Abstract—This paper discusses filter design problems for multirate systems in $H_2$ and $H_{\infty}$ settings by using the LMI machinery based on lifted models. The causality constraint on lifted filters is tackled, and the solution is provided in terms of a set of LMI conditions.

Keywords: $H_2/H_{\infty}$ filtering, multirate systems, lifting technique, causality constraint, linear matrix inequality (LMI).

I. Introduction

Due to practical limitations, it is unrealistic, or sometimes impossible, to sample all variables in a complex system with a single rate. A system where several sampling rates coexist is called a multirate system. Instead of treating general multirate systems, this paper considers the dual-rate case where the process input $w$ is sampled with period $mh$, and the process output $y$ is sampled with period $nh$. Without loss of generality, $m$ and $n$ are assumed to be coprime integers; $h$ is called the base period, and $mnh$ the frame period. This setup captures most of the essential features of multirate systems while maintaining some clarity in exposition.

$H_2$ and $H_{\infty}$ filter designs for the above mentioned multirate systems will be discussed in this paper. For simplicity, we only consider problems for single-input single-output systems. The extension to general multi-input multi-output multirate systems can be made following a similar line of research. The design objective of this work is to design a time-varying dynamic filter $\Sigma_{df}$, which, based on the multirate input $w$ and output $y$, will provide an accurate estimation $\hat{x}$ of true intersample states $x$ every base period $h$, say, $\hat{x}[mnk+i]$ $(k = 0, 1, \ldots, \infty, i = 1, \ldots, mn)$, in $H_2$ and $H_{\infty}$ formulations. The design problem in this work can be described as follows:

Problem: Given $\gamma > 0$, or $\beta > 0$, find a time-varying dynamic filter $\Sigma_{df}$, so that the $H_2$ norm, or the $H_{\infty}$ norm of the transfer function from $w$ to $e = x - \hat{x}$ is asymptotically stable and satisfies $\|G_{ew}\|_2^2 < \gamma$ ($\gamma$-suboptimal $H_2$ filtering), or $\|G_{ew}\|_{\infty} < \beta$ ($\beta$-suboptimal $H_{\infty}$ filtering).

Similar to that in [7], we will convert this time-varying design problem into an equivalent time-invariant one, by applying the lifting technique. Assume the discrete-time model with underlying period $h$ is known (otherwise it can be extracted from the multirate input-output data, see, e.g., [4]): the state $x$ is with dimension $n_x$, and the model state space realization is $\Sigma_h : [A_h, B_h, C_h, D_h]$. The corresponding lifted model for the dual-rate system can then be obtained, say, $\Sigma_l : [A_l, B_l, C_l, D_l]$. $\Sigma_l$ is with underlying period $mnh$; its state $x_l[k] = x[kh]$, its input $w_l$ and output $y_l$ are related with that of model $\Sigma_h$ in...
the following way:

\[
\begin{align*}
\mathbf{w}[k] &= \begin{bmatrix} \mathbf{w}[kT] & \mathbf{w}[kT + m] & \cdots & \mathbf{w}[kT + T - m] \end{bmatrix}^T, \\
\mathbf{y}[k] &= \begin{bmatrix} \mathbf{y}[kT] & \mathbf{y}[kT + n] & \cdots & \mathbf{y}[kT + T - n] \end{bmatrix}^T,
\end{align*}
\]

where we have defined \( T = mn \). For derivations of \( A_t, B_t, C_t, D_t \), readers can refer to [4] for more details.

