Algebraic Criteria for Consensus Problem of Networked Systems with Continuous-time Dynamics

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Abstract—This paper addresses algebraic criteria for consensus problem of continuous-time networked systems, in which both fixed and switching topology cases are considered. A special eigenvector \( \omega \) of Laplacian matrix is first constructed and correlated with the connectivity of digraph. And then, based on this tool, some necessary and/or sufficient algebraic conditions are proposed, which can directly determine whether the consensus problem can be solved or not. Furthermore, it is clearly shown that only the agents corresponding to the positive elements of \( \omega \) contribute to the group decision value and decide the collective behavior of all agents. Particularly for the fixed topology case, not only the role of each agent is exactly measured by the value of the corresponding element of \( \omega \) but also the group decision value can be calculated by such a vector and the initial states of all agents.

I. INTRODUCTION

In the last decade, due to the broad applications of networked systems in the fields of mobile robots, unmanned air vehicles(UAVs), autonomous underwater vehicles(AUVs), etc., where the coordination control of all agents is in the central position, the importance of the consensus problem in these fields has been well recognized and many results have been obtained(see [2]—[11]). For example, Jadabaie et al. [2] have shown that the states of all the jointly-connected agents converge to the same value or the value of a given leader’s state, where the neighbor-based rule is used and the information flow is bidirectional. Then, Ren and Beard [3] extend the results in [2] to the case in which consensus can be achieved if the union of interaction digraph contains a spanning tree across each bounded time interval. The similar results can be found in [4], where Moreau shows that the conditions for the discrete-time consensus in [3] are also necessary. From the above literatures, what group decision value is reached by all agents is not clear. However, this is not an issue for the average consensus problem since for this special case, the state of each agent converges to the average value of the initial states of all agents [5]—[9]. More specifically, Olfati-Saber and Murray [5] propose a continuous-time update scheme and prove that average consensus can be reached if the interaction digraph is balanced and strongly connected. Recently, Liu et al.[9] extend the results in [5] to the switching topology. Obviously, all the above results depend on the structure property of interaction digraphs, for convenience, we uniformly call them as geometrical criteria. It is noted that in order to examine whether the consensus problem can be solved or not, the algorithms are further needed to identify whether the interaction digraphs contain spanning tree or are jointly connected, etc. In addition, it is not clear that agents play the leading role in consensus procedure.

Our main objective in this paper is to develop the algebraic criteria for consensus problem of continuous-time networked systems. With this in mind, a special nonnegative left eigenvector \( \omega \) of Laplacian matrix is first constructed and correlated with the connectivity of digraph. It has been proved that the digraph is strongly connected(weakly connected with spanning tree) if and only if the vector \( \omega \) is positive(nonnegative). Further, if a connected digraph is balanced, then it must be strongly connected and meanwhile, all elements of \( \omega \) are equal. Since the vector \( \omega \) has these properties to examine the above digraphs, it is a natural tool to study the considered consensus problem. The obtained results provide a set of algebraic conditions to directly determine whether the consensus problem can be solved or not. In addition, more information about the consensus procedure are reflected. For example, it has been proved that only the agents corresponding to the positive elements of \( \omega \) decide the collective behavior of all agents and contribute to the group decision value, i.e., they are the leaders of the system. Meanwhile, for the fixed topology case, not only the role of each agent is proportional to the value of the corresponding element of \( \omega \) but also the group decision value can be calculated by \( \omega \) and the initial states of all agents. These new facts imply that if the elements of \( \omega \) are not equal, then different agent plays different role in the consensus procedure.

