. The numerical comparison between Lemma 2 and the results in [3] and [4] will be given in Section 4. Note that the Lyapunov-Krasovskii functional (14) is more general, and lead to a less conservative stability condition than that in [6]. The details will be discussed in the sequel.

By the elimination Lemma ([14], p.22), it is readily seen that if there exist matrices \( M_i \) (\( i = 1, 2, \ldots, 9 \)) that solve \( \Gamma < 0 \), if and only if
\[
\mathcal{M}_{\delta}^T \begin{bmatrix}
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3
\end{bmatrix} \mathcal{M}_{\delta} < 0
\]
holds, where \( \mathcal{M}_{\delta} \) denotes the full-rank matrix representations of the right annihilator of \( \delta \). By the Schur complement, it yields that
\[
\Pi < 0,
\]
\[
\Theta < 0,
\]
where
\[
\Pi = \begin{bmatrix}
\Pi_1 & \Pi_2 \\
\Pi_3 & \Pi_4
\end{bmatrix},
\]
\[
\Pi_1 = \begin{bmatrix}
\Pi_{11} & \Pi_{12} & \Pi_{13} & -\Pi_{12} - \Pi_{13} & \Pi_{15} \\
\Pi_{21} & \Pi_{22} & \Pi_{23} & -\Pi_{22} & \Pi_{23} \\
\Pi_{31} & \Pi_{32} & \Pi_{33} & 0 & \Pi_{35} \\
\Pi_{41} & \Pi_{42} & \Pi_{43} & -\Pi_{22} & \Pi_{23} \\
\Pi_{51} & \Pi_{52} & \Pi_{53} & -\Pi_{12} & \Pi_{13}
\end{bmatrix},
\]
\[
\Pi_{11} = \begin{bmatrix}
Q_1 + Q_2 + h_{12} Z_1 + h_{12} Z_2 \Pi_{21} + \Pi_{22} + \Pi_{23} + (P_{11} + h_{12} N_1 + h_{12} N_2)^T + C^T C - h_{12}^{-1} S_1, \\
\Pi_{12} + (P_{11} + h_{12} N_1 + h_{12} N_2) B K + C^T (D K) + h_{12}^{-1} S_1, \\
\Pi_{15} = P_{22} + A^T P_{22} - h_{12}^{-1} N_2^T, \\
\Pi_{22} = (D K)^T (D K) - h_{12}^{-1} S_1 - h_{12}^{-1} (S_1 + S_2) - h_{12}^{-1} S_2, \\
\Pi_{25} = (B K)^T P_{22} + h_{12}^{-1} N_2^T, \\
\Pi_{33} = -Q_1 - h_{12}^{-1} S_2, \\
\Pi_{44} = -Q_2 - h_{12}^{-1} (S_1 + S_2), \\
\Pi_{46} = -P_{22} - P_{23} - P_{33} - h_{12}^{-1} (N_1^T + N_2^T), \\
\Pi_{46} = -P_{22} - P_{23} - P_{33} - h_{12}^{-1} (N_1^T + N_2^T), \\
\Pi_3 = \begin{bmatrix}
-\Pi_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & \Pi_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & \Pi_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & \Pi_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & \Pi_{11} & 0 \\
0 & 0 & 0 & 0 & 0 & \Pi_{11}
\end{bmatrix},
\]
\[
\Pi_{89} = B_w^T (h_{2} S_1 + h_{12} S_2), \\
\Pi_{99} = -(h_{2} S_1 + h_{12} S_2), \\
h_{12} = h_2 - h_1.
\]

On the other hand, if \( \Pi < 0 \) holds, then it is very easy to see that
\[
\Gamma < 0 \text{ holds by taking}
\]
\[
M_1 = -(P_{11} + h_{2} N_1 + h_{12} N_2), \\
M_2 = -(h_{2} S_1 + h_{12} S_2), \\
M_6 = -P_{12}, \\
M_7 = -(P_{21} + P_{23})^T, \\
M_8 = -P_{13}, \\
M_3 = M_5 = M_9 = 0.
\]

This implies that \( M_3, M_4, M_5, M_9 \) are all redundant in \( \Gamma \).

Remark 3. From above analysis, it is shown that \( M_6 \) is redundant when \( P_{22} = 0 \), \( M_8 \) is redundant when \( P_{12} = 0 \), and \( M_6, M_7, M_8 \) are redundant when \( P_{22} = P_{23} = 0 \) in Lemma 2.

