Remarks on Tracking and Robustness Analysis for MEM Relays

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Abstract—We announce a new class of tracking controllers, applicable to both electrostatic and electromagnetic microelectromechanical (MEM) relays, that yield arbitrarily fast local exponential convergence of the tracking error to zero and uniform global asymptotic stability of the error dynamics. Our stability analysis is based on an explicit, strict, global Lyapunov function construction. Our Lyapunov approach also leads to an input-to-state stability based quantification of the effects of parametric uncertainty on the tracking performance. The MEM dynamics contain a quadratic nonlinearity that leads to constraints on the class of reference trajectories that can be tracked. We illustrate how to craft a reference trajectory that is compatible with these constraints and with a typical opening and closing relay operation. Our simulation indicates the good tracking performance of our controllers.

Key Words—MEM relays, nonlinear control, Lyapunov theory, input-to-state stability

I. INTRODUCTION

Several types of relays are used in industrial applications to close or open connections in electric circuits. Many industrial control processes use traditional mechanical relays, which are large, slow, and noisy. By contrast, solid-state relays have faster response, much longer lifetimes, and smaller sizes. However, solid-state relays have high power consumption and poor electrical isolation, because of their low off-resistance and high on-resistance. Reducing their on-resistance can increase output capacitance, which can cause other problems when there is switching of high-frequency signals [16].

Recent advances in the area of microelectromechanical systems (MEMS) have led to new opportunities for developing power and signal relays [16]. MEM relays have the same advantages as mechanical relays, viz., higher dielectric strength, lower power consumption, higher off-resistance, lower cost, and lower on-resistance. MEMS technology can also miniaturize mechanical relays, thereby treating the problems of switching time and size. Moreover, micro-relays can be used readily in conjunction with other electronic components.

There are two major classes of MEM relays, involving different methods of actuation: electrostatic and electromagnetic [2], [3], [4], [5], [12], [16]. Relays involve a control circuit and an output circuit. In electrostatic relays, the circuits use a common pair of parallel electrodes (one movable and the other fixed) which act as a capacitor; see Figure 1. Voltage applied across the electrodes yields an electric field between them, and then creates an attractive force between them. As the electrodes come together, so do the two contacts of the output circuit; this allows the flow of current and the closing of the circuit. In electromagnetic relays, the control circuit has an electromagnet whose magnetic force acts on a movable electrode located above the fixed coil post; see Figure 2. The magnetic force then attracts the electrode, and it closes the output circuit like in an electrostatic relay.

The electrostatic actuation method is more common for micro-relays [16]. This is largely due to difficulties in producing micro-size electromagnetic actuators. Another disadvantage in using electromagnetic actuation is that it requires more current, leading to larger heat generation and more power consumption. Nevertheless, [2] studies both actuation methods in their ability to produce micron and submicron precision under rapid motion, and it provides a strong rationale in favor of using magnetically-driven micro-actuators. See [5], [16] for a detailed review on the design and production of electromagnetic and electrostatic micro-relays.

Voltage-controlled MEM actuators produce an important
nonlinear phenomenon associated with a saddle-node bifurcation [7], called pull-in [13]. A physical explanation of pull-in is as follows. Assume that the voltage across the MEM actuator is incrementally raised from zero. First the (capacitive or magnetic) force will pull the movable electrode down with increasing voltage. The voltage achieves a critical value, corresponding to an electrode displacement of 1/3 the nominal (zero-voltage) gap. The movable electrode then suddenly ‘crashes’ onto the bottom electrode. This problem is especially detrimental in MEM relays, because they necessarily operate in the pull-in region when the relay closes [17]. Clearly, repeated occurrence of pull-in will damage the micro-relay. The pull-in phenomenon implies that MEM

Most MEM actuator feedback control studies have largely been devoted to the electrostatic case. In [7], [8], [9], partial-state feedback controllers were proposed using charge and position with a velocity observer. Control schemes based on backstepping, control Lyapunov functions, differential flatness, and input-to-state stabilization were provided in [18], [19]. A PD-type controller was reported in [1] for a MEM electrostatically-actuated optical switch. In [17], a common nonlinear dynamic structure was developed for voltage-controlled electromagnetic and electrostatic MEM relays. Also, [17] provided two nonlinear state feedback control schemes: a feedback linearization tracking controller and a Lyapunov-based setpoint controller.