The remainder of this paper is organized as follows. In section II, some preliminaries and background knowledge are given. Then, some algebraic criteria for connectivity of digraph are presented in section III. Furthermore, algebraic criteria for consensus problem and average consensus problem are proposed in section IV and section V, respectively. Finally, we conclude our work in section VI.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Preliminaries and Background
In this subsection, we introduce some notations in graph theory and matrix theory which are used throughout this paper. Let \( I = \{0, 1, 2, \ldots, n\} \) be an index set and \( G = (V, E, A) \) be a weighted digraph of order \( n \) with the set of nodes \( V = \{v_1, v_2, \ldots, v_n\} \), set of edges \( E \subseteq V \times V \), and a weighted adjacency matrix \( A = [a_{ij}] \) with nonnegative adjacency elements \( a_{ij} \geq 0 \) for all \( i, j \in I \), \( i \neq j \) and \( a_{ii} = 0 \) for all \( i \in I \). As \( a_{ij} > 0 \), it means that there exists an edge from the node \( v_j \) to the node \( v_i \). The set of neighbors of the node \( v_i \) is denoted by \( N_i \) and defined as \( N_i = \{v_j \in V : a_{ij} > 0\} \). The in-degree and out-degree of the node \( v_i \) are defined as \( \text{deg}_i(v_i) \) and \( \text{deg}_o(v_i) \), respectively, just as follows:

\[
\text{deg}_i(v_i) = \sum_{j=1}^{n} a_{ji}, \quad \text{deg}_o(v_i) = \sum_{j=1}^{n} a_{ij}.
\]  

(1)

The degree matrix of \( G \) is a diagonal matrix denoted by \( \Delta = [\Delta_{ij}] \) where \( \Delta_{ij} = 0 \) for all \( i \neq j \) and \( \Delta_{ii} = \text{deg}_o(v_i) \). The Laplacian matrix of a weighted digraph \( G \) is defined as

\[
L(G) = \Delta - A.
\]  

(2)

A path in a digraph \( G \) is a sequence of edges such that the terminal vertex of one edge is the initial vertex of the next. A digraph \( G \) is said to be strongly connected if and only if for every pair of distinct vertices \( v_i \) and \( v_j \) in \( V \), there is a directed path from \( v_i \) to \( v_j \). A digraph \( G \) is called weakly connected if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. A digraph \( G \) is called disconnected if it is not even weak. Meanwhile, a weighted digraph \( G \) is called balanced if its out-degree equals to its in-degree, i.e., \( 1^T L = 0 \) with \( L = [1, \ldots, 1]^T \in \mathbb{R}^{n \times 1} \). For a series of digraphs \( G_1, \ldots, G_n \) with the same vertices, we say the digraph \( G_u \) is the union of them if the vertices of \( G_u \) are the same as any one of these digraphs and the weight \( a_{ij} \) of edge \( v_i \times v_j \) of \( G_u \) is defined as follows.

\[
a_{ij} = \frac{1}{n} \sum_{k=1}^{n} a_{kj}^k
\]  

(3)

where \( a_{kj}^k \) is the weight of edge \( v_i \times v_j \) of \( G_k, k \in I \). For convenience let \( L_a \) denote the Laplacian matrix of \( G_u \) and \( C \) the collection of digraphs \( G_1, \ldots, G_n \).

To proceed, a vector \( x = [x_1, \ldots, x_n]^T \) is called positive if each element of \( x \) is positive(i.e., \( x_i > 0, \forall i \in I \)). A vector \( x \) is called nonnegative if each element of \( x \) is nonnegative(i.e., \( x_i \geq 0, \forall i \in I \)), meanwhile, there exists at least one nonzero element in \( x \). As all elements in \( x \) are zero, the vector \( x \) is called zero vector. Following the same line, a matrix \( A \in \mathbb{R}^{m \times n} \) is called positive if all its elements are positive and matrix \( A \) is called nonnegative if all its elements are nonnegative. Now, some lemmas are introduced because they will be used in the below.

**Lemma 1:** [3] Given a matrix \( S = [a_{ij}] \in \mathbb{R}^{n \times n} \), where \( a_{ii} \geq 0, a_{ij} \leq 0, \forall i \neq j \), and \( \sum_{j=1}^{n} a_{ij} = 0 \) for each \( j \), then \( S \) has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Furthermore, \( S \) has exactly one zero eigenvalue if and only if the digraph with \( S \) has a spanning tree.