Thus, we have the following result for the \( H_m \) performance analysis, which is equivalent to Lemma 2 and has fewer decision variables.

Theorem 1. For given scalars \( h_1, h_2 \) (\( h_2 > h_1 \geq 0 \)), \( \gamma > 0 \), and a matrix \( K \), the system (7)-(10) is exponentially stable with an \( H_m \) norm bound \( \gamma \) if there exist \( n \times n \) matrices \( P_{ij}, P_{ij}^T, (i = 1, 2, 3) \), \( P_{11} > 0, Q_i \geq 0, Z_i \geq 0, N_i, S_i \geq 0 \) (\( i = 1, 2 \)), and matrices \( M_i \) (\( i = 1, 2, \ldots, 5 \)), such that
\[
\Pi < 0,
\]
\[
\Theta < 0,
and
\[ P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ \ast & P_{22} & P_{23} \\ \ast & \ast & P_{33} \end{bmatrix} \geq 0, \quad H_i = \begin{bmatrix} Z_i & N_i \\ \ast & S_i \end{bmatrix} \geq 0 \quad (i = 1, 2), \] (19)
where
\[ \Theta = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \ast & \Gamma_3 \end{bmatrix} + \Theta_1 + \Theta_1^T, \]
\[ \Theta_1 = -\begin{bmatrix} M_{11}^T & M_{12}^T & 0 & 0 & 0 & M_{21}^T & M_{22}^T & M_{23}^T & 0 \\ \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \end{bmatrix}^T \alpha \omega, \]
\[ \alpha \omega = \begin{bmatrix} A & -I & BK & 0 & 0 & 0 & 0 & B_{w} \end{bmatrix}, \]
and \( \Gamma_1, \Gamma_2, \Gamma_3 \) are defined in Lemma 2.

From Theorem 1 and the equivalence of the inequality (17) and (18), we can derive the following stability conditions for the system (7)-(8).

**Corollary 1.** For given scalar \( h_1, h_2 \) (\( h_2 > h_1 \geq 0 \)) and matrix \( K \), the linear system (7)-(8) with time-varying delay \( d(t) \) satisfying (10) and \( B_{w} = 0 \) is asymptotically stable if there exist matrices \( P_{ij}, P_{ij} = P_{ji} \) (\( i, j = 1, 2, 3 \)), \( P_{11} > 0 \), \( Q_i > 0 \), \( Z_i \geq 0, N_i, S_i \geq 0 \) \((i = 1, 2)\), such that the following LMIs hold:
\[ \Omega < 0, \] (20)
and
\[ P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ \ast & P_{22} & P_{23} \\ \ast & \ast & P_{33} \end{bmatrix} \geq 0, \quad H_i = \begin{bmatrix} Z_i & N_i \\ \ast & S_i \end{bmatrix} \geq 0 \quad (i = 1, 2), \] (21)
where
\[ \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \ast & \Omega_3 \end{bmatrix}, \]
\[ \Omega_1 = \begin{bmatrix} \Omega_{11} & \Omega_{12} & -P_{12} - P_{13} & \Omega_{15} \\ \ast & \Omega_{22} & h_{11}^{-1}S_2 & h_{12}^{-1}(S_1 + S_2) \\ \ast & \ast & \Omega_{33} & 0 \\ \ast & \ast & \ast & \ast \end{bmatrix}, \]
\[ \Omega_{11} = Q_1 + Q_2 + h_2Z_1 + h_1Z_2 + P_{12} + P_{12}^T - h_1^{-2}S_1 \]
\[ \quad + (P_{11} + h_2N_1 + h_2N_2)A + A^T(P_{11} + h_2N_1 + h_2N_2)^T, \]
\[ \Omega_{12} = (P_{11} + h_2N_1 + h_2N_2)BK + h_1^{-1}S_1, \]
\[ \Omega_{15} = P_{22} + A^TP_{12} - h_2^{-1}N_2^T, \]
\[ \Omega_{22} = -h_2^{-1}S_1 - h_1^{-2}(S_1 + S_2) - h_1^{-2}S_2, \]
\[ \Omega_{25} = (BK)^TP_{12} + h_2^{-1}N_1^T, \]
\[ \Omega_{33} = -\Phi_1 - h_2^{-1}S_2, \]
\[ \Omega_{44} = -h_2^{-1}(S_1 + S_2), \]
\[ \Omega_{16} = P_{23} + A^TP_{13}, \]
\[ \Omega_{18} = A^T(h_2S_1 + h_2S_2), \]
\[ \Omega_{26} = (BK)^T(P_{13} + h_1^{-2}N_1^T) + \Omega_{28}, \]
\[ \Omega_{28} = (BK)^T(h_2S_1 + h_2S_2), \]
\[ \Omega_{46} = -P_{23} - P_{33}, \]
\[ \Omega_{48} = 0. \]

The comparison between Corollary 1 and Lemma 1 is given as follows.

