Adaptive Output Feedback Design for Actuator Failure Compensation
Using Dynamic Bounding: Output Tracking and An Application *

Xidong Tang and Gang Tao
Department of Electrical and Computer Engineering
University of Virginia
Charlottesville, VA 22903

Suresh M. Joshi
Mail Stop 132
NASA Langley Research Center
Hampton, VA 23681

Abstract
A new adaptive output feedback design using dynamic bounding is presented for actuator failure compensation of a class of nonlinear systems with unknown nonlinearities bounded by output-dependent functions and dynamic signals. The new features are that desired objectives can be achieved via a Lyapunov based design without using high gain type controllers and that it handles a larger class of systems. An adaptive scheme dealing with unknown actuator failures as well as system uncertainties is developed to achieve desired output tracking and closed-loop stability in spite of actuator failures.

1 Introduction
Compensation for actuator failures that are often unknown in terms of failure patterns, times and values is an important area in control. To design an effective adaptive control scheme achieving a desired objective in the presence of unknown actuator failures, the structure of systems and controllers, the choice of actuation schemes, and the stability of zero dynamics have to be considered with the uncertainties caused by actuator failures. There have been significant efforts in developing adaptive control algorithms for systems with failures [1], [2], [9], [10]. On the other hand, effective nonlinear feedback designs are needed to handle enlarged classes of nonlinear systems, especially for output feedback. Recently, progress has been made in nonlinear control using dynamic bounding [6]. This paper presents new results in these two aspects.

In [7], adaptive actuator failure compensation using output feedback design is studied for a class of nonlinear systems with unknown state-dependent nonlinearities bounded by a nonlinear function of the output. This paper addresses the actuator failure compensation problem using output feedback design for the class of nonlinear systems with unknown state-dependent nonlinearities, which are bounded not only by some linear parameterized static functions dependent on the output, but also by some dynamic signals generated by some linear dynamics whose inputs are functions of the output. By employing dynamic bounding, a new robust adaptive scheme is presented to solve the actuator failure problem with the stability established based on a new backstepping design and Lyapunov analysis. Compared with the control scheme designed in [7], the new control scheme achieves better results in the sense that we avoid using a high gain type controller to ensure the stability and that we can handle a larger class of systems at the same time.

2 Problem Statement
Consider the nonlinear system in the form of
\[
\dot{x}_t = x_{t+1} + \phi_t(x,t) + \varphi_t(y), \quad i = 1, 2, \ldots, r - 1
\]
\[
\dot{x}_r = x_{r+1} + \varphi_r(y) + \sum_{j=1}^{m} b_{r,j} y_j, \quad \ldots
\]
\[
\dot{x}_{n-1} = x_n + \varphi_{n-1}(x,t) + \sum_{j=1}^{m} b_{n-1,j} y_j
\]
\[
\dot{x}_n = \varphi_n(x,t) + \varphi_n(y) + \sum_{j=1}^{m} b_{n,j} y_j + \sum_{j=1}^{m} b_{n,j} \psi_j(y) u_j
\]
(2.1)
where \( u_j, j = 1, 2, \ldots, m, \) are the inputs that may fail due to system operation, \( x = [x_1, x_2, \ldots, x_n]^T \) is the vector of unmeasured states, \( y = x_1 \) is the output, \( b_{r,j}, r = 0, 1, \ldots, n = n - r, j = 1, 2, \ldots, m, \) are unknown constants, \( \varphi_t(y), i = 1, 2, \ldots, n, \) and \( \beta_j(y), j = 1, 2, \ldots, m, \) are smooth nonlinear functions and known, \( \beta_j(y) \neq 0, \forall y \in R, \) and \( \phi_t(x,t), i = 1, 2, \ldots, n, \) are unknown functions that are not necessarily smooth. The following assumption is made on \( \phi(x,t) = [\phi_1(x,t), \phi_2(x,t), \ldots, \phi_n(x,t)]^T : \)

