Unscented Filtering for Equality-Constrained Nonlinear Systems

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Abstract—This paper addresses the state-estimation problem for nonlinear systems in a context where prior knowledge, in addition to the model and the measurement data, is available in the form of an equality constraint. Three novel suboptimal algorithms based on the unscented Kalman filter are developed, namely, the equality-constrained unscented Kalman filter, the projected unscented Kalman filter, and the measurement-augmented unscented Kalman filter. These methods are compared on two examples: a quaternion-based attitude estimation problem and an idealized ﬂow model involving conserved quantities.

I. INTRODUCTION

Under Gaussian noise and linear dynamics assumptions, the equality-constrained Kalman filter (ECKF) [19] is the optimal solution to the equality-constrained state-estimation problem. ECKF takes advantage of prior knowledge of the state vector provided by an equality constraint and uses this information to obtain better estimates than would be provided by the classical Kalman ﬁlter (KF) [9] in the absence of such information.

Although it is difﬁcult to make corresponding precisely statements in the case of nonlinear systems, the same principles and objectives still apply. For example, in undamped mechanical systems, such as a system with Hamiltonian dynamics, conservation laws hold. In the quaternion-based attitude estimation problem, the attitude vector must have unit norm [4]. Additional examples arise in optimal control [6], parameter estimation [1, 22], and navigation [23].

However, the solution to the equality-constrained state-estimation problem for nonlinear systems is complicated by the fact that the random variables are not completely characterized by their ﬁrst-order and second-order moments [5]. Thus, suboptimal solutions based on the extended KF (EKF) [9] are generally used. One of the most popular techniques is based on measurement augmentation, in which a perfect “measurement” of the constrained quantity is appended to the physical measurements [22, 23]. In addition, the estimate-projection method [17] has also been considered. A two-step projection algorithm for handling nonlinear equality constraints is presented in [10].

In this context, two contributions are presented in this paper. First, we present and compare three suboptimal algorithms for state estimation for equality-constrained nonlinear systems, namely, the equality-constrained unscented KF (ECUKF), the projected unscented KF (PUKF), and the measurement-augmented unscented KF (MAUKF). These methods, which extend algorithms for equality-constrained state estimation developed for linear systems [19], are based on the unscented Kalman ﬁlter (UKF) [11, 12], which is a specialized sigma-point Kalman ﬁlter (SPKF) [21]. Recent work [7, 12, 14, 21] reports the improved performance of SPKFs compared to EKF, which is prone to numerical problems such as sensitivity to initialization, divergence, and instability for strongly nonlinear systems [16]. A quaternion-based attitude estimation problem [4] is addressed to illustrate ECUKF, PUKF, MAUKF, and UKF. Although the state of the process model satisﬁes the unit norm constraint, this constraint is violated by the state estimates obtained from unconstrained Kalman ﬁltering [4].

Second, we use equality-constrained Kalman ﬁltering techniques to improve estimation when an approximate discretized model is used to represent a continuous-time process. According to [15], constraints can also be used to correct model error. The problem of using UKF with discrete-time models obtained from black-box identiﬁcation to perform state estimation for continuous-time nonlinear systems is treated in [2]. We illustrate the application of equality-constrained unscented Kalman ﬁlter techniques to this problem through an example of a one-dimensional inviscid and compressible hydrodynamic model discretized by means of a ﬁnite-volume scheme [3]. The boundary conditions are chosen such that density and energy are conserved, and this knowledge is used to improve the state estimates. A detailed version of this paper appear as [18, 20].

II. STATE ESTIMATION FOR NONLINEAR SYSTEMS

For the stochastic nonlinear discrete-time dynamic system

\begin{align}
  x_k &= f(x_{k-1}, u_{k-1}, w_{k-1}, k - 1), \\
  y_k &= h(x_k) + v_k,
\end{align}

where \( f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{N} \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m \) are, respectively, the process and observation models, the state estimation problem can be described as follows. Assume that, for all \( k \geq 1 \), the known data are the measurements \( y_k \in \mathbb{R}^m \), the inputs \( u_{k-1} \in \mathbb{R}^p \), and the probability density functions (PDFs) \( \rho(x_0), \rho(w_{k-1}) \) and \( \rho(v_k) \), where \( x_0 \in \mathbb{R}^n \) is the initial state vector, \( w_{k-1} \in \mathbb{R}^q \) is the process noise, and
\( v_k \in \mathbb{R}^m \) is the measurement noise. Next, define the profit function
\[
J(x_k) \triangleq \rho(x_k|(y_1, \ldots, y_k)),
\]
which is the conditional PDF of the state vector \( x_k \in \mathbb{R}^n \) given the past and present measured data \( y_1, \ldots, y_k \). Under the stated assumptions, the maximization of (2.3) is the state estimation problem, while the maximizer of \( J \) is the optimal state estimate.