**Theorem 2.** If the LMI in Lemma 1 is feasible, the LMIs in Corollary 1 are also feasible.

**Proof:** Denoting
\[ \Lambda_4 = \begin{bmatrix} -h_1^{-1}I & 0 & 0 & 0 \\ -h_2^{-1}I & -h_2^{-1}I & h_1^{-1}I & 0 \\ 0 & 0 & -h_2^{-1}I & 0 \\ 0 & h_2^{-1}I & 0 & 0 \end{bmatrix} \] (22)
pre- and post-multiplying both sides of \( \Lambda \) by \[ \begin{bmatrix} I & \Delta_4 \\ 0 & I \end{bmatrix} \]
and its transpose, respectively, then it is easy to see that \( \Omega < 0 \) from \( \Lambda < 0 \) by taking \( P_{11} = P_1, P_{12} = P_{13} = P_{22} = P_{23} = P_{33} = 0 \) and \( Z_i = \varepsilon I, N_i = 0 \) \((i = 1, 2)\) with \( \varepsilon > 0 \) being sufficient small scalar.

**Remark 4.** By Theorem 2, it proves theoretically that Corollary 1 is less conservative than Lemma 1.

**B. Robust performance analysis**

Next, consider the following system with parameter uncertainties given by
\[ \dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]u(t) + B_{w} \omega(t), \] (23)
where \( \Delta A(t) \) and \( \Delta B(t) \) denote the parameter uncertainties satisfying the following condition:
\[ [\Delta A(t) \Delta B(t)] = GF(t) \begin{bmatrix} E_a & E_b \end{bmatrix}, \] (24)
where \( G, E_a \) and \( E_b \) are constant matrices of appropriate dimensions and \( F(t) \) is an unknown time-varying matrix, which is Lebesque measurable in \( t \) and satisfies
\[ FT(t)F(t) \leq I, \quad \forall \ t \geq 0. \] (25)

In this case, from Theorem 1, (18) is substituted by
\[ \Theta + (-\tilde{M})F(t) \begin{bmatrix} E_a & 0 & E_bK & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ + \begin{bmatrix} E_a & 0 & E_bK & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T F(t)(-\tilde{M})^T < 0, \] (26)
where
\[ \tilde{M} = \begin{bmatrix} M_{11}^T & M_{12}^T & 0 & 0 & M_{21}^T & M_{22}^T & M_{23}^T & M_{24}^T & 0 \end{bmatrix}^T G, \]
thus, according to the definition of robust exponential stability in [3], it is easy to get the following result.

**Theorem 3.** For given scalars \( h_1, h_2 \) (\( h_2 > h_1 \geq 0 \)), \( \gamma > 0 \), and a matrix \( K \), the system described by (23)-(25) and (8)-(10) is robustly exponentially stable with an \( H_{\infty} \) norm bound \( \gamma \) if there exist \( n \times n \) matrices \( P_{ij}, P_{ij} = P_{ji} \) (\( i, j = 1, \ldots, 5 \)), \( Q_i \geq 0, Z_i \geq 0, N_i, S_i \geq 0 \) \((i = 1, 2)\), and matrices \( M_i \) \((i = 1, 2, \ldots, 5)\), and scalar \( \varepsilon > 0 \) such that
\[ \Phi = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \ast & -\varepsilon I \end{bmatrix} < 0, \] (27)
and
\[ P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ \ast & P_{22} & P_{23} \\ \ast & \ast & P_{33} \end{bmatrix} \geq 0, \quad H_i = \begin{bmatrix} Z_i & N_i \\ \ast & S_i \end{bmatrix} \geq 0 \quad (i = 1, 2). \] (28)
where
\[ \Phi_1 = \Theta + \varepsilon \begin{bmatrix} E_a & 0 & 0 & 0 & 0 & 0 & 0 \\ E_a & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \]
\[ \Phi_2 = -\begin{bmatrix} M_1^T & M_2^T & 0 & 0 & M_1^T & M_2^T & 0 \end{bmatrix}^T G, \]
and \( \Theta \) is defined in Theorem 1.

