Nonlinear Control of the Burgers PDE—Part I: Full-State Stabilization

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Abstract—We consider the problem of stabilization of unstable “shock-like” equilibrium profiles of the viscous Burgers’ equation with actuation at the boundaries. These equilibria are not stabilizable (even locally) using the standard “radiation boundary condition.” Using a nonlinear spatially-scaled transformation (that employs three ingredients, of which one is the Hopf-Cole nonlinear integral transformation) and linear backstepping design, we design an explicit nonlinear full-state control law that achieves exponential stability, with a region of attraction for which we give an estimate. The region of attraction is not the entire state space since the Burgers equation is known not to be globally controllable, however, the stability result achieved is stronger than being infinitesimally local. In a companion paper we consider output feedback stabilization, for which we design a nonlinear observer with boundary sensing, and solve the problems of trajectory generation and tracking.

I. INTRODUCTION

We study a nonlinear control problem for the viscous Burgers equation, which is considered a basic model of nonlinear convective phenomena such as those that arise in Navier-Stokes equations. While the Burgers model is not able to capture the complexity of turbulence, its nonlinearity makes it challenging and a starting point towards developments for the more realistic Navier-Stokes equations. We consider a family of stationary solutions called “shock profiles” [9] (or “shock-like” profiles) which are unstable and not stabilizable (even locally) by simple means such as the standard “radiation boundary conditions.” We achieve exponential stabilization (in spatial $L^2$ norm) of the shock profiles using two control inputs (one at each boundary).

Early results with linear control of the Burgers equation were presented in [4], achieving local stabilization. Optimal control was considered in [7]. In [6] a linear static collocated output feedback (a.k.a. “radiation” boundary conditions) is proved to achieve local $L^2$ exponential stability. In [20] this result is improved to $L^\infty$, but remains local. Global stabilization is achieved in [13], using nonlinear boundary conditions, and extended to KdV, Kuramoto-Sivashinsky, adaptive control, and other problems in [1], [2], [15], [16], [17], [18]. References [10] and [11] study the null controllability of the Burgers equation with one and two inputs respectively, deriving bounds on the minimal time of controllability. In [3] the stabilization problem is solved using nonlinear model reduction techniques, with in-domain actuation.

Our recent design for nonlinear parabolic PDEs [22], [23] (without convective nonlinearities) is based on a feedback linearizing transformation in the form of a Volterra series and has opened the avenue for fully nonlinear designs for PDEs. In this paper we follow a conceptually similar strategy (though rather different in its execution) and find a nonlinear spatially-scaled transformation (based on three ingredients, one of which is the Hopf-Cole nonlinear transformation [8]), that transforms the system (with the help by one of the two boundary controls) into a linear reaction-diffusion PDE. This PDE is stabilized using the linear backstepping approach [19], yielding a control law that is nonlinear in the original state variable. We provide an estimate of the region of attraction for the closed-loop system. This estimate is finite because the Burgers system unfortunately is not globally controllable [10], [11].

We illustrate our theoretical results with numerical examples. It is first shown, via a study of eigenvalues and via simulations of the nonlinear system, that the “radiation feedback” is not stabilizing, even locally, for sufficiently large shock profiles. The stabilization properties of the feedback laws are illustrated in simulations.

In a companion paper [14] we pursue output feedback stabilization, for which we use a nonlinear observer that uses injection of the output estimation error at one of the boundaries. We also solve the problem of trajectory generation and tracking for the Burgers equation.

II. BURGERS EQUATION AND ITS SHOCK PROFILES

Consider the viscous Burgers’ equation

$$u_t = u_{xx} - u_x u, \quad x \in [0, 1]$$

where $u(x, t)$ is the state, with boundary conditions

$$u_x(0, t) = \omega_0(t), \quad u_x(1, t) = \omega_1(t),$$

where $\omega_0(t)$ and $\omega_1(t)$ are the control inputs. In the sequel we drop the arguments $(x, t)$ whenever the context allows to do this without harming clarity.

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ever, we concentrate on the symmetric profiles (3) as the actual size of the region of attraction but not the fundamental form of the estimate of its size. The primary effect of \( \varepsilon \) parameter, hence \( \tilde{\omega}_0(t) = \tilde{\omega}_1(t) \), is neutralizing the system (7) is also destabilizing in the vicinity of \( x = 1/2 \). This is shown in Fig. 2. The larger the value of \( \sigma > 0 \), the more positive the first eigenvalue of (12)–(14) becomes. For example, for \( \sigma = 15 \) the first eigenvalue is +0.6.