Based on the model \( \Sigma_h \) with base period \( h \), the real intersample states during the \( k \)th frame period: \( x[mnk + 1], x[mnk + 2], \ldots, x[mnk + mn] \) can be calculated and lifted. This lifted signal is written as:

\[
X[k] = \Phi x_t[k] + \Gamma \mathbf{w}[k],
\]

where \( X[k] = \begin{bmatrix} x[kT + 1] & x[kT + 2] & \cdots & x[kT + T] \end{bmatrix}^T \),

\[
\Phi = \begin{bmatrix}
\Psi \cdot A_h^n & \Psi \cdot A_h^{(n-1)m} & \cdots & \Psi \\
\Omega & \Omega & \cdots & \Omega \\
\Psi \cdot A_h^{(n-2)m} \cdot \theta & \Psi \cdot A_h^{(n-3)m} \cdot \theta & \cdots & \theta \\
A_h & A_h & \cdots & A_h \\
\end{bmatrix},
\]

\[
\Gamma = \begin{bmatrix}
B_h & B_h & \cdots & B_h \\
A_h B_h + B_h & A_h B_h + B_h & \cdots & A_h B_h + B_h \\
\cdots & \cdots & \cdots & \cdots \\
A_h^{m-1} B_h + A_h^{m-2} B_h + \cdots + B_h & \cdots & \cdots & \cdots \\
\end{bmatrix},
\]

\[
\Omega = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 \\
\end{bmatrix},
\]

\[
\theta = \begin{bmatrix}
A_h & \cdots & A_h \end{bmatrix}.
\]

Assume the dynamics of the lifted filter is \( \Sigma_{df} : [A_f, B_f, C_f, D_f] \), whose input is \( y \), output is \( \hat{X} \) (lifted states estimation during the \( k \)th frame period) and state is \( \xi \), then the lifted state estimation error \( \xi = X - \hat{X} \) during the \( k \)th frame period is:

\[
\xi[k] = \Phi x_t[k] + \Gamma \mathbf{w}[k] - C_f \xi[k] - D_f C_t x_t[k] + D_f \mathbf{w}[k].
\]

The error dynamics, with input \( \mathbf{w} \), output \( \xi \), and state \( \xi \), is \( \Sigma_{ed} : [\hat{A}, \hat{B}, \hat{C}, \hat{D}] \).

Here \( \xi = \begin{bmatrix} x_t[k] \\ \xi[k] \end{bmatrix} \), \( \hat{A} = \begin{bmatrix} A_f & 0 \\ B_f C_t & A_f \end{bmatrix} \), \( \hat{B} = \begin{bmatrix} B_f I_t \\ B_f D_t \end{bmatrix} \), \( \hat{C} = \begin{bmatrix} \Phi - D_f C_t & -C_f \end{bmatrix} \), \( \hat{D} = \Gamma - D_f D_t \).

The proposed time-varying filter design problem is now equivalent to a time-invariant filter design problem, see Lemma 1.

Lemma 1: Given \( \gamma > 0 \), \( \beta > 0 \), find a lifted filter \( \Sigma_{ed} \) so that the corresponding lifted error dynamics \( \Sigma_{ed} \) is asymptotically stable and the \( H_2 \) norm, or the \( H_\infty \) norm of the lifted error dynamics transfer function \( G_{ew} \) satisfies \( \|G_{ew}\|_2 < mn \gamma \) or \( \|G_{ew}\|_\infty < \beta \); moreover, the matrix \( D_f \) of \( \Sigma_{df} \) should satisfy the causality constraint.

Proof: Results can be obtained by using the lifting technique, and analyzing the relationship between \( H_2/H_\infty \) norms of the original time-varying but periodic system \( G_{ew} \) and the lifted linear time-invariant (LTI) system \( G_{ew} \). For a reference, see, e.g., [7], [8].