**Proof:** See the Lemma 3.3 in [3].

**Lemma 2:** [3] If a nonnegative matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) has the same positive constant row sums given by \( \mu > 0 \), then \( \mu \) is an eigenvalue of \( A \) with an associated eigenvector \( \mathbf{1} \) and \( p(A) = \mu \), where \( p(\cdot) \) denotes the spectral radius. In addition, the eigenvalue \( \mu \) of \( A \) has algebraic multiplicity equal to one, if and only if the digraph associated with \( A \) has a spanning tree.

**Proof:** See Lemma 3.4 in [3].

**B. Problem Statement**

In this subsection, we consider a protocol with continuous-time dynamics to solve consensus problem as follows.

\[
\dot{x}_i = \sum_{j \in N_i(t)} a_{ij}(x_j - x_i), \quad i, j \in I,
\]  

(4)

where \( x_i \) and \( N_i(t) \) denote the state and the neighbor set of the node \( v_i \), respectively.

We say \( x_i \) and \( x_j \) agree if and only if \( x_i = x_j \) (component-wise). And we say that system (4) has reached a consensus if and only if \( x_i = x_j \) for all \( i, j \in I \). The common value of consensus variable is called the group decision value. Let \( \chi : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) be a consensus function of vectors \( x^1, \ldots, x_n \), \( x_i \in \mathbb{R}^{m \times 1} \), and \( x(0) \) denote the initial state of agent \( v_i \), then system (4) solves the \( \chi \)-consensus problem if and only if there exists an asymptotically stable equilibrium point \( x^* = [x^*_1, \ldots, x^*_n]^T \) satisfying \( x^*_i = \chi(x(0)) \in \mathbb{R}^{m \times 1} \) for all \( i \in I \). The function \( \chi \) is called consensus function. The special cases with \( \chi(x) = \max_{\|x\|}\|x_i\|; \chi(x_i) = \min_{\|x\|}\|x_i\|; \chi(x_i) = \frac{1}{2} \sum_{j=1}^{n} a_{ij}(x_j(0)) \) are called max-consensus problem, min-consensus problem and average consensus problem, respectively.

To proceed, we first give the algebraic conditions for connectivity of digraph. These results are the basis of our method.

**III. ALGEBRAIC CRITERIA FOR CONNECTIVITY OF DIGRAPH**

Obviously, all the information about a digraph is reflected by its Laplacian matrix, \( L \). For a given node \( v_i (i \in I) \), the row \( i \) of \( L \) denotes how much the other nodes directly affect the node \( v_i \), meanwhile, the column \( i \) of \( L \) reflects how much the other nodes are affected by the node \( v_i \). In this sense, the study of Laplacian matrix is helpful for us to learn something about the digraph, especially to understand the structure information of digraph. Here, we first construct a vector \( \omega \) from the Laplacian matrix \( L \) as follows.

\[
\omega = [\omega_1, \omega_2, \ldots, \omega_n]^T
\]  

\[
= [\det(L_{11}), \det(L_{22}), \ldots, \det(L_{nn})]^T,
\]  

(5)

where \( \det(L_{ii}) \) denotes the determinant of matrix \( L_{ii} \), and \( L_{ij} \in \mathbb{R}^{(n-1) \times (n-1)} \) is obtained from \( L \) by deleting the row \( i \) and the column \( i \). In what follows, we give some algebraic results about the connectivity of digraph.
Theorem 1: Suppose that digraph $G$ contains spanning tree, $L$ is its Laplacian matrix, and let the vector $\omega$ be defined in (5), then

$$\omega^TL = 0,$$

meanwhile, $\omega$ is nonnegative.

Proof: See Appendix A.

Remark 1: For all connected digraphs with spanning tree, theorem 1 gives a uniform formula about the left eigenvector of Laplacian matrix associated with simple zero eigenvalue.

Corollary 1: Suppose that $G$ is a strongly connected weighted digraph and $L$ is its Laplacian matrix, let vector $\omega$ be defined in (5), then

$$\omega^TL = 0,$$

meanwhile, $\omega$ is positive.