However, the solution to this problem is complicated by the fact that, for nonlinear systems, \( \rho(x_k|(y_1, \ldots, y_k)) \) is not completely characterized by its mean \( \bar{x}_{k|k} \) and covariance \( P_{x|k|k} \). We thus use an approximation based on the classical Kalman filter (KF) for linear systems [9] to provide a suboptimal solution to the nonlinear case, specifically, the unscented Kalman filter (UKF) [11]. To accomplish that, UKF propagates only approximations to \( \bar{x}_{k|k} \) and \( P_{x|k|k} \) using the initial mean \( \bar{x}_{0|0} \) and covariance \( P_{x|k|0} \triangleq \mathbb{E} \left[ (x_0 - \bar{x}_{0|0}) (x_0 - \bar{x}_{0|0})^T \right] \) [5]. We thus use for an approximation based on the classical Kalman filter (KF) for linear systems [9] to provide a suboptimal solution to the nonlinear case, specifically, the unscented Kalman filter (UKF) [11]. To accomplish that, UKF propagates only approximations to \( \bar{x}_{k|k} \) and \( P_{x|k|k} \) using the initial mean \( \bar{x}_{0|0} \) and covariance \( P_{x|k|0} \triangleq \mathbb{E} \left[ (x_0 - \bar{x}_{0|0}) (x_0 - \bar{x}_{0|0})^T \right] \) of \( \rho(x_0) \).

We assume that the maximizer of \( J \) is \( \bar{x}_{k|k} \). Furthermore, we assume that the mean and covariance of \( \rho(w_{k-1}) \) and \( \rho(v_k) \) are known and equal to zero and \( Q_{k-1}, R_k \), respectively. Also, \( w_{k-1} \) and \( v_k \) are assumed to be uncorrelated.

### A. Unscented Kalman Filter

Instead of analytically linearizing (2.1)-(2.2) and using the KF equations [9], UKF employs the unscented transform (UT) [12], which approximates the posterior mean \( \bar{y} = h(x) \) and covariance \( P_{yy} \in \mathbb{R}^{m \times m} \) of a random vector \( y \) obtained from the nonlinear transformation \( y = h(x) \), where \( x \) is a prior random vector whose mean \( \mu \in \mathbb{R}^n \) and covariance \( P_{x|x} \in \mathbb{R}^{n \times n} \) are assumed known. UT yields the actual mean \( \bar{y} \) and the actual covariance \( P_{yy} \) if \( h = h_1 + h_2 \), where \( h_1 \) is linear and \( h_2 \) is quadratic [12]. Otherwise, \( \bar{y} \) is a pseudo mean and \( P_{yy} \) is a pseudo covariance.

UT is based on a set of deterministically chosen vectors known as sigma points. To capture the mean \( \bar{x}_{k|k-1} \) of the augmented state vector \( x_{k-1} | \mathbf{X}_{k}^{-1} \) of the form
\[
X_{k-1|k}^{-1} = \begin{bmatrix} x_{k-1|k-1} & w_{k-1|k-1} \end{bmatrix},
\]
where \( x_{k-1|k-1} \in \mathbb{R}^n \) and \( n_a \triangleq n + q \), as well as the augmented error covariance \( P_{x|k-1|k}^{-1} = \begin{bmatrix} P_{x|k-1|k}^{-1} & 0_{n \times q} \\ 0_{q \times n} & Q_{k-1} \end{bmatrix} \), the sigma-point matrix \( \mathbf{X}_{k-1|k}^{-1} \in \mathbb{R}^{n \times (2n+1)} \) is chosen as
\[
\mathbf{X}_{k-1|k}^{-1} = \begin{bmatrix} x_{k-1|k-1}^{-1} & \sqrt{(n_a + \lambda)} \\ 0_{(n_a + \lambda) \times 1} & \sqrt{(n_a + \lambda)} \end{bmatrix} \times \begin{bmatrix} 1 \\ 0_{(n_a + \lambda)} \end{bmatrix}^{-1/2},
\]
where
\[
\begin{align*}
\chi_0^{(m)} & \triangleq \frac{\lambda}{n_a + \lambda}, \\
\chi_i^{(c)} & \triangleq \frac{1}{n_a + \lambda} + 1 - \alpha^2 + \beta, \\
\chi_i^{(m)} & \triangleq \frac{1}{n_a + \lambda} \chi_i^{(c)}, \quad \gamma_i^{(c)} \triangleq \frac{\chi_i^{(m)}}{2(n_a + \lambda)}, \quad i = 1, \ldots, n_a,
\end{align*}
\]
with weights
\[
\begin{align*}
\lambda & \triangleq \kappa, \\
\alpha & \triangleq \kappa + \beta, \\
\beta & \triangleq \kappa + \gamma.
\end{align*}
\]
where \( (\cdot)^{1/2} \) is the Cholesky square root, \( 0 \leq \alpha \leq 1, \beta \geq 0, \kappa \geq 0, \) and \( \lambda \triangleq \alpha^2(n + n_a) - n_a > -n_a \). We set \( \alpha = 1 \) and \( \kappa = 0 \) [7] such that \( \lambda = 0 \) [11] and set \( \beta = 2 \) [7]. Alternative schemes for choosing sigma points are given in [11]. The notation \( \bar{x}_{k|k-1} \) indicates an estimate of \( x_k \) at time \( k \) based on information available up to and including time \( k - 1 \). Likewise, \( \bar{x}_{k|k} \) indicates an estimate of \( x_k \) at time \( k \) using information available up to and including time \( k \).