In this note, we announce a new class of controllers for electrostatic and electromagnetic voltage-driven MEM relays, based on the explicit construction of Lyapunov functions and state feedback. Then we use input-to-state stability (ISS) theory [14], [15] to quantify the robustness of our feedback controller to parametric uncertainty. This makes it possible to analyze the effects of parameter uncertainty on the tracking error. We validate our results in Section VI below in a simulation. For complete proofs of our results and extensions involving partial state feedback, see [10].

II. System Model

As noted in [17], electrostatic and electromagnetic micro-relays share the nonlinear dynamic model

\[
m\ddot{z} + b\dot{z} + kx = \alpha z^2 + \beta \dot{z} + \gamma (g_0 - x) z = u,
\]

in which

\[
z = \begin{cases}
q \\
\phi
\end{cases}, \quad \alpha = \begin{cases}
1/(2\varepsilon A) \\
1/(2\mu A)
\end{cases}, \quad \beta = \begin{cases}
R \\
N
\end{cases}, \quad \gamma = \begin{cases}
1/(\epsilon A) \\
R/(N\mu A)
\end{cases},
\]

and the upper (resp., lower) rows in (2) are for the electrostatic (resp., electromagnetic) micro-relay, \( x \) denotes the position of the movable electrode with \( x = 0 \) when it is in the open position, \( m > 0 \) denotes the movable electrode mass, \( b > 0 \) denotes the squeezed-film damping coefficient, \( k > 0 \) denotes the spring stiffness, \( g_0 > 0 \) is the gap when the movable electrode is in the open position, \( A \) denotes the movable electrode area, \( R > 0 \) denotes the resistance of the circuit, and \( u = v \) where \( v \) is the input voltage. For the electrostatic micro-relay case, \( q \) is the charge, and \( \epsilon \) is the gap permittivity. For the electromagnetic micro-relay case, \( \phi \) is the flux, \( \mu \) denotes the gap permeability, and \( N \) denotes the number of coil turns.

Remark 3: The structure of (1) leads to our conditions (3) on the reference trajectory, as we will show below. In
In Section VI, we build a reference trajectory that is physically compatible with typical MEM relay operations while also satisfying (3).

To specify our control objective, consider the dynamics for \( Y = (e_1, e_2, \zeta) \) in which \( e_1 = x - y_d, e_2 = e_1, \)
\[
\dot{\zeta}(t) = \sqrt{\frac{k}{m}} z(t) - R\mu(e_1(t), e_2(t), t),
\]
\[
R\mu(e_1, e_2, t) = \sqrt{y_d(t) + \kappa_2 y_d(t) + \kappa_1 y_d(t) + \mu(e_1, e_2)},
\]
and \( \mu \in C^1 \) will be specified to satisfy
\[
|\mu(e_1, e_2)| \leq 0.1 \kappa_m \quad \forall t \in [0, \infty).
\]

The following theorem from [10] shows how our goals can be realized:

**Theorem 1:** Let \( a_1 \) and \( a_2 \) be any given positive constants and set \( \sigma(s) = s/\sqrt{1+s^2} \) and
\[
\mu(e_1, e_2) = -\frac{I_{\sigma}}{2m} \left[ \sigma \left( \frac{2a_1}{\kappa_m^2} e_1 \right) + \sigma \left( \frac{2a_2}{\kappa_m^2} e_2 \right) \right].
\]

Then for any constant \( a_3 \) for which
\[
a_3 \geq \Theta \{ a_1 + a_2(1 + 2R_\mu + \kappa_1 + \kappa_2 + a_1 + a_2) \},
\]
the feedback
\[
e_1(e_1, e_2, \zeta, t) = -a_3 \zeta(1 + \zeta^2) + \frac{1}{2R_\mu(e_1, e_2)} \{ y_d(t) + \kappa_2 y_d(t) + \kappa_1 y_d(t) \}
\]
renders (8) UGAS to the origin. Moreover, for each constant \( \mathcal{L} > 0 \), we can choose values of the \( a_i, K, K \mathcal{L} > 0 \) so that all trajectories of (8) in closed loop with (14) that start in \( \mathcal{K} \mathcal{B}_3 \) satisfy (10). Hence, (7) with the choice (14) stabilizes \( x \) to \( y_d \) with arbitrarily fast local exponential convergence.