Proof: Combining Theorem 1 and Theorem 8.4.4. in [13], it is easy to be shown.

Corollary 2: Suppose that a weakly connected digraph $G$ has spanning tree and $L$ is its Laplacian matrix, let vector $\omega$ be defined in (5), then

$$\omega^TL = 0,$$

meanwhile, $\omega$ is nonnegative and has at least one zero element.

Proof: By Theorem 1 and the fact that $G$ is a weakly connected digraph and contains a spanning tree, this result can be shown easily. Due to the limitation of the space, the detail is omitted.

Theorem 2: Suppose that $G$ represents a digraph and $L$ is its Laplacian matrix, then $G$ is strongly connected if and only if the vector $\omega$ defined in (5) is positive.

Proof: Necessity: by corollary 1, the conclusion holds obviously. Sufficiency: We show the sufficiency by contradiction. Suppose that the digraph is not strongly connected, then it must be a weakly connected or a disconnected digraph. Firstly, if it is weakly connected and contains spanning tree, then by corollary 1, we know the simple zero eigenvalue of its Laplacian matrix has a nonnegative left eigenvector like (5) and in which there exists at least one zero element. Secondly, if it is weakly connected but does not contain spanning tree, then its Laplacian matrix has at least two zero eigenvalues by lemma 1, so the vector defined in (5) is a zero vector. Thirdly, if it is a disconnected digraph, by lemma 1, its Laplacian matrix also has at least two zero eigenvalues, so the vector $\omega$ is strongly connected, i.e., it is impossible that a weakly connected digraph is balanced. In other words, either a strongly connected digraph or a disconnected digraph with strongly connected components has the possibility to be a balanced digraph.

IV. Algebraic Criteria for Consensus Problem

In this section, we study the algebraic criteria for the system (4) to solve consensus problem. It is clear that equation (4) can be rewritten in a compact form as follows.

$$\dot{x}(t) = -L(t)x(t),$$

where $L(t)$ is the laplacian matrix. For the fixed topology, i.e., $L(t)$ is a constant matrix, the solution of (7) is given by

$$x(t) = \exp(-Lt)x(0).$$

Here, we are in a position to give the algebraic criterion for such a consensus problem.
Lemma 3: [8] The state $x$ in (7) approaches $\text{span}\{1\}$ and thus solves an agreement problem for all initial $x$ if and only if the interaction digraph of $L$ contains a spanning directed tree.

Proof: See the theorem 2 in [8].

Theorem 4: Suppose that the networked system is defined in (7), $L$ is Laplacian matrix of its interaction digraph $G$, then (7) solves consensus problem if and only if the vector $\omega$ defined in (5) is nonnegative.

Proof: Necessity: Because system (7) can solve the consensus problem, the digraph $G$ contains a spanning tree by lemma 3. Then, $\omega$ is nonnegative by theorem 1.

Sufficiency: Because the vector $\omega$ is nonnegative, then the digraph $G$ contains a spanning tree by theorem 1. Thus, (7) can solve the consensus problem by lemma 3.

As system (7) can solve consensus problem, it is another important and attractive question that what value is reached by the group and how much each node contributes to final group decision value. Before answer such a problem, suppose that (7) can solve consensus problem, i.e., $\omega$ is nonnegative, let

$$\omega = \frac{1}{\sum \alpha_k} \alpha, \quad \omega_r = [1, 1, \ldots, 1]^T,$$

(9)

clearly, $\omega^T L = 0, L \omega = 0$ and $\alpha_i \omega = 1$. Then the following theorem can be derived.

Theorem 5: Suppose that system (7) can solve consensus problem, $L$ is Laplacian matrix of its interaction digraph $G$, $\omega_l$ and $\omega_r$ are defined in (9), then

i): $R = \lim_{t \to \infty} \exp(-Lt) = \omega_l \omega_l^T \in \mathbb{R}^{n \times n}$,

(10)

furthermore, $R$ is a stochastic matrix.

ii) the group decision value is

$$\alpha = \frac{\sum \omega_r x_r(0)}{\sum \alpha},$$

(11)

which belongs to the convex hull of initial states of all agents.