The UKF forecast equations are given by (2.4)-(2.5) together with
\[
\begin{align*}
\bar{x}_{k|k-1} & \triangleq \sum_{i=0}^{2n_a} \chi_i^{(m)} X_{k|k-1}^{-1}, \\
\bar{y}_{k|k} & \triangleq \sum_{i=0}^{2n_a} \chi_i^{(c)} X_{k|k-1}^{-1} - \bar{x}_{k|k-1}^T, \\
Y_{i,k|k-1} & \triangleq \bar{y}_{k|k-1} - \bar{y}_{k|k-1}^T + R_k, \quad i = 0, \ldots, 2n_a, \\
\bar{y}_{k|k-1} & \triangleq \sum_{i=0}^{2n_a} \chi_i^{(c)} Y_{i,k|k-1}, \\
P_{\bar{y}y|k|k-1} & \triangleq \sum_{i=0}^{2n_a} \chi_i^{(c)} Y_{i,k|k-1}^T - \bar{y}_{k|k-1}^T, \\
X_{k|k-1}^{-1} & \triangleq \bar{x}_{k|k-1}^{-1} - \bar{x}_{k|k-1}^{-1} - \bar{y}_{k|k-1}^T + R_k, \\
P_{x|k-1|k}^{-1} & \triangleq \sum_{i=0}^{2n_a} \chi_i^{(c)} |Y_{i,k|k-1}| - \bar{y}_{k|k-1}^T.
\end{align*}
\]
where \( \mathbf{X}_i \) is the \( i \)-th column of \( \mathbf{X} \), \( \mathbf{X}_{k|k-1}^{-1} \in \mathbb{R}^{n \times (2n+1)} \), \( \mathbf{X}_{k|k-1}^{-1} \in \mathbb{R}^{n \times (2n+1)} \), \( P_{x|k|k}^{-1} \) is the forecast error covariance, \( P_{x|k|k}^{-1} \) is the innovation covariance, \( P_{x|k|k}^{-1} \) is the cross covariance, and \( P_{x|k|k}^{-1} \) is the data-assimilation error-covariance.

The data-assimilation equations are given by
\[
\begin{align*}
K_k & = P_{x|k|k}^{-1} (P_{k|k}^{-1})^{-1}, \\
\bar{x}_{k|k} & = \bar{x}_{k|k-1} + K_k (\bar{y}_{k|k} - \bar{y}_{k|k-1}), \\
P_{x|k|k}^{-1} & = P_{x|k|k}^{-1} - K_k P_{k|k}^{-1} K_k^T,
\end{align*}
\]
where \( K_k \in \mathbb{R}^{n \times m} \) is the Kalman gain matrix. Model information is used during the forecast step, while measurement data are injected into the estimates during the data-assimilation step.

### III. State Estimation for Equality-Constrained Nonlinear Systems

Assume that, for all \( k \geq 1 \), the state vector \( x_k \) satisfies the equality constraint
\[
g(x_k, k-1) = d_{k-1},
\]
where \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \) and \( d_{k-1} \in \mathbb{R}^p \) are assumed known. Then, the objective of the equality-constrained state-estimation problem is to maximize (2.3) subject to (3.1). That is, we look for the maximizer of \( J \) that satisfies (3.1).

In addition to nonlinear dynamics, the solution to this problem is complicated due the inclusion of (3.1). We thus extend algorithms derived in the linear scenario to provide a suboptimal solution to the nonlinear case.