C. Robust \( H_\infty \) controller design

Based on Theorem 3, we are now in a position to design the feedback gain \( K \), which can ensure the robustly exponentially stable of the uncertain system described by (23)-(25) and (8)-(10) with \( H_\infty \) norm bound \( \gamma \).

Obviously, (18) implies \( M_2 \) is nonsingular, so there exist matrices \( U_1, U_2, U_3, U_5 \), such that \( M_1 = M_2 U_1, M_3 = M_2 U_2, M_4 = M_2 U_3, M_5 = M_2 U_5 \).

Pre- and post-multiplying both sides of (27) by \( \text{diag}(M_2^{-1}, ..., M_2^{-1}, I, I) \) and its transpose, pre- and post-multiplying both sides of \( H_i \) (\( i = 1, 2 \)) in (28) by \( \text{diag}(M_2^{-1}, M_2^{-1}) \) and its transpose, pre- and post-multiplying both sides of \( P \) in (28) by \( \text{diag}(M_2^{-1}, M_2^{-1}, M_2^{-1}) \) and its transpose, respectively, and denoting
\[ M_2 = M_2^{-1}, \bar{P}_2 = M_2 P M_2^{-1}, \bar{Q}_2 = M_2 Q M_2^{-1}, \bar{Z}_2 = M_2 Z M_2^{-1}, \]
and using the Schur complement, we can obtain the following theorem.

Theorem 4. For prescribed scalars \( h_1, h_2 \) (\( h_2 > h_1 \geq 0 \), \( \gamma > 0 \), and some tuning matrix parameters \( U_i, (i = 1, 3, 4, 5) \), the system described by (23)-(25) and (8)-(10) is robustly exponentially stable with an \( H_\infty \) bound norm \( \gamma \) if there exist \( n \times n \) matrices \( \bar{P}_i, \bar{P}_i = \bar{P}_i^T, i = 1, 2, 3 \), \( \bar{Q}_1 > 0, \bar{Q}_2 \geq 0, \bar{Z}_1, \bar{Z}_2 \geq 0 \) (\( i = 1, 2 \)), and matrices \( Q, M_2 \) and scalar \( \mu > 0 \), such that
\[ \Psi < 0, \]
and
\[ P = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 & \bar{P}_3 \\ \bar{P}_4 & \bar{P}_5 & \bar{P}_6 \\ \bar{P}_7 & \bar{P}_8 & \bar{P}_9 \end{bmatrix} \geq 0, \quad \bar{H}_i = \begin{bmatrix} \bar{Z}_1 & \bar{S}_1 \\ \bar{S}_2 & \bar{S}_3 \end{bmatrix} \geq 0, \quad (i = 1, 2) \]

where
\[ \Psi = \begin{bmatrix} \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_4 & \Psi_5 & \Psi_6 \\ \Psi_7 & \Psi_8 & \Psi_9 \end{bmatrix} + \Psi_7 + \Psi_8 \eta + \mu \Psi_9 \eta, \]
\[ \Psi_1 = \begin{bmatrix} \Psi_11 & \Psi_12 \\ \Psi_13 & \Psi_14 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_13 + \bar{P}_12 - \bar{P}_13, \]
\[ \Psi_2 = \begin{bmatrix} \Psi_21 & \Psi_22 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_12, \]
\[ \Psi_3 = \begin{bmatrix} \Psi_31 & \Psi_32 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_13, \]
\[ \Psi_4 = \begin{bmatrix} \Psi_41 & \Psi_42 \\ \Psi_43 & \Psi_44 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_13, \]
\[ \Psi_5 = \begin{bmatrix} \Psi_51 & \Psi_52 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_12, \]
\[ \Psi_6 = \begin{bmatrix} \Psi_61 & \Psi_62 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_13, \]
\[ \Psi_7 = \begin{bmatrix} \Psi_71 & \Psi_72 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_12, \]
\[ \Psi_8 = \begin{bmatrix} \Psi_81 & \Psi_82 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_13, \]
\[ \Psi_9 = \begin{bmatrix} \Psi_91 & \Psi_92 \end{bmatrix} h_2^{-1} S_1 + \bar{P}_12, \]
and using the Schur complement, we can obtain the following theorem.