**Remark 3.1:** With “radiation boundary feedback,”
\[ \tilde{\omega}_0(t) = k\tilde{u}(0,t), \quad \tilde{\omega}_1(t) = -k\tilde{u}(1,t), \quad k > 0, \]
(15)

motivated by [6], the system (12)–(14) changes only in its boundary conditions,
\[ \zeta_x(0) = (k - \sigma \tanh(\sigma/2))\zeta(0), \]
(16)
\[ \zeta_x(1) = -(k - \sigma \tanh(\sigma/2))\zeta(1). \]
(17)
The stability properties of the system (12), (16), (17) improve as \( k \to +\infty \). However, numerical eigenvalue calculation shows that for \( \sigma > \sigma^* \), where \( \sigma^* \approx 10 \), the system always has exactly one unstable eigenvalue, so no value of \( k \) exists that stabilizes the system, see Fig. 3. The same negative result applies to the “non-linear radiation boundary feedback” in [13]. So, a more sophisticated form of feedback, either full-state feedback or dynamic output-feedback, is needed to stabilize the shock-like equilibria (even locally).

**IV. FULL STATE FEEDBACK LAW**

We perform a two-step feedback linearizing design. In the first step, we use a state transformation and a feedback law for \( \bar{\omega}_0 \) to linearize the transformed PDE and its boundary condition at \( x = 0 \). In the second step we design a feedback law for \( \bar{\omega}_1 \) to stabilize the resulting linear system using the backstepping method.

**A. Linearizing transformation and the design of \( \bar{\omega}_0 \)**

Define a new state variable \( v(x, t) \) as follows:

\[
v(x, t) = \tilde{u}(x, t)e^{-\frac{1}{2} \int_0^x \omega(y, t) + U(y) \, dy}, \tag{18}
\]
which can be written using (11) as

\[
v(x, t) = G(x)\tilde{u}(x, t)e^{-\frac{1}{2} \int_0^x \omega(y, t) \, dy}. \tag{19}
\]

**Remark 4.1:** Transformation (19) is a composition of the Hopf-Cole transformation on \( \tilde{u}(x, t) \) and the transformation \( \tilde{\omega}_x \), scaled by \( G(x) \). The former is used to study the solutions of Burgers’ equation [8]. The transformation \( \tilde{\omega}_x \) was used in [22] to design a feedback control law for a semilinear parabolic system. The scaling \( G(x) \) is the standard “gauge” transformation used to eliminate advection terms in parabolic PDEs.

The transformation (19) from \( \tilde{u} \) to \( v \) is invertible:

\[
\tilde{u}(x) = \frac{v(x)/G(x)}{1 - \frac{1}{2} \int_0^x \omega(y) \, dy}. \tag{20}
\]

Substituting the transformation (19) into the system (7)–(8) we obtain that \( v \) verifies the following equations:

\[
v_t = v_{xx} - \left(U'(x) + \sigma^2 \right) v + \frac{1}{2} \left( \bar{\omega}_0 - U(0)\tilde{u}(0) - \tilde{u}^2(0)/2 \right) v, \tag{21}
\]

\[
v_x(0) = \bar{\omega}_0 - \frac{1}{2} (\tilde{u}(0) + U(0))\tilde{u}(0), \tag{22}
\]

\[
v_x(1) = \left( \bar{\omega}_1 - \frac{1}{2} (\tilde{u}(1) + U(1))\tilde{u}(1) \right) e^{-\frac{1}{2} \int_0^1 \omega(y) + U(y) \, dy}. \tag{23}
\]

Setting the feedback law

\[
\bar{\omega}_0 = U(0)\tilde{u}(0) + \frac{\tilde{u}^2(0)}{2} = 2\sigma \tanh(\sigma/2)\tilde{u}(0) + \tilde{u}^2(0)/2, \tag{24}
\]

we obtain the following system for \( v \), in which we express all the coefficients as explicitly as possible:

\[
v_t = v_{xx} + \sigma^2 \left( \frac{2}{\cosh^2(\sigma(x - 1/2))} - 1 \right) v, \tag{25}
\]

with boundary conditions

\[
v_x(0) = \sigma \tanh(\sigma/2)v(0), \tag{26}
\]

\[
v_x(1) = \sigma \tanh(\sigma/2)v(1) + (\bar{\omega}_1 - \tilde{u}(1)^2/2) \times \left(1 - \frac{1}{2} \int_0^1 \frac{\cosh(\sigma/2)y}{\cosh(\sigma(y - 1/2))} \, dy \right). \tag{27}
\]

Notice that (25) is a linear reaction-diffusion parabolic equation with the same destabilizing reaction coefficient that the linearized system (12) had.

**B. Design of \( \bar{\omega}_1 \) using the backstepping method**

To find a feedback law for \( \bar{\omega}_1 \) that stabilizes (25)–(27), we use the backstepping method for 1-D parabolic equations [19]. We define a new state \( w(x, t) \), which is obtained from \( v \) using the transformation

\[
w(x, t) = v(x, t) - \int_0^x k(x, y)v(y, t) \, dy. \tag{28}
\]
The state \( w \) satisfies the following PDE:

\[
\begin{align*}
\dot{w}_t &= w_{xx} + \sigma^2 \left[ \frac{1}{\cosh^2(\sigma(x - 1/2))} - 1 \right] w \\
&\quad - c w, \\
\dot{w}_x(0) &= \sigma \tanh(\sigma/2) w(0), \\
\dot{w}_x(1) &= -\sigma \tanh(\sigma/2) w(1),
\end{align*}
\]

where \( c \geq 0 \) is a parameter of the controller. The system (29)–(31) will be referred to as the target system.

