Output Feedback Stabilization of Nonlinear Feedforward Systems using Arbitrarily Bounded Control

Jason Polendo† and Cheryl B. Schrader‡

Abstract—In this paper, we consider the output feedback stabilization problem of a class of nonlinear feedforward (upper triangular) systems, of which the stabilization using an arbitrarily bounded controller has not been addressed previously. This stabilization is accomplished here via nested saturation linear control and dynamic compensator. The novelty in such an approach is that the maximum controller value can be set prior to designing the control, whereas in similar design schemes the maximum controller level is required to be sufficiently small.

I. BACKGROUND AND MOTIVATION

In this paper we develop a constructive methodology to globally stabilize a nonlinear feedforward system using output feedback. The system to be stabilized is of the form

\[
\begin{align*}
\Sigma_1: \quad & \dot{x}_1 = x_2 + f_1(x_3, \ldots, x_n, u) \\
& \dot{x}_2 = x_3 + f_2(x_4, \ldots, x_n, u) \\
& \vdots \\
& \dot{x}_{n-2} = x_{n-1} + f_{n-2}(x_n, u) \\
& \dot{x}_{n-1} = x_n \\
& \dot{x}_n = u \\
& y = x_1
\end{align*}
\]

where the nonlinear terms are assumed to be Lipschitz in \(x\) and vanish at the origin. Briefly stated, systems of the form of \(\Sigma_1\) cannot be stabilized by output feedback using existing stabilization techniques for feedforward systems while maintaining an arbitrary control bound.

The problem of stabilization of feedforward systems is well known in the nonlinear controls research community and has received considerable attention in recent years. A fairly comprehensive literature review was given in [9]; however, for the sake of completeness, a brief overview of results will be addressed. In particular, state feedback controllers using nested saturation designs [1] have been presented in [3]-[7], [11], while a controller designed using the recursive technique of forwarding was presented in [8]. Forwarding and nested saturation designs were combined in [10] to alleviate restrictive growth conditions. Previous state feedback nested saturation designs such as [3], [4], [6], [7] necessitate the nonlinear term be of higher order, thus requiring the saturation levels to be sufficiently small. This not only severely limits the linear operating range of the controller, but makes the designs very susceptible to additive disturbances and modeling errors. Additionally when the initial state values are relatively large and the controller saturated, the rate of change is a very small linear constant of time inherited from the saturation level, thus substantially increasing correction time. [11] considered locally Lipschitz systems with a switching scheme controller. However, the saturation levels are again assumed to be sufficiently small to accommodate the locally Lipschitz definition, thereby also allowing the designs to be open to disturbance errors leaving the controller unable to stabilize the system. State feedback forwarding schemes [8], on the other hand, are known to be quite complex in their construction and at times necessitate numerical approximations of nonlinear ODE’s providing a difficult stabilization path. Such inherent complexities do not appear to lend themselves well to output feedback stabilization designs. [9] presented a linearization technique for feedforward systems which makes use of a forwarding-type state feedback controller and eliminates the need to solve a series of nonlinear ODE’s, though a new need is encountered in the computation of a series of integrals. As far as the authors know, the issue of output feedback stabilization of nonlinear feedforward systems has only been addressed in [12] where an unbounded high-gain scaling controller was developed; in [5], where marginally stable free dynamics were restrictively assumed; and in [2], where the aforementioned forwarding approach is applied, though the form of the final control law and its bound are unknown.

Herein we develop a controller inherently more robust to additive disturbances than existing nested saturation designs since the saturation levels are not restricted to be small, thus greatly expanding the linear operating range of our control design. And since nested saturation is used a bounded controller is guaranteed which is quite straightforward in its construction. The novelty in this approach, as opposed to existing designs, is that the maximum controller value can either be arbitrarily selected or given as a design constraint.

II. GLOBAL OUTPUT FEEDBACK STABILIZATION

In this section, it is shown that nonlinear feedforward systems of the form of \(\Sigma_1\) can be globally asymptotically stabilized using output feedback. This is first accomplished
by designing an observer and proving the error dynamics are asymptotically stable independent of the controller used.