III. Main Results

Lemma 2: 1) Let \( \gamma > 0 \), \( \|G_{ew}\|_2 < mn \gamma \) if and only if there exists \( P = P^T > 0 \) such that

\[
\begin{bmatrix}
P & \tilde{A} P & \tilde{B} \\ PA' & P & 0 \\ \tilde{B}' & 0 & I \end{bmatrix} > 0,
\]

\[
\begin{bmatrix}
mn \gamma I & \tilde{C} P & \tilde{D} \\ PC' & P & 0 \\ \tilde{D}' & 0 & I \end{bmatrix} > 0.
\]

(2)

2) Let \( \beta > 0 \), \( \|G_{ew}\|_\infty < \beta \) if and only if there exists \( P = P^T > 0 \) such that

\[
\begin{bmatrix}
P & \tilde{A} & \tilde{B} & 0 \\ \tilde{A}' & P & 0 & \tilde{C}' \\ \tilde{B}' & 0 & \beta I & \tilde{D}' \\ 0 & \tilde{C} & \tilde{D} & \beta I \end{bmatrix} > 0,
\]

(3)

Proof: (2) and (3) are standard LMI conditions for LTI \( H_2 \) and \( H_\infty \) design problems stated in Lemma 1, see, e.g., [5], [6], and the references therein.

Theorem 1: 1) Let \( \gamma > 0 \) be given, \( \Sigma_{df} \) is an admissible filter assuming \( \|G_{ew}\|_2 < mn \gamma \) if and only if there exist \( R > R' > 0 \), \( X = X' > 0 \), \( M \), \( Z \), \( N \), and \( D_f \) satisfying

\[
\begin{bmatrix}
R & X & E_1 & E_2 & E_3 \\ X & X & X & 0 & X \\ E_1 & A_t' X & 0 & X & 0 \\ E_2 & A_t' X & 0 & X & 0 \\ E_3 & B_t' X & 0 & 0 & I \\
\end{bmatrix} > 0,
\]

(4)

\[
\begin{bmatrix}
mn \gamma I & \Xi_1 & \Xi_2 & \Xi_3 \\ \Xi_1 & R & X & 0 \\ \Xi_2 & X & 0 & 0 \\ \Xi_3 & 0 & 0 & I \end{bmatrix} > 0.
\]

(5)

where \( E_1 = RA_t + ZC_t, E_2 = RA_t + ZC_t + M, E_3 = RB_t + ZD_t, \) and \( \Xi_1 = \Phi - D_f C_t, \Xi_2 = \Phi - D_f C_t - N, \Xi_3 = \Gamma - D_f D_t. \)

2) Let \( \beta > 0 \) be given, \( \Sigma_{df} \) is an admissible filter assuming \( \|G_{ew}\|_\infty < \beta \) if and only if there exist \( R > R' > 0, X = X' > 0, M, Z, N, \) and \( D_f \) satisfying

\[
\begin{bmatrix}
R & R & RA_t & RA_t & RB_t & 0 \\ R & X & K_1 & K_2 & K_3 & 0 \\ A_t' R & K_2' & R & R & 0 & \Theta_1' \\ A_t' R & K_2' & R & 0 & \Theta_2' & 0 \\ B_t' R & K_3' & 0 & 0 & \beta I & \Theta_3' \\ 0 & 0 & \Theta_1 & \Theta_2 & \Theta_3 & \beta I \end{bmatrix} > 0,
\]

(6)

where \( K_1 = XA_t + ZC_t + M, K_2 = XA_t + ZC_t, K_3 = XB_t + ZD_t, \) and \( \Theta_1 = \Phi - D_f C_t - N, \Theta_2 = \Phi - D_f C_t, \Theta_3 = \Gamma - D_f D_t. \)

Proof: Conditions (4), (5) (for \( H_2 \) filtering problem) and (6) (for \( H_\infty \) filtering problem) are derived by using similar methods shown in, e.g., [5], [6], and considering the causality constraint on \( D_f \).

Remark 1: Causality constraint on \( D_f \) in this work means that \( D_f \) should have a block lower triangular
structure as follows:

\[
D_f = \begin{bmatrix}
D_f^{i1} & 0 & \cdots & 0 \\
D_f^{i2} & D_f^{22} & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
D_f^{m1} & D_f^{m2} & \cdots & D_f^{mm}
\end{bmatrix},
\]

(7)

where \(D_f^{ij} (i, j = 1, 2, \cdots, m)\) is a full matrix with \(n \times n_h\) rows and one column since we consider single-input single-output dual-rate systems only.