### IV. Kalman Filters for Equality-Constrained Linear Systems

In this section, we briefly review three state-estimation algorithms for equality-constrained linear systems. For details the reader is referred to [19].
Assume that $f$, $h$, and $g$ are linear functions, respectively,
given by
\[ x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}w_{k-1}, \]  
\[ y_k = C_kx_k + v_k, \]  
\[ D_{k-1}x_k = d_{k-1}, \]  
where $A_{k-1} \in \mathbb{R}^{n \times n}$, $B_{k-1} \in \mathbb{R}^{n \times p}$, $G_{k-1} \in \mathbb{R}^{n \times q}$, $C_k \in \mathbb{R}^{m \times n}$, and $D_{k-1} \in \mathbb{R}^{r \times n}$. Then, the recursive solution $\hat{x}_{k|k-1}$ to the equality-constrained state-estimation problem is given by the equality-constrained Kalman filter (ECKF) [19], whose forecast step is given by
\[ \hat{x}_{k|k-1} = A_{k-1}\hat{P}^x_{k-1|k-1} + B_{k-1}u_{k-1}, \]  
\[ P^{xx}_{k|k-1} = A_{k-1}P^{xx}_{k-1|k-1}A^{T}_{k-1} + G_{k-1}Q_{k-1}G^{T}_{k-1}, \]  
\[ \tilde{y}_{k|k-1} = C_k\hat{x}_{k|k-1}, \]  
\[ P^{yy}_{k|k-1} = C_kP^{xx}_{k|k-1}C^{T}_k + R_k, \]  
\[ P^{yy}_{k|k-1} = P^{xx}_{k|k-1} + C_kP^{xx}_{k|k-1}C^{T}_k, \]  
where $P^{xx}_{k|k-1} \triangleq \mathbb{E} \left[ (x_k - \hat{x}^P_{k|k})(x_k - \hat{x}^P_{k|k})^{T} \right]$, whose data-assimilation step is given by (2.13)-(2.15), and whose projection step is given by
\[ d_{k-1|k} = D_{k-1}\hat{x}_{k|k-1}, \]  
\[ P^{dd}_{k|k} = D_{k-1}P^{xx}_{k-1|k-1}D^{T}_{k-1}, \]  
\[ P^{dd}_{k|k} = P^{dd}_{k|k}D_{k-1}, \]  
\[ K^P_k = P^{dd}_{k|k}D^{T}_{k-1}, \]  
\[ \hat{x}^P_{k|k} = \hat{x}_{k|k} + K^P_k(d_{k-1|k} - \hat{d}_{k-1|k}), \]  
\[ P^{xx}_{k|k} = P^{xx}_{k|k} - K^P_kP^{dd}_{k|k}K^{T}_k, \]  
Note that the projection step is absent in the KG algorithm.

For all $k \geq 1$, the projected Kalman filter by estimate projection (PKF-EP) [17] projects $\hat{x}_{k|k}$ given by (2.14) onto the hyperplane (4.3) yielding the projected estimate $\hat{x}^P_{k|k}$. However, unlike ECKF, $\hat{x}^P_{k|k}$ of PKF-EP is not recursively fed back in the next iteration. That is, PKF-EP equations are equal to ECKF equations except for (4.4)-(4.5), which are replaced by
\[ \hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1}, \]  
\[ P^{xx}_{k|k-1} = A_{k-1}P^{xx}_{k-1|k-1}A^{T}_{k-1} + G_{k-1}Q_{k-1}G^{T}_{k-1}, \]  
Finally, the measurement–augmentation Kalman filter (MAKF) [23] treats (4.3) as perfect measurements. That is, we replace (4.2) by the augmented observation
\[ \hat{y}_k \triangleq \hat{C}_kx_k + \tilde{v}_k, \]  
where $\hat{y}_k \triangleq \begin{bmatrix} \hat{y}_k \\ d_{k-1} \end{bmatrix}$, $\hat{C}_k \triangleq \begin{bmatrix} C_k \\ D_{k-1} \end{bmatrix}$, and $\tilde{v}_k \triangleq \begin{bmatrix} v_k \\ 0_{n \times 1} \end{bmatrix}$. With (4.17), MAKF uses (4.15)-(4.16) together with the augmented forecast equations
\[ \hat{\tilde{y}}_{k|k-1} = \hat{C}_k\tilde{x}_{k|k-1}, \]  
\[ \tilde{P}^{yy}_{k|k-1} = \hat{C}_kP^{xx}_{k|k-1}\hat{C}^{T}_k + \tilde{R}_k, \]  
\[ \tilde{P}^{yy}_{k|k-1} = P^{xx}_{k|k-1}\hat{C}^{T}_k, \]  
where $\tilde{R}_k \triangleq \begin{bmatrix} R_k & 0_{n \times m} \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix}$, and the augmented data-assimilation equations given by
\[ \tilde{K}_k = \tilde{P}^{yy}_{k|k-1}\left(\tilde{P}^{yy}_{k|k-1}\right)^{-1}, \]  
\[ \tilde{x}_{k|k} = \hat{x}_{k|k} + \tilde{K}_k(\tilde{y}_k - \tilde{\tilde{y}}_{k|k-1}), \]  
\[ \tilde{P}^{xx}_{k|k} = \tilde{P}^{xx}_{k|k-1} - \tilde{K}_k\tilde{P}^{yy}_{k|k-1}\tilde{K}^{T}_k, \]  
For the linear case, MAKF and ECKF estimates are equal [19,20].