**Remark 4:** Our proof of Theorem 1 constructs an explicit strict Lyapunov function for (8) in closed loop with (14), namely,
\[
V_3(e_1, e_2, \zeta) = V_2(e_1, e_2) + \Gamma Q(\zeta), \quad \text{where}
\]
\[
V_2(e_1, e_2) = e_1 e_2 + K V_1(e_1, e_2),
\]
\[
Q(\zeta) = \frac{1}{a_3} \left( \frac{1}{4} \zeta^2 + \frac{1}{4} \zeta^4 \right),
\]
\[
V_1(e_1, e_2) = \frac{1}{2} e_2^2 + \int_0^{e_1} \left\{ \kappa_1 l + \frac{\kappa_m a_1}{20} \sigma \left( \frac{20a_2}{\kappa_m^2} \right) \right\} dl.
\]

\( \Gamma \) and \( K \) are from (11), and the positive constants \( a_i \) will be chosen later. Strict Lyapunov functions are very useful in a variety of applications, e.g., for quantifying the effects of parametric uncertainty. For example, our Lyapunov analysis shows that the tracking is robust to suitably small uncertainties in our parameters \( k \) and \( b \), using the input-to-state stability paradigm; see Section V below.

**Remark 5:** Theorem 1 also holds (with the identical proof) with \( \sigma(s) = s/\sqrt{1+s^2} \) replaced by any \( C^2 \) function \( \sigma : \mathbb{R} \to [-1, 1]^+ \) satisfying \( \sigma(0) = 0, \sigma'(0) = 1, \) and \( 0 \leq \sigma'\leq 1 \) everywhere and (II) \( s \mapsto \sigma(s) \) is positive definite. We made our choice \( \sigma(s) = s/\sqrt{1+s^2} \) (which satisfies (I)-(II)) to make the statement of the theorem simpler. For a variant of Theorem 1 leading to partial-state (i.e., no velocity measurement) stabilizing feedbacks when only \( x \) and \( z \) can be measured, see [10].

**Remark 6:** The simpler controller \( v_1 = -\zeta + \{ y_d(t) + \kappa_2 y_d(t) + \kappa_1 y_d(t) \}/\{2R_0(t)\} \) and the choice \( \mu \equiv 0 \) also make (8) UGAS to 0. This is seen by checking that the time derivative of the positive definite function
\[
V_0(e_1, e_2, \zeta) = A e_2^2 + e_2^2 + B e_1 c_2 + C(\zeta^2 + \zeta^4), \quad \text{where}
\]
\[
A = \kappa_1 + \frac{R_0^2}{2 \kappa_2}, \quad B = \min \left\{ \frac{\kappa_2}{2}, \sqrt{\kappa_1} \right\}, \quad \text{and}
\]
\[
C = \max \left\{ \frac{1}{\kappa_2} + \frac{R_0}{4 \kappa_1}, \frac{1}{4} + \frac{16 \kappa_2 \kappa_1}{\kappa_2}, 4B \frac{m^2}{\kappa_2} \right\}
\]
along the solutions of $\dot{e}_1 = e_2$, $\dot{e}_2 = -\kappa_1 e_1 - \kappa_2 e_2 + \zeta^2 + 2\zeta R_0$, $\zeta = -\xi$ satisfies

$$V_o = -B\kappa_1 e_1^2 + (B - 2\kappa_2)e_2^2 + (2\kappa_2 + B\xi)e_1^2 + 2C(2\zeta^2 + 4\zeta^4) + C(2\zeta^2 + 4\zeta^4)$$

$$\leq -B\kappa_1 e_1^2 - \kappa_2 e_2^2 + \left\{ e_1^2 + B\kappa_2 \right\} \left\{ \frac{1}{\sqrt{\kappa_2}} \zeta^2 + 2\xi R_0 \right\}$$