Remark 2: Similar to [5], [6], the lifted \(H_2\) and \(H_\infty\) filters can be written out explicitly as follows:

1) Any matrices \(R, X, M, N,\) and \(Z\) satisfying (4) and (5) yield admissible \(H_2\) optimal lifted filter \(\Sigma_{df}\) with \(A_f = (X - R)\^{-1} M, B_f = (X - R)^{-1} Z, C_f = N\); \(D_f\) can be solved directly from LMI conditions (4) and (5).

2) Any matrices \(R, X, M, N,\) and \(Z\) satisfying (6) yield admissible \(H_\infty\) optimal lifted filter \(\Sigma_{df}\) with \(A_f = (R - X)\^{-1} M, B_f = (R - X)^{-1} Z, C_f = N, D_f\) can be solved directly from LMI condition (6).

Remark 3: Once the lifted filter \(\Sigma_{df}\) is found, i.e., matrices \(A_f, B_f, C_f,\) and \(D_f\) are determined with \(D_f\) satisfying the causality constraint, the time-varying filter to be implemented is \(\Sigma_{df} = L_{m1}^{-1} \Sigma_{df} L_m.\)

IV. Example

In this example, we assume input and output sampling periods of the considered system are \(mh\) and \(nh,\) respectively, where \(m = 2, n = 3,\) and \(h = 0.25\) seconds.\(\Sigma_h\) is given by \(A_h = [1.0168 \ 0.2059; -1.8177 \ 0.3991], B_h = [0.0317; 0.0111], C_h = [-0.8 \ 0.6,\) and \(D_h = 1.5.\)

Two states of the system are required to be estimated every base period \(h\) and the proposed LMI approach to both \(H_2\) and \(H_\infty\) designs is applied. It is found that the smallest \(H_2\) performance level is \(mny = 0.016,\) i.e., \(\gamma = 0.016/6.\) And the optimal lifted \(H_2\) filter is described by: \(A_f = [-0.5378 - 0.1579; 0.5619 - 0.1832], B_f = [-0.0300 \ 0.0359; -0.0343 - 0.1301],\)

\[
D_f = \begin{bmatrix}
0.5222 & 0.1485 \\
-0.7395 & 0.2453 \\
0.4432 & 0.2267 \\
-0.3487 & -0.1024 \\
0.1731 & 0.2059 \\
-1.3413 & -0.4516 \\
0.0514 & 0.1427 \\
-0.2622 & -0.0060 \\
-0.2288 & 0.0280 \\
-0.2510 & -0.4845 \\
-0.2833 & -0.0707 \\
0.3161 & 0.2441
\end{bmatrix}
\]

For the \(H_\infty\) design, the smallest \(H_\infty\) level calculated by LMI is \(\beta = 0.447.\) And the optimal lifted \(H_\infty\) filter is described by: \(A_f = [-0.4307 - 0.1302; 0.3951 - 0.0683], B_f = [-0.03050.0407; -0.0524 - 0.1735],\)

\[
D_f = \begin{bmatrix}
0.6278 & 0.1347 \\
-0.9369 & 0.1926 \\
0.4523 & 0.1872 \\
-1.5588 & 0.1736 \\
0.1259 & 0.1419 \\
-1.4299 & -0.3979 \\
-0.0455 & 0.0752 \\
-0.9833 & -0.0035 \\
-0.2966 & -0.0363 \\
0.2942 & -0.3074 \\
0.4307 & -0.1299 \\
0.9559 & -0.0674
\end{bmatrix}
\]

V. Conclusions

\(H_2\) and \(H_\infty\) filter design for multirate systems based on lifted models has been studied in this paper. Causality constraint on the lifted filters was tackled; and the causal solution was provided in terms of a set of LMI conditions. The effectiveness of the proposed method has been verified by a numerical example.

References