Abstract — $H_\infty$ control problem for linear discrete-time systems with instantaneous and delayed measurements is studied. A sufficient and necessary condition for the existence of the $H_\infty$ controller is derived by applying re-organized innovation analysis approach. The measurement-feedback controller is designed by performing two Riccati equations. The presented approach does not require the state augmentation.

I. INTRODUCTION

$H_\infty$ control has been one of the most important topics in control theory and has attracted much attention of numerous researchers in the past two decades. In 1981, Zames [1] originally proposed the $H_\infty$ control problem in an input-output setting. It was an important advantage that Doyle introduced the state space method to output setting. It was an important advantage that Doyle originally proposed the control theory and has attracted much attention of numerous researchers in the past two decades. In 1981, Zames [1] and Tadmor [4] extended the $H_\infty$ control problem to the time-variant, finite horizon case by maximum principle. Plenty of results in frequency domain [1], [5], [6] and time domain [2]-[4] have been achieved.

Recently, increasing attention has been paid to the problem of $H_\infty$ control for delay systems. Delays may exist in many practical control problems, such as process control. A lot of interesting results for this problem have been introduced in many practical control problems, such as process control. A lot of interesting results for this problem have been presented in [17]-[20] and references therein. However, only sufficient condition for the existence of the controller is given in most of the previous works.

In this paper, we study the $H_\infty$ measurement feedback control problem for the systems with delayed measurement. A new approach is applied to derive the $H_\infty$ controller. With the using of a re-organization innovation, we convert the delayed measurements into measurement delay-free. The $H_\infty$ control problem is equivalent to an $H_2$ estimation problem in Krein space.

This paper is organized as follows: the problem statement is presented in Section II. The main results are presented in Section III. Some concluding remarks end the paper.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Problem Statement

Consider the time-variant state-space model with instantaneous and delayed measurements

\begin{align*}
  x(t+1) &= F_1x(t) + G_1w(t) + G_2u(t) \quad (1) \\
  y(t) &= H_1x(t) + v(t) \quad (2) \\
  z_{t-d}(t) &= M_{t-d}x(t-d) + v_z(t) \quad (3) \\
  s(t) &= L_t x(t) \quad (4)
\end{align*}

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^l$, $u(t) \in \mathbb{R}^r$, $y(t) \in \mathbb{R}^m$, $z_{t-d}(t) \in \mathbb{R}^p$ and $s(t) \in \mathbb{R}^q$ are respectively the state, process noise, control inputs, instantaneous measurement, delayed measurement and signals, $v(t) \in \mathbb{R}^n$ and $v_z(t) \in \mathbb{R}^r$ are the measurement noises. The matrices $F_1, G_1, G_2, H_1, M_{t-d}$ and $L_t$ are known matrices of appropriate dimensions. For convenience, $t_d$ denotes $t - d$ ($d > 0$) and $\ast$ stands for the transpose throughout this paper.

Let $\tilde{u}(t)$ be a control strategy, we denote

\begin{equation}
  \|T(F)\|_\infty^2 = \sup_{x(0), w, v, v_z \in h_2} \frac{A}{B},
\end{equation}

where

\begin{align*}
  A &= x^*(N+1)P_{N+1}^x x(N+1) + \sum_{t=0}^N \tilde{u}^*(t)Q_{t}^w \tilde{u}(t) \\
  &\quad + \sum_{t=0}^N s^*(t)R_{t}^i s(t) \\
  B &= x^*(0)\Pi_0^{-1} x(0) + \sum_{t=0}^N w^*(t)Q_{t}^w w(t) \\
  &\quad + \sum_{t=0}^N v^*(t)(R_{t}^{v_i})^{-1} v(t) + \sum_{t=d}^N v_z^*(t)R_{t_d}^{v_z} v_z(t)
\end{align*}

The matrices $\Pi_0$, $P_{N+1}^x Q_{t}^w Q_{t}^v R_{t}^i$, $R_{t_d}^{v_z}$ are positive (semi) definite weighting matrices.

Now we formulate the problem which will be addressed in this paper.