$$-C(2\zeta^2 + 4\zeta^4) \quad \text{(since } B \leq \kappa_2)$$

$$\leq -B\kappa_1 e_1^2 - \kappa_2 e_2^2 + \left[ \frac{1}{e_1^2} + \frac{B}{\kappa_1} \right] \left( \zeta^4 + 4\zeta^2 R_0^2 \right)$$

$$\leq -B\kappa_1 e_1^2 - \kappa_2 e_2^2 - \frac{1}{2} \zeta^2,$$

where we used (6) and $pq \leq \frac{1}{2}p^2 + \frac{1}{2}q^2$ on the terms in braces. On the other hand, this simpler controller would not guarantee arbitrarily rapid local exponential convergence. The $\zeta^2$ term in (1) gives rise to the constraint (5) on the feedback, which restricts the global convergence rate for $e_1$. Therefore, it would be useful for the control to, at least, produce arbitrarily rapid local convergence. For the proof that the feedback from Theorem 1 produces arbitrarily fast local exponential convergence, see [10].

IV. PROOF OF UGAS OF (8) IN CLOSED LOOP WITH (14)

We prove that (8) with the feedback (14) has the Lyapunov function (15) if the $a_i$’s satisfy (13); see [10] for the rest of the proof of the theorem. Along the solutions of the \textit{reduced dynamics}

$$\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= -\kappa_1 e_1 - \frac{\kappa_1 m \sigma}{2\kappa_1} \left( \frac{3\alpha m_1}{\kappa_1 m_1} e_1 \right) - \kappa_2 e_2 \quad \text{(16)}
\end{align*}$$

the function $V_1$ in (15) has time derivative $\dot{V}_1 \leq -2\kappa_2 e_2^2$. By (I)-(II) from Remark 5 and simple calculations, the derivative of $T(e_1, e_2) = e_1 e_2$ along solutions of (16) yields

$$\dot{T} \leq e_2^2 - \kappa_1 e_1^2 + \left\{ \frac{(\kappa_1 + \kappa_2)\xi_1}{\sqrt{\kappa_1}} \right\} \left\{ \sqrt{\kappa_1} e_1 \right\}$$

$$\leq \left( 1 + \frac{(\kappa_1 + \kappa_2)^2}{2\kappa_1} \right) e_2^2 - \frac{1}{2} \kappa_1 e_1^2,$$

by using the triangle inequality $pq \leq \frac{1}{2}p^2 + \frac{1}{2}q^2$ on the terms in braces. Thus, $\dot{V}_2 \leq -W_2(e_1, e_2)$ along the solutions of (16), where $V_2$ is from (15) and $W_2(e_1, e_2) = e_2^2 + \kappa_1 e_1^2/2$. Also, $V_2(e_1, e_2) \geq e_1 e_2 + K(e_2^2 + \kappa_1 e_1^2)/2 \geq K e_2^2/4 + K \kappa_1 e_1^2/2$ everywhere.

Using $\frac{\partial V_2}{\partial e_1}(e_1, e_2) = e_1 + K e_2$, we get

$$\dot{V}_2 \leq -W_2(e_1, e_2)$$

$$+ e_2^2 - \kappa_1 e_1^2 + \left\{ \frac{(\kappa_1 + \kappa_2)\xi_1}{\sqrt{\kappa_1}} \right\} \left\{ \sqrt{\kappa_1} e_1 \right\}$$

$$\leq -W_2(e_1, e_2)$$

$$+ \sqrt{\frac{W_2(e_1, e_2)}{2\kappa_1}} \left\{ \kappa_2 + 2\xi R_\mu(e_1, e_2, t) \right\}$$

$$\leq -\frac{1}{2} W_2(e_1, e_2) + \frac{1}{\sqrt{\kappa_1}} \left\{ \kappa_2 + 2\xi R_\mu(\xi) \right\}$$

along the solutions of the full system (8), where we again used $pq \leq \frac{1}{2}p^2 + \frac{1}{2}q^2$ on the terms in braces. Recalling that $|e'| \leq 1$ everywhere and (9) gives

$$\frac{1}{2} |\dot{\xi}| \leq T(e_1 + e_2) + |\xi| + \zeta^2,$$

$$T := \frac{1}{2\kappa_2} \left( a_1 + a_2 (1 + \kappa_2) \right).$$

Hence, along the closed loop trajectories of (8), the time derivative for $Q$ from (15) gives