(A1) There exists a vector of functions \( \psi(y) = [\psi_1(y), \psi_2(y), \ldots, \psi_n(y)]^T \in R^d, \psi_i(y) \geq 0, i = 1, 2, \ldots, l, \) a diagonal matrix of functions \( \Psi(y) = \text{diag} \{ \Psi_1(y), \Psi_2(y), \ldots, \Psi_n(y) \} \in R^{l \times l}, \Psi_i(y) \geq 0, i = 1, 2, \ldots, l, \) and a vector of signals \( \rho(t) = [\rho_1(t), \rho_2(t), \ldots, \rho_l(t)]^T \in R^l \) with \( \rho_i(t) = F_i(s) [\eta_i(y)], \eta_i(y) \geq 0, F_i(s) \) is stable and strictly proper, and \( F_i(s) \) with some \( \delta > 0 \) is sable, \( i = 1, 2, \ldots, l, \) such that
\[
\| \phi(x,t) - \theta_1^T \psi(y) + \theta_2^T \Psi(y) \rho(t) \| \leq \theta_3 \| \psi(y) \| + \theta_4 \| \Psi(y) \| \rho(t), \quad (2.2)
\]
where \( \psi(y), \Psi(y), \) and \( \eta_i(y) = [\eta_1(y), \eta_2(y), \ldots, \eta_l(y)]^T \in R^l \) are smooth and known, a positive constant \( \delta \) such that \( \delta \leq \min \{ \delta \} \) is known, and \( \delta \leq \delta \leq \delta \leq \delta \leq \delta \leq \delta \leq \delta \leq \delta \leq \delta \) are unknown constants with \( \delta_i \geq 0, \theta_2 \geq 0, i = 1, 2, \ldots, l, \) and \( \theta_3 \geq 0, i = 1, 2, \ldots, l. \)

*This research was supported by the NASA Langley Research Center under grant NCC-1-02006.
As in [7], the actuator failures are modeled as
\[ u_j(t) = \bar{u}_j, \quad t \geq t_j, \quad j \in \{1, 2, \ldots, m\}, \quad (2.3) \]
where the failure value \( \bar{u}_j \), the failure time instant \( t_j \), and the failure pattern, that is, the index \( j \) is unknown. More general failures modeled as \( u_j(t) = \bar{u}_j + \sum_{j \neq j_i}^{m} d_{ji} f_j(t) \), \( j = 1, 2, \ldots, m \), with unknown constants \( \bar{u}_j \) and \( d_{ji} \) and known signals \( f_j(t) \), can also be handled similarly.

Suppose that there are \( p_k \) actuators failed at a time instant \( t_k, k = 1, 2, \ldots, q \), and \( l_0 < t_1 < t_2 < \ldots < t_q < \infty \). Due to up to \( m - 1 \) failures, it follows that \( \sum_{k=1}^{q} p_k \leq m - 1 \). In other words, at time \( t \in (t_k, t_{k+1}) \), \( k = 0, 1, \ldots, q \), with \( t_{q+1} = \infty \), there are \( p = \sum_{k=1}^{q} p_k \) failed actuators, that is, \( u_j(t) = \bar{u}_j, j = j_1, \ldots, j_p, \)
\( 0 \leq p \leq m - 1 \), and \( u_j(t) = v_j(t), j \neq j_1, \ldots, j_p, \)
where \( v_j(t) \), \( j = 1, 2, \ldots, m \), are applied control inputs to be designed.

The control objective is to design an output feedback scheme for the nonlinear plant (2.1) with \( p \) unknown actuator failures, when \( p \) changes at time instants \( t_k, k = 1, 2, \ldots, q \), such that the plant output \( y(t) \) tracks a given reference signal \( y_r(t) \) with up to \( p \) th order derivative bounded as close as possible and all closed-loop signals are bounded despite unknown plant parameters and state-dependent nonlinearities.

### 3 Adaptive Compensation Control

The zero dynamics of (2.1) with \( p \) actuator failures are dependent on the failure pattern. For closed-loop stability, zero dynamics need to be developed to achieve the control objective.

(A2) The polynomials \( \sum_{j \neq j_1, \ldots, j_p} \text{sign}(b_{n-r,j})B_j(s) \) are stable, \( \forall \{j_1, \ldots, j_p\} \subset \{1, 2, \ldots, m\}, \) \( \forall p \in \{0, 1, 2, \ldots, m - 1\} \), where
\[ B_j(s) = b_{n-r,j} s^n + b_{n-1-r,j} s^{n-1} + \cdots + b_{1,j} s + b_{0,j}, \quad (3.1) \]

For an adaptive design, another assumption is required:

(A3) The sign of \( b_{n-r,j} \) is known, \( j = 1, 2, \ldots, m \).