**Problem 1:** Consider the system (1)-(4), find a suboptimal measurement-feedback $H_\infty$ control strategy $\tilde{u}(t) = F_t(y(t), \ldots, y(t), z_0(d), \ldots, z_{t-d}(t))$ that achieves

\begin{equation}
  \|T(F)\|_\infty < \gamma,
\end{equation}

where $\gamma$ is a given positive scalar.
B. Preliminaries

In view of (5), it is not difficult to observe that Problem 1 is equivalent to \( J_N \) which has a minimum \( J_N^* \) over the variables \( \{x(0), w(0), \cdots, w(N)\} \) and \( \bar{u}(t) \) can be chosen such that \( J_N^* > 0 \), where

\[
\begin{align*}
J_N &= x^*(0)\Pi_0^{-1}x(0) + \sum_{t=0}^{N} w^*(t)Q_t^w w(t) \\
&+ \sum_{t=0}^{N} (y(t) - H_t x(t))^* (R_t^{-1}) (y(t) - H_t x(t)) \\
&+ \sum_{t=d}^{N} (z_{td}(t) - M_{td} x(td))^* (R_{td}^{-1}) (z_{td}(t) - M_{td} x(td)) \\
&\times (z_{td}(t) - M_{td} x(td)) - \gamma^{-2} \left( \sum_{t=d}^{N} s^*(t) R_{td} s(t) \right) \\
&+ x^*(N+1) P_{N+1}^c x(N+1) + \sum_{t=d}^{N} \bar{u}^*(t) Q_t^c \bar{u}(t). \\
\end{align*}
\]

Let us rewrite \( J_N \) in the following fashion

\[
\begin{align*}
J_N &= x^*(0)\Pi_0^{-1}x(0) + \sum_{t=0}^{N} (y(t) - H_t x(t))^* (R_t^{-1}) \times (y(t) - H_t x(t)) \\
&\times (R_{td}^{-1})(z_{td}(t) - M_{td} x(td)) - \gamma^{-2} \tilde{J}_N \\
\end{align*}
\]

where

\[
\tilde{J}_N = x^*(N+1) P_{N+1}^c x(N+1) + \sum_{t=0}^{N} s^*(t) R_{td} s(t) + \sum_{t=d}^{N} \bar{u}^*(t) \left[ -\gamma^2 Q_t^w \ 0 \right] \left[ w(t) \ \bar{u}(t) \right] \tag{6}
\]

According to the discussion in [7], we get

\[
\begin{align*}
\tilde{J}_N &= x^*(0) P_{0}^c x(0) \\
&+ \sum_{t=0}^{N} \left[ w(t) - \bar{w}(t) \right]^* \left[ w(t) - \bar{w}(t) \right] R_{c,t} \left[ \bar{w}(t) - \bar{u}(t) \right] \tag{9}
\end{align*}
\]

where \( \bar{w}(t) \) and \( \bar{u}(t) \) are given by

\[
\begin{align*}
\left[ \bar{w}(t) \ \bar{u}(t) \right] &= -K_{c,t} x(t) = -\begin{bmatrix} K_{w,t} \\ K_{u,t} \end{bmatrix} x(t) \\
&= -K_{c,t} x(t) = -\begin{bmatrix} G_{1,t} \ G_{2,t} \end{bmatrix} \begin{bmatrix} G_{1,t}^* \ G_{2,t}^* \end{bmatrix} \begin{bmatrix} P_{t+1}^c F_t \
G_{2,t} P_{t+1}^c G_{1,t} \end{bmatrix} \tag{10}
\end{align*}
\]

with

\[
K_{c,t} = (R_{c,t}^{-1}) \begin{bmatrix} G_{1,t}^* \\ G_{2,t}^* \end{bmatrix} P_{t+1}^c F_t \tag{11}
\]

\[
R_{c,t} = \begin{bmatrix} -\gamma^2 Q_t^w + G_{1,t}^* P_{t+1}^c G_{1,t} & G_{1,t}^* P_{t+1}^c G_{2,t} \\
G_{2,t}^* P_{t+1}^c G_{1,t} & G_{2,t}^* P_{t+1}^c G_{2,t} \end{bmatrix} \tag{12}
\]

and \( P_t^c, t = 0, \cdots, N \) satisfies the backwards Riccati equation as

\[
P_t^c = F_t P_{t+1}^c F_t + L_t R_{t}^c L_t - K_{c,t} R_{c,t} K_{c,t} \tag{13}
\]

Now (9) allows us to write \( J_N \) as follows

\[
\begin{align*}
J_N &= x^*(0)(\Pi_0^{-1} - \gamma^{-2} P_0^c)x(0) \\
&+ \sum_{t=0}^{N} \left[ w(t) - \bar{w}(t) \right] \left[ w(t) - \bar{w}(t) \right] R_{c,t} \left[ \bar{w}(t) - \bar{u}(t) \right] \\
&\times \left[ -\gamma^{-2} R_{c,t} (1,1) R_{c,t}^{-1}, (1,2) \right] \left[ 0 \ 0 \right] \\
&\times \left[ 0 \ 0 \right] \left[ -\gamma^{-2} R_{c,t} (2,1) R_{c,t}^{-1}, (2,2) \right] \\
&\times \left[ y(t) - H_t x(t) \right] \left[ y(t) - H_t x(t) \right] \\
&\times \left[ z_{td}(t) - M_{td} x(td) \right], \tag{14}
\end{align*}
\]

where the \( R_{c,t}^{ij} (i,j), (i,j) = 1,2 \) denote the \( (i,j) \) block entries of \( R_{c,t}^{ij} \) and \( z_{td}(t) = M_{td} = R_{td}^{ij} = 0 \) for \( 0 \leq t < d \).