Table I

<table>
<thead>
<tr>
<th>Methods</th>
<th>( h_1 )</th>
<th>0</th>
<th>0.5</th>
<th>0.8</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>He et al. [6]</td>
<td>( h_2 )</td>
<td>1.2817</td>
<td>1.4407</td>
<td>1.5719</td>
<td>1.6626</td>
<td>2.1071</td>
</tr>
</tbody>
</table>

Corollary 1

Example 2. Consider the following system [3]:
\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t). \]
TABLE II
ALLOWABLE UPPER BOUND OF $h_2$ WITH $h_1 = 0$

<table>
<thead>
<tr>
<th>Methods</th>
<th>Allowable upper bound of $h_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yue et al. [3]</td>
<td>0.8871</td>
</tr>
<tr>
<td>Yue et al. [4]</td>
<td>0.8695</td>
</tr>
<tr>
<td>Naghshtabrizi et al. [5]</td>
<td>0.8695</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>1.0081</td>
</tr>
</tbody>
</table>

For this example, we employ the same feedback controller as in [3], that is, $K = [\begin{array}{cc} -3.75 & -11.5 \end{array}]$. It is found that the maximum allowable value of $h_2$ with $h_1 = 0$ can be 1.0081 by Theorem 1. For convenience of comparison, the allowable upper bounds of $h_2$ obtained by various methods are listed in Table 2. For the case of $h_2 - h_1 = 0.3$, we find that the maximum allowable value of $h_1$ is 0.7501 by Corollary 1, and corresponding maximum allowable value of $h_1$ was 0.6916 in [3].

Next, we consider the effect of the external perturbation on the system. Just as shown in [3], (34) can be expressed as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \omega(t),$$

$$z(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + 0.1 u(t).$$

(35)

When $h_2 = 0.8695$, we find that the minimum allowable $H_{\infty}$ norm bound $\gamma_{\min}$ is 1.00 with $h_1 = 0$, and the $H_{\infty}$ norm bound $\gamma_{\min}$ is 0.85 with $h_1 = 0.5695$ by Theorem 1, while corresponding values of $\gamma_{\min}$ were 6.82 and 1.26 in [3], respectively.

**Example 3.** Consider the following uncertain system controlled over a network:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & -0.5 \\ 1 & -0.5 & 0 \end{bmatrix} x(t) + \Delta A(t) x(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega(t),$$

$$z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + 0.1 u(t),$$

(36)

where $||\Delta A(t)|| \leq 0.01$, $u(t) = K x(t - d(t))$.

This example was also used in [3], where $h_1 = 0.1$, and the $H_{\infty}$ norm bound $\gamma_{\min}$ was 1.9 when $h_2 = 0.5$. Using Theorem 4 with $U_1 = 1.2 I$, $U_3 = U_4 = U_5 = 0$, it is found that, the $H_{\infty}$ norm bound $\gamma_{\min}$ is 1.7 for $h_2 = 0.5$. For convenience, supposing $U_3 = U_4 = U_5 = \lambda$, where $\lambda$ is a scalar, by applying a numerical optimization algorithm which is similar to the one in [11], it yields that

$$U_1 = \begin{bmatrix} 1.3323 & -0.1090 & -0.0455 \\ 0.0152 & 1.2057 & -0.0118 \\ 0.2177 & -0.2262 & 1.0809 \end{bmatrix}, \quad \lambda = -0.0166,$$

and $\gamma_{\min}$ is 1.6, $\bar{M}_2$ and $\bar{Q}$ are given by

$$\bar{M}_2 = \begin{bmatrix} -4.7194 & -3.3291 & 4.1586 \\ 3.7084 & -3.3387 & -2.6421 \\ 1.5100 & 0.9831 & -3.2380 \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} -3.0629 & 1.5339 & 3.7297 \end{bmatrix},$$

respectively. Thus, the state feedback gain is given by

$$K = \begin{bmatrix} -0.6177 \\ -0.0048 \\ -1.4414 \end{bmatrix}.$$

**V. CONCLUSIONS**

In this paper, a new type of Lyapunov functionals is exploited to derive sufficient conditions for guaranteeing the robust exponential stability and $H_{\infty}$ performance of the continuous-time networked control systems (NCSs). A method of eliminating redundant variables to reduce computation complexity is given, and it is shown that the new result is less conservative than the existing corresponding ones. A robust $H_{\infty}$ controller design method is also presented. Numerical examples are given to illustrate the effectiveness of the proposed methods.

**REFERENCES**