We apply a proportional actuation scheme
\[ v_j = \text{sign}(b_{n-r,j}) \frac{1}{b_{n-r,j}(y)} v_0, \quad j = 1, 2, \ldots, m, \quad (3.2) \]
where \( v_0 \) is a control signal generated via a backstepping design procedure to be given next.

Applying the actuation scheme (3.2) and defining
\[ \phi(y) = [\phi_1(y), \phi_2(y), \ldots, \phi_n(y)]^T, \quad (3.3) \]
\[ k_{1,r} = \sum_{j \neq j_1, \ldots, j_p} \text{sign}(b_{n-r,j}) b_{r,j}, \quad r = 0, 1, \ldots, n^*, \]
\[ k_{2,rj} = b_{r,j} \mu_r, \quad r = 0, 1, \ldots, n^*, \quad j = j_1, \ldots, j_p, \]
\[ k_{2,rj} = 0, \quad r = 0, 1, \ldots, n^*, \quad j \neq j_1, \ldots, j_p, \quad (3.4) \]
we express the system (2.1) with \( p \) actuator failures as
\[ \dot{x} = Ax + \phi(x,t) + \phi(y) + \sum_{r=0}^{n^*} e_{n-r} \sum_{j=1}^{m} k_{2,rj} \beta_j(y) + \sum_{r=0}^{n^*} e_{n-r} k_{1,r} v_0, \]
\[ y = c^T x, \quad (3.5) \]
where \( A \in \mathbb{R}^{n \times n} \) is a canonical form describing a chain of \( n \) integrators, \( c = [1, 0, \ldots, 0]^T \), and \( e_i \) is the \( i \)th coordinate vector in \( \mathbb{R}^n \). It follows from Assumption (A2) that the polynomial \( k_{1,r} s^n + k_{1,n-r-1} s^{n-1} + \cdots + k_{1,1} s + k_{1,0} \) is stable, and, in addition, from (3.2), that \( k_{1,n-r} > 0 \).

#### 3.1 State Observation

Choose a vector \( l \in \mathbb{R}^n \) such that \( A_0 = A - l c^T \) is stable. With knowledge of \( k_{1,r} \) and \( k_{2,rj} \), \( j = 1, 2, \ldots, m \), \( r = 0, 1, \ldots, n^* \), we have a nominal state observer for \( x \) as
\[ x^* = \xi + \xi^* + \sum_{r=0}^{n^*} \sum_{j=1}^{m} k_{2,rj} \beta_j + \sum_{r=0}^{n^*} k_{1,r} u, \quad (3.6) \]
with the filters defined as
\[ \xi = A_r \xi + l y + \phi(y), \quad \xi^* = A_r \xi^* + \phi(x,t), \]
\[ \dot{\xi}_{rj} = A_r \xi_{rj} + e_{n-r} \beta_j, \quad 0 \leq r \leq n^*, \quad 1 \leq j \leq m, \]
\[ \mu_r = A_r \mu_r + e_{n-r} v_0, \quad 0 \leq r \leq n^*. \]

With \( \varepsilon = x - x^* \), it follows from (3.5), (3.6) and (3.7) that \( \dot{\varepsilon} = A_0 \varepsilon \) and \( \lim_{t \to \infty} \varepsilon(t) = 0 \) exponentially. Hence \( x^* \) is the nominal observation of \( x \) in the sense that the error \( x(t) - x^*(t) \) converges to zero exponentially. However, the vector \( \xi^* \) is not available, as the system state in \( x \) in \( \phi(x,t) \) is not available, while the second component of \( \xi^* \), i.e., \( \xi^* \) appears in the backstepping design procedure (e.g., added to \( \phi_1(x,t) \) in (3.16)).