$$\dot{Q} \leq -[\zeta^2 + \zeta^4 + \zeta^6] + \frac{1}{Q \alpha_3} [\xi^2 + |\xi| + |\zeta|]$$

$$\leq -[\zeta^2 + \zeta^4 + \zeta^6] + \frac{1}{Q \alpha_3} [\xi^2 + e_2^2 + 2\zeta^2 + |\zeta|^2]$$

$$\leq -\left( 1 - \frac{6}{Q \alpha_3} \right) [\zeta^2 + \zeta^4 + \zeta^6] + \frac{1}{Q \alpha_3} [e_2^2 + e_2^2]$$

$$\leq -\frac{1}{4} [\zeta^2 + \zeta^4] + \frac{1}{Q \alpha_3} \kappa_1 e_1^2 + \frac{1}{Q \alpha_3} e_2^2,$$

by considering the possibilities $|\zeta| \geq 1$ and $|\zeta| < 1$, and then using $pq \leq \frac{1}{2}p^2 + \frac{1}{2}q^2$ and the fact that $\alpha_3$ satisfies (13). Hence, our positive definite proper function $V_2$ in (15) gives $V_3 \leq -\left( W_2(e_1, e_2) + e_2^2 + e_2^2 \right)/4$ and therefore is a Lyapunov function for the system, which gives the UGAS assertion in the theorem.

V. ROBUSTNESS ANALYSIS

Next assume that the constant $k$ in (1) is uncertain. This can be handled by replacing $k$ with $k + \varepsilon m$ in (1), where $k$ indicates an estimated or nominal value, $\varepsilon$ is the uncertainty, and $m$ is the electrode mass as before. We stipulate the admissible values of $\varepsilon$ later. Our goal is to quantify the impact of $\varepsilon$ on the tracking; see Remark 7 for the case where the parameter $b$ is uncertain. To simplify arguments, we assume $\varepsilon$ is constant, but analogous arguments apply for a general measurable essentially bounded uncertainty $\varepsilon$. With $k$ replaced by $k + \varepsilon m$ in (1) and the same change of variables we used before, the feedback

$$v_1 = -\lambda_0 \zeta (1 + \zeta^2)$$

$$+ \frac{1}{\sqrt{\kappa_1}} \frac{1}{\kappa_1} \left\{ \sqrt{\kappa_1} \right\} \left\{ \sqrt{\kappa_1} \right\} e_1$$

with $\lambda > 0$ a constant we specify below (and with $\zeta$, $\mu$, $R_\mu$, and $y_d$ defined as before) gives the $Y = (e_1, e_2, \zeta)$ system

$$\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= -\kappa_1 e_1 - \kappa_2 e_2 + \mu(e_1, e_2) + \xi^2 + 2\xi R_\mu(e_1, e_2, t) - e_2(e_1 + y_d) \\
\dot{\zeta} &= -\lambda_0 \zeta (1 + \zeta^2) - \frac{1}{\sqrt{\kappa_1}} \frac{1}{\kappa_1} \left\{ \sqrt{\kappa_1} \right\} e_1 + K e_2$$
\end{align*}$$

where the function $\mu$ will be chosen later to satisfy $|\mu(e_1, e_2)| \leq 0.5 \kappa_1 m_1$ everywhere and we calculate $\mu$ along the $c$-subdynamics of (21). We aim to show that $\mu$ can be selected so that the dynamics (21) is \textit{input-to-state stable (ISS) with respect to $\varepsilon$ with arbitrarily small linear overflows}. The relevant definitions are as follows.

We call (21) (with prescribed $\lambda$ and $\mu$) ISS with respect to $\varepsilon$ provided that there are functions $\beta \in K_{L}$ and $\Delta \in K_{\infty}$ so that all solutions $Y(t)$ of (21) for all initial values $Y(t_0)$ and
all $\varepsilon$ satisfy $|Y(t)| \leq \beta(|Y(t_0)|, t - t_0) + \Delta(|\varepsilon|)$ when $t \geq t_0 \geq 0$. When $\Delta$ has the form $\Delta(r) = \tau r$ for a constant $\tau > 0$, we call $\tau$ an overflow rate. The ISS inequality becomes the standard UGAS condition when $\varepsilon$ is zero, but is far more general since its overshoot term $\Delta(|\varepsilon|)$ quantifies the impact of $\varepsilon$. Notice that ISS of (21) implies that $|e_1(t)| = |x(t) - y_d(t)| \to 0$ as $t \to +\infty$ with a small overshoot term when the uncertainty $\varepsilon$ is suitably small.