Note that \( J_N \) is an indefinite quadratic form and includes the information of instantaneous and delayed measurement.

A new approach termed as re-organized innovation analysis in Krein space shall be developed to deal with such a problem in the following discussion.

III. MAIN RESULTS

Denote

\[
\tilde{x}(t) = \begin{cases} x(t), & 0 \leq t < d \\
\left[ x(t), x(td) \right], & t \geq d \\
y(t), & 0 \leq t < d \end{cases}, \quad Y_s(t) = \begin{cases} y(t), & 0 \leq t < d \\
z_{td}(t), & t \geq d \end{cases}
\]

and

\[
\begin{bmatrix} \Delta_t^{-1} \ S_t \ \\
S_t^{-1} (\Delta_t')^{-1} \end{bmatrix} = \begin{bmatrix} R_{c,t}^{1,1}, R_{c,t}^{1,2} \\
R_{c,t}^{2,1}, R_{c,t}^{2,2} \end{bmatrix}^{-1}. \tag{15}
\]

Then (14) is easily rewritten as

\[
\begin{align*}
J_N &= x^*(0)(\Pi_0^{-1} - \gamma^{-2} P_0^c)x(0) \\
&+ \sum_{t=0}^{N} \left[ \bar{u}(t) - \bar{u}(t) \right] \left[ w(t) - \bar{w}(t) \right] R_{c,t} \left[ \bar{u}(t) - \bar{u}(t) \right] \\
&\times \left[ \bar{u}(t) - \bar{u}(t) \right] \left[ 0 \ 0 \right] \left[ -\gamma^{-2} \bar{u}(t) \right] \\
&\times \left[ 0 \ 0 \right] \left[ -\gamma^{-2} \bar{u}(t) \right] \\
&\times \left[ y(t) - H_t x(t) \right] \left[ y(t) - H_t x(t) \right] \\
&\times \left[ z_{td}(t) - M_{td} x(td) \right], \tag{16}
\end{align*}
\]

where

\[
\begin{align*}
\bar{K}_{c,t} &= \begin{bmatrix} K_{w,t} \ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \ 0 \end{bmatrix}, \quad 0 \leq t < d \\
&= \begin{bmatrix} K_{w,t} \\ K_{u,t} \end{bmatrix} \\
&= \begin{bmatrix} K_{w,t} \ 0 \end{bmatrix}, \quad t \geq d \\
&= \begin{bmatrix} K_{w,t} \ 0 \end{bmatrix}, \quad 0 \leq t < d \\
&= \begin{bmatrix} K_{w,t} \ 0 \end{bmatrix}, \quad t \geq d
\end{align*}
\]

\[
\bar{H}_t = \begin{bmatrix} H_t \ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \ M_{td} \end{bmatrix}, \quad t \geq d \\
&= \begin{bmatrix} -\gamma^{-2} \Delta_t^{-1} \ \\
\bar{S}_t^{-1} \bar{S}_t \end{bmatrix} \\
&= \begin{bmatrix} -\gamma^{-2} \bar{S}_t \ \\
\bar{S}_t \end{bmatrix}, \quad 0 \leq t < d
\]

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where quadratic form (17)

Recalling the discussion in [12], the minimum of

\[ \begin{bmatrix} H_x(t) & 0 \\ 0 & M_{t,d} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_d) \end{bmatrix} + v_s(t), t > d \right) \]

which is given as

\[ Y_s(t) = \begin{cases} H_x(t) + v_s(t), & 0 \leq t < d \\ 0 & t \geq d \end{cases} \]

where

\[ v_s(t) = \begin{cases} v(t), & 0 \leq t < d \\ v(t)v_s(t), & t \geq d \end{cases} \]