V. EQUALITY-CONSTRAINED UKFs

In this section, by using UT, we extend PKF-EP, ECKF, and MAKF to the nonlinear case. Unlike the linear case (Section IV), these approaches do not guarantee that the nonlinear equality constraint (3.1) is exactly satisfied by the state estimates, but they provide approximate solutions.

A. Equality-Constrained Unconstrained Kalman Filter

The projection step of ECKF given by (4.9)-(4.14) is now extended to the nonlinear case by means of UT.

Using (2.4)-(2.5) to choose sigma points and associated weights, we have
\[ x_{i,k} = \hat{x}_{i,k|k} + 1 \times (2n+1) + \sqrt{(n + \lambda)} \begin{bmatrix} 0_{n \times 1} \\ \left( P^{dd}_{k|k}\right)^{1/2} - \left( P^{dd}_{k|k}\right)^{1/2} \end{bmatrix}, \]  
where $\hat{x}_{i,k}$ and $P^{dd}_{k|k}$ are given by (2.14) and (2.15), and with weights given by (2.5), replacing $n_h$ by $n$. Then the sigma points $x_{i,k} \in \mathbb{R}^n$, $i = 0, \ldots, 2n$, are propagated through (3.1) yielding
\[ D_{i,k|k} = g(X_{i,k|k} - 1), \quad i = 0, \ldots, 2n, \]  
not such that $\hat{d}_{k-1|k}, P^{dd}_{k|k}$, and $P^{dd}_{k|k}$ are given by
\[ \hat{d}_{k-1|k} = \sum_{i=0}^{2n} \alpha_i (x_{i,k|k} - 1), \]  
\[ P^{dd}_{k|k} = \sum_{i=0}^{2n} \gamma_i (x_{i,k|k} - 1)(x_{i,k|k} - 1)^{T}, \]  
and $K^P_k$, $\hat{x}^P_{k|k}$, and $P^{xx}_{k|k}$ are respectively given by (4.12), (4.13), and (4.14).

Now define $\hat{x}_{k|k}^{ap} \triangleq \begin{bmatrix} \hat{x}_{k|k} \\ w_{k|k-1} \end{bmatrix}$, and $P^{xx}_{k-1|k-1} \triangleq \begin{bmatrix} P^{xx}_{k-1|k-1} & 0_{n \times q} \\ 0_{q \times n} & Q_{k-1} \end{bmatrix}$, such that the sigma points
\[ x_{i,k|k-1} = \hat{x}_{i,k|k-1} + 1 \times (2n+1) + \sqrt{(n + \lambda)} \begin{bmatrix} 0_{n \times 1} \\ \left( P^{dd}_{k|k}\right)^{1/2} - \left( P^{dd}_{k|k}\right)^{1/2} \end{bmatrix}, \]  
are chosen based on $\hat{x}^P_{k-1|k-1}$ (4.13). Then, by appending the projection step given by (5.1)-(5.5), (4.12)-(4.14) to (5.6), (2.5)-(2.15), we obtain the equality-constrained unscented Kalman filter (ECUKF).

B. Projected Unscented Kalman Filter

If, unlike ECUKF, we append the projection step given by (5.1)-(5.5),(4.12)-(4.14) to UKF equations given by (2.4)-(2.15) without feedback recursion, we obtain the projected unscented Kalman filter (PUKF). PUKF is the nonlinear extension of PKF-EP.
C. Measurement-Augmentation Unscented Kalman Filter

To extend the MAKF to the nonlinear case, we replace (2.2) by the augmented observation

\[ \hat{y}_k \triangleq h(x_k, k) + \nu_k, \quad (5.7) \]

where \( \hat{y}_k \) is the measurement-augmented unscented Kalman filter (MAUKF) combines (2.4)-(2.8) with the augmented forecast equations

\[ \hat{y}_k|k-1 = \sum_{i=0}^{2n_s} \gamma_i (\hat{y}_k|k-1 - \hat{y}_k|k-1) \]

and the data-assimilation equations given by (4.21)-(4.23).