**Lemma 2.1:** The states of Σ1 can be asymptotically estimated by output feedback. 

**Proof.** The observer implemented is of the form

\[
\begin{align*}
\dot{x}_1 &= \dot{x}_2 + \frac{1}{L} a_1(x_1 - \hat{x}_1) + f_1(\hat{x}_3, \ldots, \hat{x}_n, u) \\
\dot{x}_2 &= \dot{x}_3 + \frac{1}{L} a_2(x_2 - \hat{x}_1) + f_2(\hat{x}_4, \ldots, \hat{x}_n, u) \\
&\vdots \\
\dot{x}_{n-1} &= \dot{x}_n + \frac{1}{L^{n-1}} a_{n-1}(x_{n-1} - \hat{x}_1) \\
\dot{x}_n &= u + \frac{1}{L^n} a_n(x_{n} - \hat{x}_1)
\end{align*}
\]

where \( L > 1 \) is a constant to be determined later and \( a_j > 0, j = 1, \ldots, n \), are coefficients of the Hurwitz polynomial \( p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \).

Defining \( \xi_i = \frac{x_i - \hat{x}_i}{\epsilon}, i = 1, \ldots, n \), it is straightforward to verify that

\[
\dot{\xi} = \frac{1}{L} A \epsilon + \left[ \begin{array}{c} \frac{1}{L^{n-1}} (f_1 - \hat{f}_1) \\ \vdots \\ \frac{1}{L^n} (f_{n-2} - \hat{f}_{n-2}) \\ 0 \end{array} \right] = \frac{1}{L} A \epsilon + \psi
\]

where \( \epsilon = \left[ \begin{array}{c} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{array} \right] \) and \( A = \left[ \begin{array}{c} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{array} \right] \).

Since \( A \) is a Hurwitz matrix by construction, the existence of a positive definite symmetric matrix \( P \) such that \( A^T P + PA = -I \) is implicit. The time derivative of the Lyapunov function \( V = \epsilon^T P \epsilon \) yields

\[
\dot{V} = \frac{1}{L} \epsilon^T (AT P + PA) \epsilon + 2 \epsilon^T P \psi \\
= -\frac{1}{L} \| \epsilon \|^2 + 2 \epsilon^T P \psi.
\]

Examining an arbitrary component of \( \psi \),

\[
\left| \frac{1}{L^{n-1}} (f_i(\cdot) - \hat{f}_i(\cdot)) \right| \leq \frac{1}{L^{n-1}} k_i \left\| \begin{array}{c} L^{n-2} \xi_{i+2} \\ \vdots \\ L \xi_{n-1} \\ \xi_n \end{array} \right\|
\leq \frac{1}{L^{n-1}} k_i \left\| \begin{array}{c} L^{n-2} \xi_{i+2} \\ \vdots \\ L^{n-2} \xi_{i+3} \\ \cdots \\ L^{n-2} \xi_n \end{array} \right\|
\leq k_i \| \epsilon \| / L^2
\]

since \( L < L^2 < \cdots < L^{n-i-2} \). Taking into account (3), the Lyapunov function under consideration becomes

\[
\dot{V} \leq -\frac{1}{L} \| \epsilon \|^2 + \frac{2}{L^2} \epsilon^T P k_i \| \epsilon \|
\leq -\frac{1}{L} \| \epsilon \|^2 + \frac{2k}{L^2} \| P \| \| \epsilon \|^2.
\]

Choosing \( L > 2 \| P \|, k = \text{max} \{ k_i \} \), will guarantee that the Lyapunov function is negative definite and the error dynamics go to zero asymptotically, thereby proving \( \dot{x} \to x \) asymptotically.

**Definition 2.1:** Given a positive constant \( M_i \), a function \( \sigma_i : \mathbb{R} \to \mathbb{R} \) is said to be a simple saturation if it is a continuous, nondecreasing function with \( \sigma_i(s) = s \) for all \( |s| < M_i \) and \( \sigma_i(s) = M_i \text{sign}(s) \) for all \( |s| \geq M_i \).

As in [3], a change of coordinates will be utilized, albeit with a simpler form.

**Theorem 2.1:** Systems of the form \( \Sigma_2 \) are globally stabilizable by output feedback using a nested saturation controller of the form

\[
u = -\sigma_n(K_n z_n + \epsilon_n + \lambda_{n-1} \sigma_{n-1}(K_{n-1} z_{n-1} + \epsilon_{n-1} + \cdots + \lambda_1 \sigma_1(K_1 z_1 + \epsilon_1)) \cdots)
\]

where \( z = T \dot{x} \) is an invertible linear coordinate transformation given by \( z_i = \bar{x}_i + \hat{x}_n \) and \( z_n = \bar{x}_n \), \( \lambda_i \) and \( K_j \) are positive constants, and \( \epsilon_j \) represents perturbations due to modeling errors, uncertainties, or deviations in the output feedback implementation, \( i = 1, \ldots, n-1, j = 1, \ldots, n \).