For the convenience of discussion, we introduce the Krein space state-space model associated with estimation quadratic form (17)

\[ x(t + 1) = (F_t - G_{1,t}K_{w,t})x(t) + G_{1,t}(w(t) - \tilde{w}(t)) + G_{2,t}\tilde{u}(t) \]

\[ \begin{bmatrix} \tilde{u}(t) \\ Y_s(t) \end{bmatrix} = \begin{bmatrix} -\tilde{K}_{u,0} \\ \hat{H}_t \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{v}(t) \end{bmatrix} \]

where

\[ \hat{v}(t) = \begin{bmatrix} v(t) \\ v_s(t) \end{bmatrix} \]

and \( x(0), w(t) - \tilde{w}(t) \) and \( \hat{v}(t) \), in bold face, are Krein space variables with

\[ \begin{bmatrix} x(0) \\ w(t) - \tilde{w}(t) \\ \hat{v}(t) \end{bmatrix} \]

\[ \begin{bmatrix} w(r) - \tilde{w}(r) \\ \hat{v}(r) \end{bmatrix} \]

\[ = \left( (\Pi_0^{-1} - \gamma^2 P_0^{-1})^{-1} 0 \\ 0 \begin{bmatrix} Q_t^{w} & S_t \end{bmatrix} \delta_r \right). \]

Recalling the discussion in [12], the minimum of \( J_N \) is given by

\[ J_N = \sum_{i=0}^{N} \begin{bmatrix} \tilde{u}(t) + K_{u,0}\tilde{x}(t | t - 1) \\ Y_s(t) - \hat{H}_t\tilde{x}(t | t - 1) \end{bmatrix} \]

\[ \times Q_{w,s}^{-1}(t) \begin{bmatrix} \tilde{u}(t) + K_{u,0}\tilde{x}(t | t - 1) \\ Y_s(t) - \hat{H}_t\tilde{x}(t | t - 1) \end{bmatrix} \]

where

\[ \tilde{x}(t | t - 1) = \begin{bmatrix} \tilde{x}(t | t - 1), & 0 \leq t < d \\ \tilde{x}(t_d | t - 1), & t \geq d \end{bmatrix} \]

The value of \( \tilde{x}(t | t - 1) \) and \( \tilde{x}(t_d | t - 1) \) in (25) are obtained from the projection of \( x(t) \) and \( x(t_d) \) onto the linear space

\[ \mathcal{L} \left\{ \begin{bmatrix} \tilde{u}(i) \\ Y_s(i) \end{bmatrix} \right\}_{i=0}^{t-1}, \]

respectively. In (24), \( Q_{w,s}(t) \) is the covariance matrix of innovation \( W_s(t) \), which is given as

\[ W_s(t) = \begin{bmatrix} \tilde{u}(t) \\ Y_s(t) \end{bmatrix} - \begin{bmatrix} \tilde{u}(t | t - 1) \\ Y_s(t | t - 1) \end{bmatrix} \]

\[ = \begin{bmatrix} -K_{u,0} \\ 0 \\ H_t \end{bmatrix} \begin{bmatrix} x(t) - \tilde{x}(t | t - 1) \\ x(t) - \tilde{x}(t | t - 1) \\ x(t_d) - \tilde{x}(t_d | t - 1) \end{bmatrix} \]

\[ + \begin{bmatrix} \tilde{v}(t) \\ \tilde{v}(t) \end{bmatrix}, \quad 0 \leq t < d \]

\[ = \begin{bmatrix} -K_{u,0} \\ 0 \\ H_t \end{bmatrix} \begin{bmatrix} x(t) - \tilde{x}(t | t - 1) \\ x(t) - \tilde{x}(t | t - 1) \\ x(t_d) - \tilde{x}(t_d | t - 1) \end{bmatrix} \]

\[ + \begin{bmatrix} \tilde{v}(t) \\ \tilde{v}(t) \end{bmatrix}, \quad t \geq d \]

where \( \tilde{x}(t | t - 1) \) (\( \tilde{x}(t_d | t - 1) \)) is the projection of \( x(t) \) (\( x(t_d) \)) onto the linear space \( \mathcal{L} \left\{ \begin{bmatrix} \tilde{u}(i) \\ Y_s(i) \end{bmatrix} \right\}_{i=0}^{t-1} \). It should be seen that the estimator \( \tilde{x}(t | t - 1) \) and innovation covariance matrix \( Q_{w,s}(t) \) play important role for designing the controller. Note the observation \( \begin{bmatrix} \tilde{u}(i) \\ Y_s(i) \end{bmatrix} \) contains time delay, the standard Kalman filtering formulation is not applicable to compute \( \tilde{x}(t | t - 1) \) and \( Q_{w,s}(t) \). To deal with such problems we shall re-organize the delayed measurements and define re-organization innovation. The estimator \( \tilde{x}(t | t - 1) \) and innovation covariance matrix \( Q_{w,s}(t) \) can be calculated by using innovation analysis method.