VI. NUMERICAL EXAMPLES

A. Attitude Estimation

Consider an attitude estimation problem [4], whose kinematics is modeled as

\[ \dot{e}(t) = -\frac{1}{2} \Omega(t)e(t), \quad (6.1) \]

where the state vector is the quaternion vector \( e(t) = [e_0(t) \ e_1(t) \ e_2(t) \ e_3(t)]^T \), the matrix \( \Omega(t) \) is given by

\[ \Omega(t) = \begin{bmatrix} 0 & -r(t) & p(t) & q(t) \\ r(t) & 0 & -q(t) & p(t) \\ -p(t) & q(t) & 0 & -r(t) \\ -q(t) & -p(t) & r(t) & 0 \end{bmatrix}, \quad (6.2) \]

and the angular velocity vector \( u(t) = [p(t) \ q(t) \ r(t)]^T \) is a known input. Since \( ||e(0)||_2 = 1 \) and \( \Omega(t) \) is a skew-symmetric matrix, it follows that, for all \( t > 0 \),

\[ ||e(t)||_2 = 1. \quad (6.3) \]

We set \( e(0) = [0.9603 \ 0.1387 \ 0.1981 \ 0.1387]^T \), and \( u(t) = [0.03 \sin (\frac{2\pi}{3}) \ 0.03 \sin (\frac{2\pi}{3} t - 300) \ 0.03 \sin (\frac{2\pi}{3} t - 600)]^T \).

To perform attitude estimation, we assume that

\[ u_k = u((k-1)T) \beta_k + w_k \quad (6.4) \]

is measured by rate gyro, where \( T \) is the discretization step, \( w_k \in \mathbb{R}^3 \) is zero mean, Gaussian noise, and \( \beta_k \in \mathbb{R}^3 \) is drift error. The discrete-time equivalent of (6.1) augmented by the gyro drift random-walk model [4] is given by

\[ e_k = A_k e_{k-1} + 0_{4 \times 3} u_k \]

where \( e_k \triangleq e(kT) \), \( u_k \in \mathbb{R}^3 \) is process noise associated to drift-error model, \( x_k \triangleq e_k / \beta_k \in \mathbb{R}^3 \) is the state vector, \( w_k \in \mathbb{R}^3 \) is the process noise, and

\[ A_k \triangleq \cos(s_{k-1}) I_{4 \times 4} - \frac{1}{2} \frac{T}{s_{k-1}} \sin(s_{k-1}) \Omega_k, \quad (6.6) \]

We assume that two directions are available [4, 13], which can be obtained using either a star tracker or a combined three-axis accelerometer/three-axis magnetometer. We set \( r^{[1]} = [1 \ 0 \ 0]^T \), \( r^{[2]} = [0 \ 1 \ 0]^T \), and \( \hat{R}_k = 10^{-4} \hat{I}_{6 \times 6} \). These direction measurements are assumed to be provided at a lower rate, specifically, at 1 Hz, which corresponds to a sample interval of 10T s.

We implement Kalman filtering using UKF, ECUKF, PUKF, and MAUKF with (6.5), (6.10), and constraint (6.9). We initialize these algorithms with \( \hat{x}_0 = [1 \ 0 \ 0 \ 0 \ 0 \ 0]^T \) and \( P_{0|x} = \text{diag}(0.5 I_{3 \times 3} \ 0.01 I_{3 \times 3}) \). Table 1 shows the results obtained from a 100-run Monte Carlo simulation. Note that, though the state vector of the model (6.5) satisfies (6.9), UKF estimates do not satisfy (6.9). Nevertheless, with the usage of prior knowledge, more informative (smaller trace of error covariance) estimates are produced compared to the unconstrained estimates given by UKF. However, a slight increase in the root-mean-square (RMS) error is

![Table 1: Average of percent RMS constraint error, trace of error covariance matrix, and RMS estimation error for 100-run Monte Carlo simulation for attitude estimation using UKF, ECUKF, PUKF, and MAUKF.](image)

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\[ s_{k-1} = \frac{T}{2} ||\hat{u}_{k-1} - \beta_{k-1} - w_{k-1}||_2, \quad (6.7) \]

\[ \Omega_{k-1} = \Omega((k-1)T), \quad (6.8) \]

The constraint (6.3) also holds for \( t = kT \), that is,

\[ x^T_{1,k} + x^T_{2,k} + x^T_{3,k} + x^T_{4,k} = 1. \quad (6.9) \]

