Minimax Polynomial Optimization by using Sum of Squares Relaxation and its Application to Robust Stability Analysis of Parameter-dependent Systems

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I. INTRODUCTION

Recently, a new method for solving a relaxed polynomial optimization problem (POP) by semidefinite programming (SDP) has been developed using sum of square (SOS) decomposition of positive semidefinite polynomials [1], [2]. It is possible to determine whether or not a polynomial function has SOS decompositions, and to compute global lower bounds for the polynomial function by solving SDP [3], [4]. Robust control using SOS optimization has been investigated in [5], [6], and positive polynomial representations via real algebraic geometry are summarized in [7].

The purpose of this paper is to investigate a robust control problem for parameter-dependent systems on the basis of Lagrangian dual of parametric-POPs. A robust stability problem is represented as a minimax-POP, and it is shown that the problem has a simple dual problem by using a distinguished representation of positive polynomials [8].

The following notation is used. $\mathbb{R}[x_1, \ldots, x_n]$ is the ring of real polynomials, which is abbreviated as $\mathbb{R}[x]$. The set of SOS polynomials is denoted by $\Sigma[x]$. $\mathbb{S}^n$ is the set of $n \times n$ symmetric matrices. $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n | X \succeq 0\}$. $\mathbb{S}^n_0 = \{X \in \mathbb{S}^n | X = 0\}$. $\mathbb{S}^n_0$ means $A + A^T$. $\mathbb{S}^n_{01} = \{\alpha_{ij} \in \mathbb{R}^{n \times n} | \alpha_{ij} = 0\}$. $\mathbb{S}^n_{01}$ is the monomial vector of $x$ whose maximum degree is $N$.

II. LAGRANGIAN DUAL AND SOS RELAXATION

Consider the minimax-POP

$$\max_p \min_x \quad f_0(x, p) = \sum \alpha f_{0\alpha}(p)x^\alpha$$

subject to $f_i(x) \geq 0, \quad i = 1, \ldots, M,$

$$f_0(x, p), f_i(x) \in \mathbb{R}[x], \quad f_{0\alpha}(p) = f_{0\alpha}(0) + c_{\alpha}p,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \sum_{i=1}^n \alpha_i \leq 2d, \quad p \in \mathbb{R}^m.$$  

where $S = \{x \in \mathbb{R}^n | f_i(x) \geq 0, \quad i = 1, \ldots, M\}$. 

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The optimal value of the SOS optimization problem (SOSOP)

$$\max_{\xi \in [x], p \in \mathbb{R}^m} \min_{S \subset \mathbb{S}^n} \quad L(x, s, p) = f_0(x, p) - r(x)$$

$$\text{s.t.} \quad L(x, s, p) - \zeta = s_0(x) \in \mathbb{S}^n[x],$$

is strictly positive, then the optimal value of the minimax-POP (1) is also strictly positive. $\square$

The degree of $s_0(x)$ has to be fixed as $2N$ in advance. Assume that $s_0(x) = z_{N}^{T}Qz_{N}$, then $s_0(x) = z_{[N-\nu]}^{T}R_{\nu}z_{[N-\nu]}$, where $Q \in \mathbb{S}_+^{N}$, $R_{\nu} \in \mathbb{S}_+^{N-\nu}$ and $\nu_i = \lfloor(\deg\phi(x))/2\rfloor$. When $N$ is given, the SOSOP (4) can be solved by a SDP.

Lemma 1: Consider the SDP

$$\max_{\xi, Q, R_{\nu}} \quad \zeta$$

$$\text{s.t.} \quad \text{trace} A_{\nu}Q + \sum \text{trace} B_{\nu}R_{\nu} - c_{\alpha}p + \zeta = f_0(0) \quad \forall \alpha,$$

where $\zeta_\alpha$ is 1 ($\alpha = 0$), and 0 ($\alpha \neq 0$). $A_{\nu} \in \mathbb{S}_+^{N}$ and $B_{\nu} \in \mathbb{S}_+^{N-\nu}$ satisfy $z_{[N]}^{T}z_{[N]} = \sum_{\alpha}A_{\nu}x^\alpha$ and $\phi(x_i)(x) = \sum_{\alpha}(\text{trace} B_{\nu}R_{\nu})x^\alpha$, respectively. Then, the optimal value $\zeta$ is equivalent to the optimal value of the SOSOP (4). $\square$
compact. Assume that for every constraint, that is, constraints and two quadratic constraints. Moreover, high-order system (5) is exponentially stable.