From (15), it is easy to verify that

\[ \mathcal{L} \left\{ \begin{bmatrix} \tilde{u}(i) \\ Y_s(i) \end{bmatrix} \right\}_{i=0}^{t-1} = \]

\[ = \left\{ \begin{bmatrix} \tilde{u}(0) \\ y_f(0) \\ \tilde{u}(t_d + 1) \end{bmatrix} \right\} \begin{bmatrix} x(i) + v_f(i), & i = 0, \ldots, t - d \\ y(i) \\ y(t_d + 1) \end{bmatrix} \]

where

\[ y_f(i) = \begin{bmatrix} y(i) \\ z(i + d) \end{bmatrix} = \begin{bmatrix} H_i \\ M_i \end{bmatrix} x(i) + v_f(i), \quad i = 0, \ldots, t - d \]

with

\[ v_f(i) = \begin{bmatrix} -v(i) \\ -v_s(i + d) \end{bmatrix}, \quad 0 \leq i < t - d. \]

When \( 0 \leq t < d \), we have the relationships

\[ \begin{bmatrix} \tilde{u}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -K_{u,t} \\ H_t \end{bmatrix} x(t) + \begin{bmatrix} v_u(t) \\ v(t) \end{bmatrix}. \]

When \( t \geq d \), we have the relationship for \( 0 \leq i < t - d \) and \( t - d \leq i \leq t \)

\[ \begin{bmatrix} \tilde{u}(i) \\ y_f(i) \end{bmatrix} = \begin{bmatrix} -K_{u,i} \\ H_i \end{bmatrix} x(i) + \begin{bmatrix} v_u(i) \\ v_f(i) \end{bmatrix}, \]

\[ \begin{bmatrix} \tilde{u}(i) \\ y(i) \end{bmatrix} = \begin{bmatrix} -K_{u,i} \\ H_i \end{bmatrix} x(i) + \begin{bmatrix} v_u(i) \\ v(i) \end{bmatrix}, \]

respectively.
It should be noted that, after reorganizing the measurements, the new observation $\check{y}_i(i)$ contains measurements of the state $x(i)$ at time instants $i$ and $i+d$. In other word, the delayed measurement has been applied to our design. $\check{u}(i)$ and $\check{y}(i)$ are delay-free and termed as reorganized observation. (20) and (30) or (20), (31) and (32) give a standard state space model without delay. Now define the innovation sequence associated with the re-organized observations for $i > 0$ and $i = 0$:

$$ W(t + i, t) = \begin{bmatrix} \hat{u}(t + i) - \hat{u}(t + i | t + i - 1, t) \\ \hat{y}(t + i) - \hat{y}(t + i | t + i - 1, t) \end{bmatrix}, $$

$$ W(t, t) = \begin{bmatrix} \hat{u}(t) - \hat{u}(t | t - 1, t - 1) \\ \hat{y}(t) - \hat{y}(t | t - 1, t - 1) \end{bmatrix} $$

(33)

(34)

where

$$ e(t + i) = x(t + i) - \hat{x}(t + i | t + i - 1, t), i > 0, $$

$$ e(t, t) = x(t) - \hat{x}(t | t - 1, t - 1), $$

while $\hat{x}(t | t + i, t)(i \geq 0)$ is the estimate of $x(t | t + i, t)$ given $\left\{ \check{u}(0) \right\}, \left\{ \check{y}(0) \right\}, \left\{ \check{u}(t) \right\}, \left\{ \check{y}(t) \right\}, \ldots, \left\{ \check{u}(t + i) \right\}, \left\{ \check{y}(t + i) \right\}$. It is easy to verify that $\left\{ W(\cdot, \cdot) \right\}$ is the mutually uncorrelated sequence $[11].$ $\left\{ W(\cdot, \cdot) \right\}$ is termed as re-organization innovation for observations $\check{u}(\cdot)$ and $\check{y}(\cdot)$.

A. **Innovation Covariance Matrix and Optimal Estimator**

In this subsection, we shall present the general optimal estimator $\hat{x}(t \mid t, t_k)$ ($l$ is a positive integer) and innovation covariance matrix $Q_w(t, i)$ by using the re-organized innovation defined by (26). From (33)-(34), the covariance matrix for re-organization innovation

$$ Q_w(t + i, t) = \langle W(t + i, t), W(t + i, t) \rangle, i \geq 0 $$

is calculated by

$$ Q_w(t, t) = \begin{bmatrix} -K_{u,t} \\ H_t \\ M_t \end{bmatrix} P_{t,t-1} \begin{bmatrix} -K_{u,t} \\ H_t \\ M_t \end{bmatrix} + \begin{bmatrix} Q_{v_u}(t) \\ 0 \\ 0 \end{bmatrix} Q_{v_f}(t) $$