Proof: i). The proof is similar to theorem 3 in [5].

For convenience, we do not omit it. Because system (7) can solve consensus problem, $\omega_l$ exists by theorem 4. Let $H = -L$ and $J$ be the Jordan form associated with $H$, i.e., $H = JSJ^{-1}$. We have $\exp(Ht) = \exp(Jt)S^{-1}$ and as $t \to \infty$, $\exp(Jt)$ converges to a matrix $Q = [a_{ij}]$ with a single nonzero element $a_{11} = 1$. The fact that other blocks in the diagonal of $\exp(Jt)$ vanish is due to the property that $\text{Re}(\lambda_k(H)) < 0$ for all $k \geq 2$ by lemma 1, where $\lambda_k(H)$ is the k-th largest eigenvalue of $H$ in terms of magnitude $|\lambda_k|$. Notice that $R = SJS^{-1}$. Since $HS = SJ$, the first column of $S$ is $\omega_l$, similarly $S^{-1}H = JS^{-1}$ that means the first row of $S^{-1}$ is $\omega_l^T$. Due to $S^{-1} = I_n$, $\omega_l^T \omega_l = 1$ is satisfied just as equation (9) states. So $R = SJS^{-1} = \omega_l \omega_l^T$. By a simple calculation, all entries in $R$ are nonnegative and for a given row, the sum of all elements is equal to 1, thus $R$ is a stochastic matrix.

ii). From the above, due to $\omega_l^T L = 0$, we have $\omega_l^T \dot{x} = \omega_l^T u = \omega_l^T (-Lx) = 0$. So $\omega_l^T x$ is an invariant quantity. Thus, we have $\omega_l^T x = \omega_l^T x(0)$, i.e., $x = \alpha I$, where $\alpha$ has the form as (11).

Remark 6: Theorem 5 is an extension of the theorem 3 in [5]. Because we have found the vector $\omega$, the importance of this theorem is greatly enhanced.

In what follows, we extend theorem 4 to the switching topology case. To proceed, let $\Upsilon$ be a infinite set of dwell time $\tau_i = t_{i+1} - t_i$, $i \in I$, which is closed under addition and multiplication by integers, let $G_u$ denote the union of the finite interaction digraphs and the elements of its Laplacian matrix $L_u$ be defined by equation (3), then the algebraic characterization of theorem 3.12 in [3] can be given as follows.

Theorem 6: Let $t_1, t_2, \cdots$ be an infinite time sequence at which the interaction digraph for weighting factors switch and $\tau_i = t_{i+1} - t_i \in \Upsilon$, $i \in I$. Let $G(t_i) \in \mathcal{G}$ be a switching interaction digraph at time $t = t_i$, where $\mathcal{G}$ is the set of all possible interaction digraphs. Suppose that there exists an infinite sequence of uniformly bounded, non-overlapping time intervals $[t_j, t_{j+1}]$, $j = 1, 2, \cdots$, starting at $t_0$, with the property that interval $[t_j, t_{j+1}]$ is uniformly bounded. Let $G_u(j)$ denote the union of the interaction digraphs across interval $[t_j, t_{j+1}]$ and $L_u(j)$ denote the Laplacian matrix of $G_u(j)$. Then the continuous-time update scheme (4) achieves consensus asymptotically if the vector $\omega(j)$ of $L_u(j)$ is nonnegative, where $\omega(j)$ is defined by (5). Furthermore, if $\omega(j) = 0$ for all time interval $j$, then consensus can not be achieved asymptotically.

Proof: Combined the theorem 3 and theorem 3.12 in [3], the conclusion holds obviously.

Remark 7: For the switching topology case, we can not directly use the element of $\omega$ to estimate the contribution of each agent to the group decision value. However, if the element $\omega_l(j)$ is zero for all time interval $j$, then agent $v_j$ must make no contribution to the group decision value. This is a new fact that can not be reflected by theorem 3.12 in [3].