Under this construction, the saturation level of the \( i^{th} \) function, \( \sigma_i(\cdot) \), can be arbitrarily set (not necessarily small), thus giving a pre-determined and well-defined maximum control value.

**Proof.** Under the linear coordinate transformation \( z = T \dot{x} \) the new system is given as

\[
\Sigma_2
\]

Before proceeding some preliminaries need to be addressed. We must first note that since the nonlinear term is Lipschitz, then it is also bounded by a linear growth in \( x \) and therefore by a linear growth in \( z \). It is simple to verify that since \( |f_i| \leq k_i \| \bar{x}_{i+2} \|, \) where \( \bar{x}_{i+2} = [x_{i+2}, \ldots, x_n]^T, \) then \( |f_i| \leq c_{f_i}(|x_{i+2}| + \cdots + |x_n|) \), and \( |f_i| \leq c_{f_i}(|z_{i+2}| + \cdots + |z_{n-1}| + |z_n|) \), for some positive constants \( c_{f_i}, c_{\gamma} \).

In designing the gain values we first let \( p_1 = 1, p_2 = 3, \) and \( p_i = \text{max}(c_{f_{i-2}}, \ldots, c_{f_1}) + 2, i = 3, \ldots, n \). Define \( K_n = p_n \) and

\[
K_i = p_i/(\lambda_{n-1} \cdots \lambda_i), \quad i = 1, \ldots, n-1.
\]
Choose $\lambda_{n-1} = 2p_n$, let $\lambda_{n-2} = 2p_{n-1}$ and

$$\lambda_i = \frac{2p_{i+1}}{\lambda_{i+1}}, \quad i = 1, \ldots, n-3.$$  \hspace{1cm} (5)

Note that the gains are selected in such a way that at any certain level, $\lambda_{n-1}\lambda_i = \lambda_{n-1}(2p_{i+1}) > 2p_{i+1} + p_{i+1} \geq \lambda_{n-1}K_{i+1}$.

For all saturation levels, select

$$M_{i-1} < \frac{1}{4\lambda_{i-1}} M_i, \quad i = 1, \ldots, n$$ \hspace{1cm} (6)

and suppose $|\epsilon_i| \leq \frac{\lambda_i}{2n} M_i \forall i = 1, \ldots, n$.

The proof is iterative in its approach and is accomplished by first considering the $n$th state $z_n = x_n$. Define $\Omega_n = \{z_n : K_n|z_n| \leq \frac{1}{2} M_n\}$, where $M_n$ is the controller maximum value. As can be seen from the state equation $\dot{z}_n = u$, the evolution of $z_n$ depends solely on the controller. Suppose $z_n \notin \Omega_n$ and note that $|\epsilon_n| < \frac{1}{M_n} M_n$. In the instance when $K_n z_n > \frac{1}{2} M_n$, $K_n z_n + \epsilon_n + \lambda_n-1\sigma_n(-1) > \frac{1}{2} M_n - \frac{1}{2} M_n - \lambda_n-1 M_n = b_n > 0$, since by definition $\sigma_{n-1}(-1) \leq M_n$ and from (6) $M_{n-1} < \frac{1}{\lambda_{n-1}} M_n$. Therefore, $\dot{z}_n = -\sigma_n K_n z_n + \epsilon_n + \lambda_n-1\sigma_n(-1) \leq \sigma(b_n)$ which shows that the trajectory of $z_n$ is a strictly decreasing function of time in this range, giving that $z_n \in \Omega_n$ in finite time.

For the case when $K_n z_n < -\frac{1}{2} M_n$, $K_n z_n + \epsilon_n + \lambda_n-1\sigma_n(-1) < -\frac{1}{2} M_n + \frac{1}{2} M_n-\lambda_n-1 M_n = b_n > 0$. So, $\dot{z}_n = -\sigma_n(0)(K_n z_n + \epsilon_n + \lambda_n-1\sigma_n(-1)) \geq \sigma(b_n)$ which shows that the trajectory of $z_n$ is a strictly increasing function of time in this range, giving that $z_n \in \Omega_n$ in finite time. Thus, after some time $T_1$, $z_n \in \Omega_n$ and stays thereafter. The state equations for $z_n$ and $z_{n-1}$ are now

$$\dot{z}_n = -K_n z_n - \epsilon_n - \lambda_n-1\sigma_n(-1)$$ \hspace{1cm} (7)

$$\dot{z}_{n-1} = -\epsilon_n + \lambda_n-1\sigma_n(-1)$$ \hspace{1cm} (8)

where $\sigma_{n-1}(-1) = \sigma_{n-1}(K_n z_{n-1} + \epsilon_n + \lambda_n-2\sigma_{n-2}(-1))$. Define $\Omega_{n-1} = \{z_{n-1} : K_n|z_{n-1}| = \frac{1}{2} M_{n-1}\}$ and suppose $z_{n-1} \notin \Omega_n$. Note $|\epsilon_{n-1}| < \frac{1}{2n} M_{n-1}$.