(36)

$$ Q_w(t + i, t) = \begin{bmatrix} -K_{u,t+i} \\ H_{t+i} \end{bmatrix} P_{t+i,t+i} \begin{bmatrix} -K_{u,t+i} \\ H_{t+i} \end{bmatrix} + \begin{bmatrix} Q_{v_u}(t) \\ 0 \end{bmatrix} Q_{v_f}(t), i > 0 $$

(37)

with $Q_{v_u}(t) = -\gamma^{-2} (A_i)_{t-1}^{-1}$, $Q_{v_f}(t) = R_{t,t}^v$, $Q_{v_f}(f) = \begin{bmatrix} Q_{v_f}(t) & 0 \\ 0 & Q_{v_f}(t) \end{bmatrix}$, and $P_{t+t+1}^i = \langle e(t + i, t), e(t + i, t) \rangle (t \geq 0)$ is the covariance matrix of estimation error $e(t + i, t)$, which can be computed by the following Lemma.

**Lemma 1**: Let $\hat{f}_t = F_t - G_{t,i}K_{w,t}$. The cross-covariance matrix $P_{t+t+1}^i$ can be calculated as

1. For $i = 1$, $P_{t+t+1}^i$ is calculated recursively by the following Riccati equation

$$ P_{t+t+1}^i = \phi_t P_{t+t,i}^{t+i} \phi_t^t + G_{t,i} Q_{w}(t + i) + G_{t,i} Q_{w}(t) G_{t,i}^t $$

$$ \times P_{t+t+1}^{i+1} \phi_t^t, \quad P_{t+1}^0 = (P_{t+1}^0)^{-1} $$

(38)

where $Q_w(t, i)$ is the same as in (36).

2. For $i > 1$, $P_{t+t+1}^i$ is calculated as

$$ P_{t+t+1}^i = \phi_t P_{t+t,i}^{t+i} \phi_t^t + G_{t,i} Q_{w}(t + i) + G_{t,i} Q_{w}(t) G_{t,i}^t $$

$$ \times P_{t+t+1}^{i+1} \phi_t^t, \quad P_{t+1}^0 = (P_{t+1}^0)^{-1} $$

(39)

where $P_{t+t,1}^i$ and $Q_w(t, i)$ are as in (38) and (37).

**Proof**: The proof is straightforward from [11].

Further, let

$$ R_{t+i,1}^j \triangleq \langle x(t, j), e(t + i, t) \rangle, i, j \geq 0 $$

(40)

be the cross-covariance matrix of the state $x(t + j)$ and the state estimation error $e(t + i, t)$, then we have following result

$$ R_{t+i,1}^j = \begin{bmatrix} P_{t+j,1}^i A(t + j, t) \cdots A(t + i, t) \cdots A(t + i - 1, t) \\ \phi_{t+j-1} \cdots \phi_{t+i} P_{t+j-i,1}^{t+i} \phi_{t+i} \cdots \phi_{t+j-1} \end{bmatrix}, i < j $$

(41)

where $A(t + k, t), k > 0$ is given by

$$ A(t + k, t) = \phi_{t+k} \times \begin{bmatrix} I_n - P_{t+k,t}^{t+k} \begin{bmatrix} -K_{u,t+k} \\ H_{t+k} \end{bmatrix} Q_w(t + k, t) \begin{bmatrix} -K_{u,t+k} \\ H_{t+k} \end{bmatrix} \end{bmatrix} $$

(42)

For $k \leq 0$, $P_{t+k,t}^{t+k} = P_{t+k,t+k}^t K_{w,t+k}$ and $A(t + k, t) = A(t + k, t + k)$, the matrix $A(t + k, t + k)$ is given by

$$ A(t + k, t + k) = \phi_{t+k} \begin{bmatrix} I_n - P_{t+k,t+k}^{t+k} \begin{bmatrix} -K_{u,t+k} \\ H_{t+k} \end{bmatrix} Q_w(t + k, t + k) \begin{bmatrix} -K_{u,t+k} \\ H_{t+k} \end{bmatrix} \end{bmatrix} $$

(43)
Next we will calculate the optimal estimator \( \hat{x}(t_l \mid t, t_d) \) and innovation covariance matrix \( Q_w(t) \).