We then say that (21) can be rendered ISS with respect to $\varepsilon$ with arbitrarily small linear overflows provided that we can find a function $\beta \in \mathcal{KL}$ such that: For each choice of the constant $\tau > 0$, we can select our function $\mu$ and a constant $\lambda > 0$ (which, in general, both depend on $\tau$) such that all solutions $Y(t)$ of (21) for all choices of the initial state $Y(t_0)$ satisfy $|Y(t)| \leq \beta(|Y(t_0)|, t - t_0) + \tau|\varepsilon|$ whenever $t \geq t_0 \geq 0$. This means that we can pick the controller to guarantee that the overflow rate $\tau$ is as small as desired.

To prove our ISS conditions, we use the Lyapunov function from Remark 6 except with $\zeta \equiv 0$, namely,

$$V_0(e_1, e_2) = Ae_1^2 + e_2^2 + B e_1 e_2, \quad A = \kappa_1 + \frac{B_0}{\sqrt{m}} \quad \text{and} \quad B = \min\left\{\kappa_2, \sqrt{\kappa_1}\right\}. \tag{22}$$

Separately considering the possibilities $B = \kappa_2$ and $B = \sqrt{\kappa_1}$ leads easily to the global inequalities

$$\omega_1(e_1^2 + e_2^2) \leq \dot{V}_0(e_1, e_2) \leq \omega_2(e_1^2 + e_2^2), \quad \omega_1 = \frac{1}{2} \min\left\{\kappa_1, 1\right\} \quad \text{and} \quad \omega_2 = \max\left\{A, 1\right\} + \frac{B}{2}. \tag{23}$$

To prescribe the allowable values of $\lambda$ and $\varepsilon$, put

$$\theta = \frac{c_{\omega}}{20(4 + \beta^2)}, \quad \varepsilon = \min\left\{\frac{B_0}{B_0 + 2B_2 \max\{A, 1\}}\right\}, \quad M = \frac{1}{\theta} \max\left\{1, 4\bar{R}^2\right\}, \quad \text{and} \quad \bar{L} = \kappa_1 m_1 \left\{2(\kappa_1 |m_1 + 1| + \kappa_2 + 2\bar{R}) + 3B\kappa_1 m_2 + 2m_2 + 2 + B\right\}. \tag{24}$$

where the $m_is$ are from (3) and $\bar{R}$ is from (6). We assume that $\varepsilon$ satisfies

$$|\varepsilon| \leq \min\left\{\frac{B_0}{20(B + 1)}, \frac{B_2}{20}, \frac{2\kappa_1 m_1}{5m_2}\right\} \tag{25}$$

but see Remark 7 for robustness results under less restrictive bounds on $\varepsilon$, and Section VI for an example where we compute our bounds explicitly. In [10], we prove:

**Theorem 2**: The choices $\sigma(s) = s/\sqrt{1 + s^2}$ and

$$\mu(e_1, e_2) = -\frac{9}{20} \kappa_1 m_1 \sigma\left(\lambda \frac{\partial V_0}{\partial e_2}(e_1, e_2)\right) \tag{26}$$

with any constant $\lambda > 2 + \frac{2\xi}{\kappa_1} (1/\sqrt{\kappa_1} + 1)(M + 1)$ render (21) ISS with respect to uncertainties $\varepsilon$ satisfying (25). In fact, by choosing $\mu$ as in (26) and $\lambda$ appropriately, we can make (21) ISS with respect to $\varepsilon$ with arbitrarily small linear overflows. Moreover, $V(e_1, e_2, \zeta) := V_0(e_1, e_2) + (M + 1)(\zeta^4/4 + \zeta^2)/2$

is an ISS Lyapunov function for (21) when the uncertainty $\varepsilon$ is constrained to satisfy (25).