We set \( T = 0.1 \) s, \( \beta_{k-1} = [-0.001 -0.001 0.0005]^T \) rad/s, and \( Q_k = \text{diag}(10^{-5} I_{3 \times 3} \ 10^{-10} I_{3 \times 3}) \). Attitude observations \( y_k \in \mathbb{R}^3 \) for a direction sensor are given by [4]

\[ [i] = C_{k}[i] + v_i, \quad (6.10) \]

where \( v[i] \in \mathbb{R}^3 \) is a reference direction vector to a known point, and \( C_k \) is the rotation matrix from the reference to the body-fixed frame

\[ C_k = \begin{bmatrix} 2(x^2_{1,k} - x^2_{2,k} - x^2_{3,k} + x^2_{4,k}) & 2(x_{1,k} x_{2,k} - x_{3,k} x_{4,k}) & -2(x^2_{1,k} + x^2_{2,k} + x^2_{4,k}) \\ 2(x_{1,k} x_{3,k} - x_{2,k} x_{4,k}) & 2(x_{1,k} x_{2,k} + x_{3,k} x_{4,k}) & -2(x_{1,k} x_{2,k} + x_{3,k} x_{4,k}) \\ 2(x_{1,k} x_{4,k} + x_{2,k} x_{3,k}) & 2(x_{1,k} x_{2,k} + x_{3,k} x_{4,k}) & -2(x^2_{1,k} + x^2_{2,k} + x^2_{3,k}) \end{bmatrix}, \quad (6.11) \]

We assume that two directions are available [4, 13], which can be obtained using either a star tracker or a combined three-axis accelerometer/three-axis magnetometer. We set \( r^{[1]} = [1 \ 0 \ 0 \ 0 \ 0 \ 0]^T \), \( r^{[2]} = [0 \ 1 \ 0 \ 0 \ 0 \ 0]^T \), and \( \hat{R}_k = 10^{-4} \hat{I}_{6 \times 6} \). These direction measurements are assumed to be provided at a lower rate, specifically, at 1 Hz, which corresponds to a sample interval of 10T s.

We implement Kalman filtering using UKF, ECUKF, PUKF, and MAUKF with (6.5), (6.10), and constraint (6.9). We initialize these algorithms with \( \hat{x}_0 = [1 \ 0 \ 0 \ 0 \ 0 \ 0]^T \) and \( P_{0|x} = \text{diag}(0.5 I_{3 \times 3} \ 0.01 I_{3 \times 3}) \). Table 1 shows the results obtained from a 100-run Monte Carlo simulation. Note that, though the state vector of the model (6.5) satisfies (6.9), UKF estimates do not satisfy (6.9). Nevertheless, with the usage of prior knowledge, more informative (smaller trace of error covariance) estimates are produced compared to the unconstrained estimates given by UKF. However, a slight increase in the root-mean-square (RMS) error is

![Table 1: Average of percent RMS constraint error, trace of error covariance matrix, and RMS estimation error for 100-run Monte Carlo simulation for attitude estimation using UKF, ECUKF, PUKF, and MAUKF.](image)
observed for algorithms ECUKF and MAUKF implying loss of accuracy around 5% compared to UKF. The equality constraint is more closely tracked whenever a constrained filter is employed; see Figure 1. Percent errors around $10^{-4}$ are obtained and ECUKF exhibits the smallest error.

**B. One-Dimensional Hydrodynamics**

We consider state estimation for one-dimensional hydrodynamic flow. The flow of an inviscid, compressible fluid along a one-dimensional channel is governed by Euler’s equations [8]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z} (\rho v) = 0,
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial}{\partial z} (\rho v^2) = -\frac{\partial p}{\partial z},
\]

\[
\frac{\partial E}{\partial t} + \frac{\partial}{\partial z} (\rho v E) = \frac{\partial}{\partial z} (p v),
\]

where $\rho \in \mathbb{R}$ is the density, $v \in \mathbb{R}$ is the velocity, $p \in \mathbb{R}$ is the pressure of the fluid, $z \in \mathbb{R}$ is the spatial coordinate, and $\gamma = \frac{5}{3}$ is the ratio of specific heats of the fluid. A discrete-time model of hydrodynamic flow is obtained by using finite-volume-based spatial and temporal discretization [3].

Assume that the channel consists of $l$ identical cells (see Figure 2). For $i = 1, \ldots, l$, let $\rho_i$, $v_i$, and $p_i$ be the density, velocity, and pressure in the $i$th cell, and define $U_i = [\rho_i, v_i, E_i]^T \in \mathbb{R}^3$, where $m_i = \rho_i v_i$ is the momentum and $E_i = \frac{1}{2} (\rho_i v_i)^2 + \frac{p_i}{\gamma - 1}$ is the energy in the $i$th cell. We use a second-order Rusanov scheme [8] to discretize (6.12)-(6.14) and obtain a discrete-time model that updates the flow variables at the center of each cell.

The discrete-time update equation [8] is given by

\[
U_{k+1} = U_k - \frac{T}{\Delta z} \left[ F_{Rus,k} - F_{Rus,k-1} \right],
\]

where $T > 0$ is the sampling time, $\Delta z$ is the width of each cell, and $F_{Rus,k}$ is a nonlinear function of $U_{k-1}^{[i-1]}, \ldots, U_{k-1}^{[i+2]}$ whose equation is in [8]. Hence, $U_{k+1}^{[i]}$ depends on $U_k^{[i-1]}, \ldots, U_k^{[i+2]}$, as expected for a spatially second-order scheme.