**Theorem 1**: For the Krein space state space model (20)-(21), the innovation covariance matrix \( Q_w(t) \) and the optimal estimator \( \hat{x}(t_l \mid t) = \hat{x}(t_l \mid t, t_d) \) (\( l \) is an integer) are given by

1) The innovation covariance matrix \( Q_w(t) = \langle W_s(t), W_s(t) \rangle \) is calculated by

\[
Q_w(t) = \begin{pmatrix}
Q_w(1,1)Q_w(1,2)
Q_w(2,1)Q_w(2,2)
\end{pmatrix},
\]

where

\[
Q_w(1,1) = K_{u,t}P_{t,t-d}tK_{u,t} - \gamma^2(\Delta t)_1^{-1},
\]

\[
Q_w(1,2) = -K_{u,t}P_{t,t-d}rK_{u,t},
\]

\[
Q_w(2,1) = -H_tP_{t,t-d}rH_t^* + R^u_t,
\]

\[
Q_w(1,3) = -K_{u,t}(R_{t,t-d}^u)^* M_{t,\bar{d}},
\]

\[
Q_w(2,3) = H_t(R_{t,t-d}^u)^* M_{t,\bar{d}},
\]

\[
Q_w(3,1) = -M_{t,\bar{d}}R_{t,t-d}^uK_{u,t},
\]

\[
Q_w(3,2) = M_{t,\bar{d}}R_{t,t-d}^uH_t^* + R^v_t,
\]

\[
P(t_d) = \begin{pmatrix}
P_{t,t-d-1} - \sum_{i=0}^{d-1} R_{t+i,t-d+i}^u H_{t+i}^* \end{pmatrix},
\]

\[
\times Q_w^1(t + i, t + d - 1) H_{t+i} + (R_{t+d+i,t-d+i}^u)^* M_{t,\bar{d}},
\]

2) The optimal estimator \( \hat{x}(t_l \mid t, t_d) \) can be calculated as:

\[\hat{x}(t_l \mid t, t_d) = \hat{x}(t_l \mid t_l - 1, t_d) + \sum_{i=0}^{d-1} R_{t_i,t+t_i-d+i}^u \]

\[
\times Q_w^1(t + i, t + d - 1) H_{t+i}^* \]

where

\[
\hat{x}(t_l \mid t_i + i - 1, t_d), i = 0, \ldots, l \]

in the above equation is calculated recursively as

\[
\hat{x}(t_l + i + 1 \mid t_l + i, t_d) = \phi_{t_l+i} \hat{x}(t_l + i \mid t_l + i - 1, t_l) + \phi_{t_l+i}P_{t_l+i}^u \]

\[
\times \begin{pmatrix}
-K_{u,t+i}^* H_{t+i}^* \n Q_w^1(t + i, t + d) \end{pmatrix},
\]

\[
\times \begin{pmatrix}
\hat{u}(t_l + i) \n y(t_l + i)
\end{pmatrix},
\]

\[
\hat{x}(t_l + i \mid t + i + 1, t_d), i = 0, \ldots, d-1, \]

respectively. The initial value \( \hat{x}(t_d + 1 \mid t_d, t_d) \) can be computed recursively as:

\[
\hat{x}(t_d + 1 \mid t_d, t_d) = \phi_{t_d} \hat{x}(t_d \mid t_d - 1, t_d - 1) + \phi_{t_d}P_{t_d}^u \]

\[
\times \begin{pmatrix}
H_{t_d}^* \n M_{t_d} + \hat{x}(t_d \mid t_d - 1, t_d - 1)
\end{pmatrix},
\]

where \( Q_w(t_d, t_d) \) and \( P^u_{t_d, t_d-1} \) are as in (36) and (38) respectively.

(b) For \( l < 0 \), the optimal estimator \( \hat{x}(t_l \mid t, t_d) \) is given by

\[
\hat{x}(t_l \mid t, t_d) = \phi_{t_l-1} \cdots \phi_{t+1} \hat{x}(t+1 \mid t, t_d),
\]

where \( \hat{x}(t+1 \mid t, t_d) \) has been given by a).

**Proof**: The proof can be obtained by applying a similar discussion as in [12] and [11].

---

**B. Solution to H∞ Control Problem**

Now we are in the position to present the main result of this paper.

**Theorem 2**: Consider the state-space model (1)-(4). Then, for any given \( \gamma > 0 \), a measurement-feedback \( H_\infty \) controller \( \hat{u}(t) = F_\gamma(y(0), \ldots, y(t), z_0(d), \ldots, z_{t-d}(t)) \) that achieves \( \|T(F)\|_\infty < \gamma \) exists if and only if

1) \( \Pi_0^{-1} - \gamma^{-2}P_0^u > 0 \),

2) \( \Delta t = -\gamma^2Q_w^1 + G_{t,t}P_{t+1}^u G_{t,t}^* - G_{t,t}^* P_{t+1}^u G_{t,t} < 0 \),

3) the matrices \( Q_w^1 - S_t(Q_w^1)^{-1} S_t \) and \( Q_w(t) \) have the same inertia for all \( t = 0, 1, \ldots, N \), where \( P_{t+1}^u \) satisfies (13) and

\[
R_{G_{t,t}^*} = Q_w^1 + G_{t+1, t}^* P_{t+1}^u G_{t+1, t}.
\]