To deal with the unavailable component \( \xi^* \), we introduce dynamic bounding signals \( \chi_0(t) \in \mathbb{R}^d \) and \( \chi(t) \in \mathbb{R}^L, L = l_1 + l_2 + 1 \), which are generated from
\[ \dot{\chi}_0(t) = -\delta_0 \chi_0(t) + \eta(y), \quad \dot{\chi}(t) = -\tilde{\delta} \chi(t) + \Phi, \quad (3.8) \]
where \( \Phi = [1, \psi^T(y), \chi_{00}^T(y)]^T, \) and \( \tilde{\delta} > 0 \) is chosen such that \( A_0 + \tilde{\delta} I \) is stable. Recall Assumption (A1) and notice that \( \delta_0 \leq \min \{\delta_i\} \) such that \( F_i(s - \delta_0) \), \( i = 1, 2, \ldots, l_2 \), are stable. It can be concluded from Lemma 2.6 in [8] that
\[ |\rho_i(t)| \leq \delta_{fi} \frac{1}{s + \delta_0} \eta_i(y) = \delta_{fi} \chi_{ui}(t), \quad i = 1, 2, \ldots, l_2, \quad (3.9) \]
where \( \delta_{fi} = ||f_{\delta_0}||_1 \) with \( f_{\delta_0} \) denoting the impulse response function of \( F_i(s - \delta_0) \), and \( \chi_{ui}(t) \) is the \( i \)th component of \( \chi_0(t) \). From (3.9), it in turn follows that
\[ \theta_{30}^T \Psi(y) \rho(t) \leq \theta_{30}^T \Psi(y) \chi_0(t), \quad (3.10) \]
where \( \theta_3 \in \mathbb{R}^{l_2} \) is an unknown constant vector whose components \( \theta_{3i} = \theta_{30 i} \delta_{fi} \geq 0, \quad i = 1, 2, \ldots, l_2 \).

From the dynamic equations of \( \xi^* \) in (3.7), it follows that
\[ \xi_2^* = H_1^*(s) [\phi_1(x,t)] + \cdots + H_n^*(s) [\phi_n(x,t)], \quad (3.11) \]
where \( H_i^*(s) = \frac{N_i(s)}{D_i(s)} \) is the transfer function from \( \phi_i(x,t) \) to \( \xi_2^*, \quad i = 1, 2, \ldots, n \), with \( D(s) = \text{det}(s l - A_0) \). Since \( A_0 + \tilde{\delta} I \) is stable, we have that \( D(s - \tilde{\delta}) \) is stable. It is also noted that \( H_i^*(s) \) is strictly proper. With the definition that \( H_i(s) = \frac{N_i(s)}{D_i(s)} \)}
The backstepping technique [4] is applied with a design procedure compensation scheme for the output tracking objective.

1: Defining the tracking error

\[ \tilde{x}_2(t) = H_1(s) \frac{1}{s+\delta} \left[ \phi_1(x,t) + \cdots + H_n(s) \frac{1}{s+\delta} \phi_n(x,t) \right], \]  

(3.12)

where \( H_i(s) \) is a proper transfer function for \( i = 1, 2, \ldots, n \).

Based on Lemma 2.6 in [8], it follows from (3.12) that

\[ |\tilde{x}_2(t)| \leq \frac{1}{s+\delta} \left( |\phi_1(x,t)| + \cdots + |H_n(s) \frac{1}{s+\delta} \phi_n(x,t)| \right) \leq \delta \Theta^T \chi(t), \]  

(3.13)

with \( \Theta = [\theta_1 \theta_2^T] \in \mathbb{R}^d \), and \( \delta = \|h_{\Omega}(\cdot)\|_1 + \cdots + \|h_{\Omega^T}(\cdot)\|_1 \), where \( h_{\Omega}(t) \) is the impulse response function of \( H_i(s+\delta), i = 1, 2, \ldots, n \).

### 3.2 Adaptive Design for Output Tracking

The backstepping technique [4] is applied with a design procedure consisting of \( p \) steps to derive an adaptive actuator failure compensation scheme for the output tracking objective.

Define unknown constant \( k = [k_{1,n}^T, k_{2,n}^T]^T \) and \( k_3 \) with

\[ k_1 = [k_{1,0}, k_{1,1}, \ldots, k_{1,n-1}]^T, \]
\[ k_2 = [k_{2,0}, k_{2,1}, \ldots, k_{2,m-1}]^T, \]
\[ k_3 = [\theta_1^T, \theta_2^T, \theta_3^T, \theta_4^T]^T, \]  

(3.14)

and \( \kappa = \frac{1}{k_{1,n}} \), and vectors of signals \( \omega_i = [\mu_{1r}, \omega_i^T, e_i^T]^T \), \( \zeta \), with

\[ \omega_i = [\mu_{i0}, \mu_{i1}, \ldots, \mu_{im-1}]^T, i = 1, 2, \ldots, n, \]
\[ e_1 = [\zeta_0, \zeta_1, \ldots, \zeta_{i-1}, \zeta_i, \zeta_{i+1}, \ldots, \zeta_{n+i-1}]^T, \]
\[ \zeta = [\chi_0^T, \chi_1^T, \chi_2^T, \chi_3^T]^T, \]  

(3.15)

where \( \mu_{i0}, \zeta_0, \zeta_i, i = 1, 2, \ldots, n \), are the \( i \)th components of \( \mu_r \) and \( \zeta_r \), \( r = 0, 1, \ldots, n-1 \), \( j = 1, 2, \ldots, m \). Let \( \kappa, \tilde{k}, k_{1,n}, \tilde{k}_1, \tilde{k}_2, \) and \( k_3 \) denote the estimates of \( \kappa, k, k_{1,n} \), \( k_1, k_2, \) and \( k_3 \).