Next, define the state vector $x_k \triangleq \left[ (U_k^{[3]})^T, \ldots, (U_k^{[l-2]})^T \right]^T \in \mathbb{R}^{3(l-4)}$. Furthermore, we assume reflective boundary conditions at cells 1, 2, $l-1$, and $l$ so that these mimic walls reflecting incident waves. Let $l = 54$ so that $n = 3(l - 4) = 150$. It follows from (6.15) that the second-order Rusanov scheme yields a nonlinear discrete-time update model of the form $x_k = f(x_{k-1})$, where $f : \mathbb{R}^n \times \mathbb{R}^n$. To account for disturbances, we assume that the truth model is given by $x_k = f(x_{k-1}) + Gw_{k-1}$, which is a special case of (2.1), where $w_{k-1} = 0_{p \times 1}$, $w_{k-1} \in \mathbb{R}$, $Q_{k-1} = I_{6 \times 6}$, and $G \in \mathbb{R}^{150 \times 1}$, where

\[
\text{row}_j(G) = \begin{cases} 0.7, & \text{if } j = 7, 22, 37, 67, 97, 127, \\ 0, & \text{otherwise}. \end{cases}
\]

It follows from (2.1) and (6.16) that although the flow variables in cells 5, 10, 15, 25, 35, and 45 are directly affected by $w_{k-1}$, the total density and the energy of cells $3, \ldots, l-2$, is not altered by $w_{k-1}$, that is,

\[
\sum_{i=3}^{l-2} \rho_i = \sum_{i=2}^{l-2} \rho_i, \quad \sum_{i=3}^{l-2} E_i = \sum_{i=3}^{l-2} E_i.
\]

Consequently, the linear equality constraint in (6.17) can be expressed as (4.3) with

\[
D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}, \quad d = \left[ \sum_{i=3}^{l-2} \rho_i, \sum_{i=3}^{l-2} E_i \right].
\]

Next, for $i = 3, \ldots, l-2$, define $C_i \triangleq \begin{bmatrix} 0_{3 \times 3(i-1)} \ 1_{3 \times 3} \ 0_{3 \times 3(l-i-1)} \end{bmatrix} \in \mathbb{R}^{3 \times 3(l-4)}$, so that the linear measurement model given by (4.2), for $y_k \in \mathbb{R}^m$, $m = 6$, corresponding to density, momentum and energy at cells 24 and 26, has $C = \begin{bmatrix} (C_24)^T \ (C_26)^T \end{bmatrix}^T$ and $R = 0.01 I_{6 \times 6}$. That is, we have a nonlinear process model with additive noise, a linear observation model, and a linear equality constraint.

Next, we compare the performance of UKF, ECUKF, PUKF, and MAUKF. Note that since the constraint in (6.18)
and the observation model are linear, ECUKF and MAUKF estimates are equal [19]. The RMS estimation error at cells 1, \ldots, 54 is shown in Figure 3. The error in the energy estimates from data-free simulation is shown for comparison. Note that the performance of ECUKF, PUKF, and MAUKF is better than the performance of UKF because of the enforcement of the constant total density and energy constraint. Figure 4 shows the total density estimated by UKF, ECUKF, PUKF, and MAUKF. The actual total density of the truth model is also plotted for comparison. The total density of the estimates of ECUKF, PUKF, and MAUKF is very close to the truth model, but UKF does not conserve total density or the total energy.

VII. CONCLUDING REMARKS

We have addressed the equality-constrained state-estimation problem for nonlinear systems. Three novel nonlinear extensions of equality-constrained linear state-estimation algorithms based on the unscented Kalman filter (UKF) were presented, namely, the equality-constrained UKF (ECUKF), the projected UKF (PUKF), and the measurement-augmented UKF (MAUKF). These methods were compared on two examples, including a quaternion-based attitude estimation problem, as well as an idealized flow model involving conserved quantities.

Numerical results suggest that, in addition to exactly (for linear constraints) or very closely (nonlinear constraints) satisfying a known equality constraint of the system, the proposed methods can be used to produce more accurate and more informative estimates. Considering the examples investigated, we recommend the user to test ECUKF, MAUKF, and PUKF in this order for a given equality-constrained state-estimation application. Recall that, since these methods are approximate, their performance depends on the application. Also, all equality-constrained approaches have required similar processing time, which was competitive to UKF processing time.

Finally, we have also addressed the case where an approximate discretized model is used to represent a continuous-time process in state estimation. Improved estimates were obtained when equality-constrained Kalman filtering algorithms were employed to enforce conserved quantities of the original continuous-time model, but without the higher computational burden required by more accurate integration schemes.

REFERENCES