V. ALGEBRAIC CRITERIA FOR AVERAGE CONSENSUS PROBLEM

In this section, we study the average consensus problem of networked system with continuous-time dynamics. Just as [5] have reported, the balanced digraph plays crucial role in solving such a problem. Here inspired by theorem 5, an algebraic result about average consensus problem can be given as follows.

Theorem 7: System (7) with invariant topology solves average consensus problem if and only if the vector $\omega$ defined in (5) is positive and meanwhile, all elements of $\omega$ are equal.

Proof: Combined the theorem 4 and 5, the conclusion holds obviously.

Remark 8: Theorem 7 is equivalent to the theorem 4 in [5]. However, we prove it directly by theorem 4 and 5. Following this line, it is clear that the average consensus problem is a special case of the general consensus problem, which is not clarified by the results reported in [5].

Just as the discussion in section 4, let $G_u$ denote the union of finite interaction digraphs and $\mathcal{G}$ denote the set of all...
possible interaction digraphs, then we give an algebraic result about average consensus problem with switching topology. This result is the algebraic version of the theorem 4 in [9].

Theorem 8: Assume that $G(t_i) \in \mathcal{G}(i \in I)$ and $L_n$ is laplacian matrix of $G_n$, suppose that there exists an infinite sequence of uniformly bounded, non-overlapping time intervals $[t_i, t_{i+1})$, $j = 1, 2, \cdots$, starting at $t_i = t_0$. Let $G_n(j)$ denote the union of interaction digraphs during time interval $[t_i, t_{i+1})$ and $L_n(j)$ be its Laplacian matrix. Then continuous-time update scheme (7) asymptotically achieves average consensus if i). $\mathbf{1}^\top L_n = 0$, $\forall t_i \in [t_i, t_{i+1})$ ii). The vector $\omega(j)$ of $L_n(j)$ is positive, where $\omega(j)$ is defined by (5).

Proof: Combined theorem 2 and theorem 4 in [9], the conclusion holds obviously.

Remark 9: It is clear that the algebraic conditions in theorem 8 are more stricter than that in theorem 6. Further, it is easy to verify that all elements of $\omega(j)$ in theorem 8 are equal.

Remark 10: Combined with corollary 4, theorem 8 and remark 7, we know that each connected component of interaction digraphs is strongly connected and balanced at each time $t$ if system (7) solves average consensus problem. In this sense, all agents make the same contribution to the group decision value if each of them is always in a connected and balanced component.

VI. CONCLUSIONS

In this paper, we have established the algebraic criteria for consensus problem of continuous-time networked systems, in which both fixed and switching topology cases are considered. The proposed results not only can algebraically determine whether the consensus problem can be solved or not but also clearly show that the average consensus problem is a special case of the general consensus problem. Furthermore, more information about consensus procedure is revealed. For example, it has been shown that only the agents corresponding to the positive elements of $\omega$ decide the collective behavior of all agents and contribute to the group decision value, i.e., the agents corresponding to the zero elements of $\omega$ make no any contribution to the group decision value except for converging to it. Particularly for the fixed topology case, the consensus procedure is distinctly clarified. All these new facts give us a deep insight into the consensus procedure.

APPENDIX

A. Proof of theorem 1

Proof: Because digraph $G$ may be reducible, we prove it directly. The proof is divided into four steps as follows.

Step 1: Because the digraph $G$ has a spanning tree, $\text{rank}(L) = n-1$ by lemma 1. Therefore, square matrix $L$ contains at least one nonsingular submatrix of order $n-1$. Without loss of generality, assume that $L_{ij}$ is such a submatrix which is obtained by deleting the row $i$ and the column $j$ of $L$. So the row vectors of $L$ except for the row $i$ are linearly independent. Now, let $L^*$ denote the submatrix of $L$ obtained by deleting the row $i$ of $L$ as follows.