For the issue of robustness, assume that the failure value \( \bar{u}_j \) is bounded by a known \( U_j \), i.e., \( |\bar{u}_j| \leq U_j, \) \( j = 1, 2, \ldots, m \), and 0 < \( b_{2j}^g \leq |\bar{u}_j| \leq b_{2j}^p \), where \( b_{2j}^g, b_{2j}^p \) are known, for \( r = 1, 2, \ldots, n \), \( j = 1, 2, \ldots, m \), and \( ||\Theta||_\infty \leq \Theta \) with \( \Theta \) known. It follows that \( |\kappa| \leq M_0, ||\kappa|| \leq M_0, ||k|| \leq M_3 \), where

\[ M_0 = \frac{1}{\sum_{j=1}^{n} b_{2j}^g}, M = \max\{ \sum_{j=1}^{n} b_{2j}^g, \sum_{j=1}^{n} r_{j}, \sum_{j=1}^{n} b_{2j}^p \}, M_3 = \|l_1 + l_2 + l_3\| \Theta^2 \}. \]

Step 1: Defining the tracking error \( z_t = y_t - y_r \), we have

\[ z_t = y_2 + \omega_2^T + \zeta_2 + \xi_1(x,t) + \xi_1(y,t) - y_r. \]  

(3.16)

Choose the auxiliary signal

\[ z_2 = \mu_{r,n} - \tilde{k}_y r - \alpha_1, \]  

(3.17)

Choosing the auxiliary signal

\[ \alpha_1 = -c_1 z_1 - d_1 z_1 - \omega_2 \tilde{k}_y - \lambda_4 z_1 \tilde{k}_y + \xi_3(y,t) - \xi_3(y,t) - \xi_4(y). \]  

(3.18)

Substituting (3.17) into (3.16) results in

\[ z_1 = - c_1 z_1 - d_1 z_1 - \lambda_4 z_1 \tilde{k}_y + \xi_3(y,t) - \xi_3(y,t) - \xi_4(y). \]

(3.18)

where \( k_{1,n} = k_{1,n} - \tilde{k}_y - \tilde{k}_k, \) \( k_1 = k_1 - \tilde{k}_k, \) \( k_2 = k_2 - \tilde{k}_k, \) \( k_3 = k_3 - \tilde{k}_k, \) and \( \tilde{k} = \kappa - \tilde{k} \) and \( c_1, d_1, \) and \( \lambda_4 \) are some positive constants to be chosen.

Consider the partial Lyapunov function \( V_1 = \frac{1}{2} \tilde{k}_y^2 + \frac{1}{2} \tilde{k}_k^2, \) where \( k_{1,n} > 0 \) due to Assumption (A2) and \( \gamma_t > 0 \) is given. Choose the adaptive law for \( \tilde{k} \) as

\[ \dot{\tilde{k}} = -\gamma_1 z_1 (y_t + \alpha_1) - \gamma_2 \sigma_{k_1} \tilde{k}, \]  

(3.19)

where \( \sigma_{k_1} \) is the switching signal defined as

\[ \sigma_{k_1}(\tilde{k}) = \begin{cases} 0 & \text{if } |\tilde{k}| \leq M_0, \\ (|\tilde{k}| - M_0)^{20} & \text{if } |\tilde{k}| \geq M_0, \end{cases} \]  

(3.20)

and the tuning functions for \( \tilde{k} \) and \( \tilde{k}_3 \)

\[ \tau_1 = z_1 (\mu_{r,n} - \tilde{k}_y - \alpha_1), \]  

(3.21)