When $K_n z_{n-1} > \frac{1}{2} M_{n-1}$, $K_n z_{n-1} + \epsilon_n + \lambda_n-2\sigma_{n-2}(-1) > \frac{1}{2} M_{n-1} - \frac{1}{2} M_{n-2} - \lambda_n-2 M_{n-2} = b_n > 0$, since by definition $\sigma_{n-2}(-1) \leq M_{n-2}$ and from (6) $M_{n-2} < \frac{1}{\lambda_{n-2}} M_{n-1}$. Therefore, $\dot{z}_n = -K_n z_n - \epsilon_n - \lambda_n-1\sigma_n(-1)$ which shows that after some time $T_1$, $z_n$ is approximately less than or equal to $-\frac{z_n}{K_n} - \frac{1}{\lambda_n} \sigma_n(-1)$. The trajectory for $z_{n-1}$ is now governed by $\dot{z}_{n-1} \leq \frac{z_n}{K_n} + (K_n - 1)\frac{1}{2n} \sigma_{n-1}(b_{n-1}) - \lambda_n-1\sigma_{n-1}(b_{n-1}) = -c_{n-1}$ for some constant $c_{n-1}$, since $\epsilon_n$ can be shown to satisfy $|\epsilon_n| < |b_{n-1}|$. As before, this implies that $z_{n-1} \in \Omega_{n-1}$ in finite time.

A similar argument can be made for the instance when $K_n z_{n-1} < -\frac{1}{2} M_{n-1}$, thus proving that after some time $T_2 > T_1 + T_1$, $z_{n-1} \in \Omega_{n-1}$ and remains thereafter. The controller equation is now

$$u = -p_n z_n - \epsilon_n - p_{n-1} z_{n-1} - \lambda_n-1\epsilon_n-1 - \lambda_n-1\lambda_n-2\sigma_{n-2}(-1)$$ \hspace{1cm} (9)

As with $z_n$, after some time $T_2$, $z_{n-1}$ can be approximated by $\frac{-c_{n-1}}{K_n} - \frac{1}{\lambda_n} \sigma_{n-2}(-1)$, while $z_n$ can now be approximated by $\frac{\lambda_n-1\epsilon_n-1}{K_n} \in \Omega_n$.

Examining the state equation for $z_{n-2}$ yields

$$\dot{z}_{n-2} = z_{n-1} - z_n + f_{n-2} + u \leq \dot{z}_{n-1} - z_{n-1} + c_{n-2}|z_n| + u$$

$$= -(p_n - 1) z_{n-1} - \epsilon_n z_n - \lambda_n-1\lambda_n-2\sigma_{n-2}(-1) - c_{n-1} - \lambda_n-1\epsilon_n-1$$

where $c_n$ is a positive constant. Substituting in the approximations for $z_n$ and $z_{n-1}$ gives $z_{n-2} = -c_{n-1} \epsilon_n - c_{n-1} \epsilon_n-1 - \epsilon_n-2\sigma_{n-2}(-1)$ for constants $c_{n-1}$ and $c_{n-1}$ and a positive constant $c_{n-2}$ and the previous arguments can be recycled. $n-3$ repetitions of this process yields a linear controller with all states proven to be globally stable, i.e. in $\Omega_i$.

**Remark 2.1:** In the case when the modeled perturbations $\epsilon_i$ are vanishing, then all the states of $\Sigma_n$ are proven to be globally asymptotically stable.

**Remark 2.2:** If state feedback stabilization is the control goal for systems of the form of $\Sigma_1$, then the dynamic equation for $x_i$ can be allowed to contain nonlinear terms with $x_{i+1}$ in them under the restriction that the term must either vanish at $u = 0$ or at $(x_{i+1}, 0, \ldots, 0)$. For output feedback, however, this term cannot be present.

**REFERENCES**