$$L^* = \begin{bmatrix}
  a_{11} & -a_{12} & \cdots & -a_{1n} \\
  \vdots & \vdots & \ddots & \vdots \\
  -a_{i-1,1} & -a_{i-1,2} & \cdots & -a_{i-1,n} \\
  -a_{i+1,1} & -a_{i+1,2} & \cdots & -a_{i+1,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  -a_{n1} & -a_{n2} & \cdots & a_{nn}
\end{bmatrix}.$$  

It is clear that $L^* \in R^{(n-1)\times n}$ is a row full rank matrix (i.e., $\text{rank}(L^*) = n-1$), meanwhile the sum of each row of $L^*$ is zero because $L$ is a Laplacian matrix. Therefore, any $n-1$ column vectors of $L^*$ must be linearly independent. Thus, the first $(n-1) \times (n-1)$ leading principal submatrix $A$ of $L$ is nonsingular if we renumber the node $v_i$ as the node $v_n$ and the node $v_n$ as the node $v_i$. To proceed, let $B = [a_{1n}, a_{2n}, \cdots, a_{n-1,n}]$ and $C = [-a_{n1}, -a_{n2}, \cdots, -a_{n,n-1}]$, then there must exist an inverse column permutation matrix $P \in R^{n \times n}$ such that

$$LP = L^*[A^{-1} 0 \quad 0 \quad 1] = \begin{bmatrix} A & B \\ C & a_{nn} \end{bmatrix}^{-1} \begin{bmatrix} I_{n-1} & B \\ a_{nn}^{-1} & DB \end{bmatrix} = L^*,$$

where $I_{n-1}$ is identity matrix of order $n-1$.

Step 2: Let

$$D = CA^{-1} = [-a_{1n}, -a_{2n}, \cdots, -a_{n-1,n}]A^{-1}$$

$$= [-b_1, -b_2, \cdots, -b_{n-1}],$$

substitute it into $L^*$, and then perform some elementary column transformations on matrix $L^*$, we have

$$L^* \sim \begin{bmatrix} I_{n-1} & 0 \\ D & a_{nn}^{-1}DB \end{bmatrix} = L_{i1}^*.$$

Since the above operations on $L$ are invertible, $\text{rank}(L_{i1}^*) = \text{rank}(L) = n-1$. Thus, we directly have $a_{nn}^{-1}DB = 0$ and get the following two equivalent equations

$$\omega^\top L = 0 \iff \omega^\top L_{i1}^* = 0 \quad (12)$$

In what follows, we calculate the value of $b_i$ for all $i = 1, \cdots, n-1$. Due to $A^{-1} = \frac{1}{\text{det}(A)}A^*$, where $A^*$ is the adjoint matrix of $A$, we have

$$D = \frac{CA^*}{\text{det}(A)} = \frac{C}{\text{det}(A)} \begin{bmatrix} A_{11}^* & A_{21}^* & \cdots & A_{n-1,1}^* \\ A_{12}^* & A_{22}^* & \cdots & A_{n-1,2}^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,n-1}^* & A_{2,n-1}^* & \cdots & A_{n-1,n-1}^* \end{bmatrix},$$

where $A_{ij}^* = (-1)^{i+j}\text{det}(A_{ij})$. $A_{ij}$ is the submatrix of $A$ by deleting the row $i$ and the column $j$. Then combined with the vector $C$ defined in the above, we have

$$-b_i = \frac{1}{\text{det}(A)}(-a_{11}A_{11}^* - a_{22}A_{22}^* - \cdots - a_{n,n-1}A_{n-1,n-1}^*)$$