Thus, where $\sum_{i=1}^{q} \theta_i P_i$. The following condition is well-known.

Lemma 2 ([11]): For all $\theta \in \Omega$, if there exists $P_0, P_1, \ldots, P_q \in S^q$ such that $\text{He}(P_0)A \{0, P_1 \} > 0$, then the system (5) is exponentially stable.

Applying Theorem 1 to Lemma 2, a corollary is obtained.

Corollary 1: Given $\epsilon > 0$. Suppose that in SOSOP (4), $s_i(x) (1 = i, \ldots, M)$, $s_j k(x)$ ($1 \leq j < k \leq M$), $\ldots \in \sum[x]$, $p = \{ \text{vec} P_0 \}^T$, $\ldots$, $\{ \text{vec} P_q \}^T \in \mathbb{R}^{(a+1)(a+1)}/2$, $x \in [\phi^T \theta^T]^T \in \mathbb{R}^{a+q}$, and

$$f_0(x,p) = \phi^T \left[ \text{He}(A^T P) \right] \phi$$

Then, if there exists strictly positive $\zeta$, the parameter-dependent system (5) is exponentially stable.

Observing the above constraints, there are several affine constraints and two quadratic constraints. Moreover, highest degree’s homogeneous part of one of the quadratic constraints, that is, $-\phi^T \phi$, is negative for every $x$. Then, the quadratic version of distinguished representations of positive polynomials is applicable to these constraints.

Lemma 3 ([8][Theorem 4.1]): Let $f_i(x) \in \mathbb{R}[x]$, $S$ is compact. Assume that for every $x \in \mathbb{R}^a \setminus \{0\}$ one of $f_i(x)$ is negative, where $f_i$ is the homogeneous part of $f_i$ of highest degree. Rearrange the sequence $f_1, \ldots, f_M$ in two sequences $f_1, \ldots, f_o$ of odd degree and $f_{o+1}, \ldots, f_M$ of even degree. Then every $f_o(x) \in \mathbb{R}[x]$ which is strictly positive on $S$ has a representation

$$f_0(x) = s_0(x) + \sum_{i=1}^{M-o} s_i(x) f_i(x) + \sum_{1 \leq j < k \leq o} s_j k(x) f_j(x) f_k(x),$$

where $s_0(x), s_i(x), s_j k(x) \in \sum[x]$.

Owing to this representation, it becomes clear that linear terms with all constraints and only second order product terms with affine constraints are needed. Thus, $L(x,s,p) = f_0(x,p) + \sum_{i=1}^{2q+1} s_i(x) f_i(x) + \sum_{1 \leq j < k \leq 2q} s_j k(x) f_j(x) f_k(x)$ is adopted instead of (3).

IV. A Numerical Example

In the system (5), suppose that $\theta_i = -\theta_i = 1, \epsilon = 1.0$.

$$A_0 = \begin{bmatrix} 0 & 1 \\ a_0 & a_0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and compute the maximum $a_0$ by using Corollary 1 as a feasible problem with the constraint $\zeta > 0$ and dichotomy (scale width $1.0 \times 10^{-6}$). When $N = 2, w_i = 1 (i = 1, \ldots, 6)$. The computation has been executed by using SOSTOOLS [12] and SeDuMi [13]. From eigenvalue analysis, the system is exponentially stable for $a_0 < -1.0$. On the other hand, $\zeta = 2.41 \times 10^{-5} (> 0)$ and the maximum $a_0 = -1.0000372( < -1.0)$ are obtained by Corollary 1. This result is close to the exact value. Note that it is infeasible with $a_0 = -1.0$. This example shows how SOS relaxation technique gives a tight condition without relying on the convexity of the problem.

V. Conclusions

This paper gives a sufficient condition for robust stability of parameter-dependent systems in terms of a minimax-P. The minimax problem is investigated by using Lagrangian dual and sum of squares relaxation of parametric-Ps. A simpler condition than P-satz is obtained from the quadratic version of distinguished representation of positive polynomials. This approach could be applicable to controller design problems.

References