**Remark 7**: The bound (25) implies arbitrarily small linear overflows in the ISS condition. If we only want to prove ISS of the dynamics (21) with respect to the uncertainty $\varepsilon$ (with no restriction on the overflow rate for the ISS estimate), then we can use the less restrictive bound

$$|\varepsilon| \leq \min\left\{\frac{B_0}{20(B + 1)}, \frac{B_2}{20}\right\} \tag{27}$$

with the same $\mu$ from (26). For details, see [10].

Using similar reasoning, we can handle cases where the parameter $b$ from (1) is uncertain. We model this case by replacing $b$ with the uncertain value $b + \varepsilon m$ in (1), where $\varepsilon$ represents the uncertainty and $m$ is the movable electrode mass. Then we can show that the resulting closed loop dynamics with (26) satisfies the ISS condition with respect to additive uncertainties $\varepsilon$ on $b$ satisfying

$$|\varepsilon| \leq \min\left\{\frac{\kappa_2}{5}, \frac{9\kappa_1}{20B}\right\}, \tag{28}$$

provided $\lambda > 0$ is a large enough constant; see [10] for the proof. Analogous results hold when the model has two independent (suitably small) parametric uncertainties, one on $k$ and one on $b$.

**VI. SIMULATIONS**

To illustrate the efficacy of our approach, we simulated (1) with the feedback $u$ from (7), $\mu$ given by (12), and $v_1$ from (14). We chose $a_1 = a_2 = 1$ and $a_3 = 100$. We used the parameters $m = 1$, $k = 2.5$, $\gamma = 1$, $b = 1$, $\alpha = 0.5$, $\beta = 0.001$, and $y_0 = 1$, and the periodic $C^3$ reference trajectory

$$y_d(t) = \begin{cases} 0.01 + e_1 [I(500 + \min\{t, 50\}) - I(\min\{\max\{t, 450\}, 550\})] + I(\max\{t, 950\} - 500) \quad \text{for} \quad 0 \leq t \leq 1000, \\
0.01 - e_1 (t - 1000) \quad \text{for} \quad t \geq 1000, 
\end{cases} \tag{29}$$

in which $I(r) = \int_{500}^{550}(s - 450)^3(550 - s)^3ds$ and $e_1 = 0.99/I(550) = 1.386 \times 10^{-12}$. The function (29) is a smoothed standard square wave with offset 0.01 as shown in Figure 3. This mimics the periodic closing and opening of the relay. Conditions (3) are satisfied with $m_1 = 0.01$, $m_2 = 1$, $m_3 = 0.0216$, and $m_4 = 0.00074386$, which give $\bar{R} = 0.01249$ and $\bar{R} = \sqrt{5}$.

For the initial state $(x, x, x, z)(0) = (0, 0, 10)$, we report our simulated error $e_1(t) = x(t) - y_d(t)$ in Figure 4. In Figures 5-6, we show the control $u$ in its steady state and transient state. They illustrate how the tracking error rapidly converges to zero. The convergence also enjoys a desirable robustness to uncertainties in the parameters $k$ and $b$. The bound in (27) is 0.45, which implies ISS convergence with respect to additive uncertainties $\varepsilon$ on $k$, as long as $|\varepsilon|$ is kept below 18% of the value $k = 2.5$. Also, the bound from (28) is 0.2, which implies ISS with respect to uncertainties $\varepsilon$ on $b$ provided $\varepsilon$ is below 20% of our nominal value $b = 1$.  


VI. CONCLUSIONS

We used Lyapunov function theory to design a family of nonlinear tracking feedback controllers for electromagnetic and electrostatic MEM relays. We established explicit conditions on the reference trajectory that guarantee that the trajectories can be tracked, and which are compatible with typical, alternating off-on relay operations. We then used ISS to quantify the robustness of our controllers to parametric uncertainty. The structure of the relay model precludes the construction of a tracking controller that gives global arbitrarily rapid exponential decay for the tracking error, but our proposed state feedback control has this property locally. Our numerical simulation illustrates the efficacy of our feedback control for a smoothed periodic square wave that mimics the periodic closing and opening of the relay.

REFERENCES