Then the central controller is given by

\[
\hat{u}(t) = -K_{u,t} \hat{x}(t \mid t-1) - K_{k,t} \]

\[
\times Q_w^{-1}(t) (Y(t) - H_t \hat{x}(t \mid t-1)).
\]

**Proof**: According to the preliminaries in Section II, we know that \( \|T(F)\|_\infty < \gamma \) equivalent to \( J_N \) has a minimum \( J_N^m \) over the variables \( \{x(0), w(0), \ldots, w(N)\} \).
and \( \tilde{u}(t) \) can be chosen such that \( J_N^\infty > 0 \). From (17), a necessary condition for \( J_N \) to be positive for all variables \( \{x(0), w(0), \ldots, w(N)\} \) is 1), 2) and 3). The minimum of the \( J_N \) is given by (24). By using UDL factorization of \( Q_w(t) \), \( J_N^\infty \) can be written as

\[
J_N^\infty = \sum_{t=0}^{N} (\tilde{u}(t) - \check{u}(t))^* \Delta^{-1}_{R,t} (\tilde{u}(t) - \check{u}(t)) + \sum_{t=0}^{N} \left( Y_s(t) - H_I \hat{x}(t \mid t-1) \right)^* Q_w^{-1}(t) Y_s(t)
\]

\[
\times \left( Y_s(t) - H_I \hat{x}(t \mid t-1) \right), \tag{47}
\]

where

\[
\Delta_{R,t} = \begin{cases}
-\gamma^2 \left( \Delta_t' \right)^{-1} + K_{u,t} P_{t,t-1} K_{u,t}^* - K_{u,t} P_{t,t-1}^* H_I^t \\
-\gamma^2 \left( \Delta_t' \right)^{-1} + K_{u,t} P_{t,t-1} K_{u,t}^* \left( \hat{R}_{u,t}^d \right)^* M_{u,d} + \hat{G}_t^2 \left( \hat{G}_t^2 \right)^{-1} + G_{2,t}^* P_{t+1} G_{2,t} \\
- K_{u,t} P_{t,t-1}^* H_I^t \left( \hat{R}_{u,t}^d \right)^* M_{u,d} + \hat{G}_t^2 \left( \hat{G}_t^2 \right)^{-1} \hat{G}_t^2 G_{2,t}^* \left( \hat{G}_t^2 \right)^{-1} - G_{2,t}^* P_{t+1} \left( \hat{G}_t^2 \right)^{-1}, \\
0 \leq t < d,
\end{cases}
\]

and where \( \Delta_{R,t} \) is the Schur complement of \( \check{Q}_w(t) \) in \( Q_w(t) \). We have

\[
\bar{u}(t) = -K_{u,t} \check{x}(t \mid t-1) - K_{k,t} \check{u}(t) \\
\check{Q}_w(t) = \left( Y_s(t) - H_I \hat{x}(t \mid t-1) \right)^* \\
\check{Q}_w(t) \check{Q}_w(t) + \left( \hat{G}_t^2 \right)^{-1} - G_{2,t}^* P_{t+1} G_{2,t}^* \left( \hat{G}_t^2 \right)^{-1}, \\
t \geq d,
\]

with \( K_{k,t} \) as in (45). In view of condition 3) on \( Q_w(t) \), we have \( Q_w(t) > 0 \) and \( \Delta_{R,t} < 0 \).

Thus we choose the control signal to be \( \tilde{u}(t) = \bar{u}(t) \) which renders \( J_N^\infty \) positive. At the same time, the above necessary condition is also sufficient. From (25), we use re-organized innovation sequence to calculate the value of \( \check{x}(t \mid t-1) \) by Theorem 1. This control strategy \( \tilde{u}(t) \) which satisfies the \( H_\infty \) performance requirement is referred to as the central controller.

IV. CONCLUSION

The \( H_\infty \) measurement-feedback control problem for linear discrete-time systems with delayed measurement has been studied in this paper. By introducing a Krein space state model, the delayed \( H_\infty \) control problem has been transformed into a full information control and an \( H_2 \) optimal estimation problem for measurement delayed systems. A necessary and sufficient condition is derived by using the re-organized innovation sequence to transform two Riccati equations with the same dimension as the original system. The presented approach can be easily extended to the \( H_\infty \) control problems for the continuous-time systems with multiple-time delays and for continuous-time systems with delayed measurements. The necessary and sufficient condition for the existence of the \( H_\infty \) controller will be derived by using a similar discussion.

REFERENCES