$$= \frac{(-1)^i}{\text{det}(A)}\left[(-1)^2a_{11}\text{det}(A_{1,1}) + (-1)^3a_{22}\text{det}(A_{i,2}) + \cdots + (-1)^n a_{n,n-1}\text{det}(A_{i,n-1})\right].$$
Let $\triangle_i$ denote the $(n-1) \times (n-1)$ submatrix of $L$ as follows
\[
\triangle_i = BL
\begin{bmatrix}
1, 2, \ldots, i - 1, i + 1, \ldots, n \\
1, 2, \ldots, i - 1, i, i + 1, \ldots, n - 1
\end{bmatrix},
\]
ie, $\triangle_i$ is obtained by deleting the row $i$ and the column $n$ of $L$. Noting that $\triangle_i$ is a square matrix of order $n-1$, we have
\[
det(\triangle_i) = (-1)^{n+1}a_{n1}\det(A_{11}) + (-1)^{n+2}a_{n2}\det(A_{12}) + \cdots + (-1)^n a_{nn-1}\det(A_{n-1,n-1})
\]
where $\triangle_{ij}$ is submatrix of $\triangle_i$ by deleting the row $i$ and the column $j$ of $\triangle_i$. From equations (13) and (15), we have $det(\triangle_{n1}) = det(A_{11}), det(\triangle_{n2}) = det(A_{12}), \ldots, det(\triangle_{n,n-1}) = det(A_{n-1,n-1})$. Thus, equation (15) becomes
\[
det(\triangle_i) = (-1)^{n+1}a_{n1}\det(A_{11}) + (-1)^{n+2}a_{n2}\det(A_{12}) + \cdots + (-1)^n a_{nn-1}\det(A_{n-1,n-1})
\]
= $(-1)^{n-3}det \begin{bmatrix}
a_{11} & -a_{12} & \cdots & -a_{1,n-1} \\
a_{31} & -a_{32} & \cdots & -a_{3,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n2} & \cdots & -a_{nn-1}
\end{bmatrix}
\]
= $(-1)^{n-3}det \begin{bmatrix}
a_{11} & -a_{13} & \cdots & -a_{12} \\
a_{31} & a_{33} & \cdots & -a_{32} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n3} & \cdots & a_{n2}
\end{bmatrix}
\]
= $(-1)^{n-4}det \begin{bmatrix}
a_{11} & -a_{13} & \cdots & -a_{1n} \\
-a_{31} & a_{33} & \cdots & -a_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{n1} & -a_{n3} & \cdots & a_{nn}
\end{bmatrix}
\]
where the third step is obtained by some elementary column transformations; the fourth step is obtained by adding the first $n-2$ columns to the last column; the fifth step used the properties of Laplacian matrix $L$, $\sum_{j=1}^{n}a_{ij} = 0$ for each $i$; and the last step is obtained by using the fact that $n$ is even. It is clear that $L_{22}$ is the principal submatrix of $L$.

By step 2, we know $det(A) > 0$ and $det(L_{22}) \geq 0$, which directly implies $b_2 \geq 0$. Just as the same analysis, we can show $b_1 = det(L_{11})/det(A) > 0$ for $i \in \{1, 3, \ldots, n-1\}$. Thus, we have
\[
b_1 = \frac{det(L_{11})}{det(A)}; b_2 = \frac{det(L_{22})}{det(A)}; \ldots; b_{n-1} = \frac{det(L_{n-1,n-1})}{det(A)};
\]

Substitute (19) into the right equation of (12) and let the free variable $\omega_b = det(A) = det(L_{nn})$, we have
\[
\omega_1 = det(L_{11}); \omega_2 = det(L_{22}); \ldots; \omega_n = det(L_{nn}).
\]
such that $\omega^T L = 0$.

Step 4: In this step, we show $\omega$ is nonnegative. To proceed, we denote eigenvalues of $L_{ii}$ as $\lambda_i^{\pm}$, where $i \in I$ and $k \in \{1, 2, \ldots, n-1\}$. Because each principal minor $L_{ii}$ of Laplacian matrix $L$ is diagonally dominant and main diagonal element of it is nonnegative, and then by Geršgorin disc theorem, each eigenvalue $\lambda_i^{\pm}$ of $L_{ii}$ is in the right open plane, i.e., $\lambda_i^{\pm} > 0$ for all $k$. Therefore, $det(L_{ii}) = \lambda_1^{\pm} \lambda_2^{\pm} \cdots \lambda_k^{\pm} > 0$ for all $i \in I$. Combined with Step 1 and Step 3 $\omega$ is nonnegative.

Combining all of the above, the conclusion holds.

REFERENCES