**Remark 4.2:** The reaction term in (29) is nonpositive, so the \( w \)-system is exponentially stable. Note that we have not completely eliminated the reaction term from the original system (25) but only lowered it to eliminate its positive part, without wasting control effort on changing its negative part, see Fig. 4.

**Remark 4.3:** The coefficient \( c \) in (29) can be set to zero whenever \( \sigma > 0 \). When \( \sigma = 0 \), we need \( c > 0 \) because the resulting system \( \dot{w} = w_{xx} - c w, \dot{w}(0, t) = w_x(1, t) = 0 \) would be only neutrally stable for \( c = 0 \).

We need to determine the kernel \( k(x, y) \) in (28) so that \( w \) verifies (29)–(31). Following (19), we find that the kernel \( k \) has to verify the following equation:

\[
\begin{align*}
k_{xx} &= k_{yy} + \sigma^2 \left[ 1 - 2 \tanh^2(\sigma(y - 1/2)) \right] + \tanh^2(\sigma(x - 1/2)) k + c k, \\
k(x, x) &= -\sigma/2 \tanh(\sigma(x - 1/2)) + \tanh(\sigma/2) - c x/2, \\
k_y(x, 0) &= \sigma \tanh(\sigma/2) k(x, 0),
\end{align*}
\]

which is a linear hyperbolic PDE in the domain \( T = \{(x, y) : 0 \leq y \leq x \leq 1\} \). In [19] it is shown that (32)–(34) is well-posed and that \( k \in C^2(T) \). The kernel \( k \) can be computed numerically or symbolically [19].

From (31), (28) and (27) we find the control law

\[
\begin{align*}
\ddot{u}_1(t) &= \frac{\ddot{u}(1, t)^2}{2} + (k(1, 1) - 2 \sigma \tanh(\sigma/2)) \ddot{u}(1, t) \\
&\quad + \int_0^1 (k_x(1, y) + \sigma \tanh(\sigma/2) k(1, y)) dy, \\
&\quad \times G(y) e^{\int_y^1 \dot{u}(\xi, t) dx} \ddot{u}(y, t) dy.
\end{align*}
\]
The nonlinearities prevent finite time blow-up (rather than causing it) and the closed-loop solutions merely converge to a different equilibrium (rather than the desired “shock profile” equilibrium). In Fig. 6 one can see the convergence of $u$ to two different non-symmetric equilibria, which seem to (nearly) match the desired shock profile at the boundaries but not elsewhere.

In simulations we find that for odd initial conditions around $x = 1/2$ radiation feedback (15) achieves convergence to $U(x)$. However, for non-symmetric initial conditions, the system goes to other stationary solutions.

C. Backstepping feedback

Before showing the results with our nonlinear design, we show the results with a linearized backstepping controller

$$\ddot{\omega}_0(t) = 2\sigma \tanh(\sigma/2) \dot{u}(0, t)$$

$$\ddot{\omega}_1(t) = -(3\sigma \tanh(\sigma/2) + c/2) \dot{u}(1, t)$$

$$- \int_0^1 \rho(y)G(y)\dot{u}(y, t)dy.$$

The kernel $-\rho(x)$ and a closed-loop solution with the controller (42), (43) are shown in Figure 7. The radiation feedback does not stabilize the desired equilibrium profile in this case, as shown in Figure 6.

In Fig. 8 we show solutions from various initial conditions under the nonlinear backstepping controller.

VII. CONCLUSIONS

We have solved the problem of full-state stabilization for the fully nonlinear viscous Burgers equation (see a companion paper [14] where we solve the problems of observer design, output feedback stabilization, trajectory generation, and tracking). Our results are based on a nonlinear feedback linearizing transformation, which allows us to use the linear backstepping control design.
method [19]. Due to the explicit nature of the transformation and our methods, we were able to derive formulas for the feedback laws. Since our transformation is not globally invertible, our results are not global, which is consistent with lack of global controllability shown in [10], [11], however we derive a non-infinitesimal region of attraction, using the largest level set of our Lyapunov function within the regions of feasibility of the control law and invertibility of the transformation.

Particularly interesting problems for future research include a possible design using only one control input, \( \hat{\omega}_1 \) (with \( \hat{\omega}_0 \) set to zero), an extension to more general nonlinear parabolic PDEs with convective nonlinearities and to convective nonlinearities of more general form, to Burgers equations in higher dimensions, and to more challenging PDEs with convective nonlinearities such as Kuramoto-Sivashinsky and Navier-Stokes.

Fig. 8. Convergence of the closed-loop system under the nonlinear full-state feedback law for various initial conditions and \( \sigma = 15 \).

### References