Then the time-derivative of \( V_1 \) is

\[ \dot{V}_1 = -c_1 \tilde{k}_y^4 - d_1 \tilde{k}_y^4 - \lambda_4 \tilde{k}_y^4 \zeta_3 + z_1 \zeta_2 + z_1 \tilde{k}_y + \tilde{k}_1 \tilde{k}_2 \zeta_3 + k_{1,n} \sigma_{k_1} \tilde{k}_k + \tau_1^2 \tilde{k}_y + \tau_1^2 \tilde{k}_k + \tau_1^2 \tilde{k}_3, \]  

(3.22)

where \( \tilde{k} = -\tilde{k}, \) \( \tilde{k}_1, \) \( \zeta_3 \) will be cancelled at the next step and \( z_1 \zeta_2, \) \( z_1 \zeta_3 + \zeta_1 (y,t) \) will be handled later on.

Step 1: Define the time-derivative \( z_t \), \( i = 2, 3, \ldots, p \) as

\[ z_i = \mu_{n,i} - \tilde{k}_y (y_t^{i-1} - \alpha_{i-1}). \]  

(3.23)

Choose the stabilizing function \( \alpha_i \) as

\[ \alpha_i = -a_i z_t - c_i z_t - d_i \frac{\partial \alpha_i}{\partial y} z_t - \lambda_i \frac{\partial \alpha_i}{\partial y} z_t \tilde{k}_3 + l_i \mu_{n,i}, \]  

(3.24)

where \( a_2 = \tilde{k}_{1,n}, \) and \( a_1 = 1, 3, 4, \ldots, p, \) \( c_i, d_i, \) and \( \lambda_i > 0, i = 2, 3, \ldots, p, \) \( \Gamma = \Gamma^T > 0 \) and \( \Gamma_3 = \Gamma^T > 0 \) are the adaptive gains, \( \sigma(\tilde{k}) \) and \( \sigma_3(\tilde{k}) \) are the switching signals given by

\[ \sigma_1(\tilde{k}) = \begin{cases} 0 & \text{if } |\tilde{k}| \leq M_0, \\ (|\tilde{k}| - M_0)^{20} & \text{if } |\tilde{k}| \geq M_0, \end{cases} \]  

(3.25)

\[ \sigma_3(\tilde{k}_3) = \begin{cases} 0 & \text{if } ||\tilde{k}_3|| \leq M_3, \\ (||\tilde{k}_3|| - M_3)^{20} & \text{if } ||\tilde{k}_3|| \geq M_3, \end{cases} \]  

(3.26)
and $\tau_i$ and $\varpi_i$, $i = 2, 3, \ldots, \rho$, are the tuning functions:

$$\tau_i = \tau_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y} \omega_2 z_i, \varpi_i = \varpi_{i-1} + \lambda_i \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 z_i^2. \quad (3.26)$$

Considering the partial Lyapunov candidate function $V_i = V_{i-1} + \frac{1}{2}\zeta_i^2$, with $z_{i+1} = \mu_{i+1} - k \zeta^{(i)} - \alpha_i$, we have

$$\dot{V}_i = -\sum_{q=1}^{i} \zeta_q^2 - d_1 \zeta_i^2 + z_i \lambda_i - k \zeta^2 k_3 + \zeta_1 \zeta_2^2$$

$$+ z_i \phi_1(x, t) - \sum_{q=2}^{i} d_q \left(\frac{\partial \alpha_{q-1}}{\partial y}\right)^2 \zeta_q^2 - \sum_{q=2}^{i} \zeta_q \frac{\partial \alpha_{q-1}}{\partial y} \zeta_2$$

$$+ \frac{\sum_{q=2}^{i} \lambda_q \left(\frac{\partial \alpha_{q-1}}{\partial y}\right)^2 \zeta_q^2}{k_3} - \sum_{q=2}^{i} \zeta_q \frac{\partial \alpha_{q-1}}{\partial y} \zeta_2$$

$$- \sum_{q=2}^{i} \frac{\partial \alpha_{q-1}}{\partial y} \phi_1(x, t) + z_i \zeta_i + k \sigma_k(\bar{k}) \bar{k} \bar{k}$$

$$+ \varpi_i \sigma_k(\bar{k}) \bar{k} + \zeta_i \frac{\partial \alpha_{i-1}}{\partial \sigma} \left(\Gamma_k \varpi_i - \Gamma_k \sigma_k(\bar{k}) \bar{k} \bar{k}\right). \quad (3.27)$$

Controller. The control signal $v_0(t)$ in (3.2) is designed as

$$v_0 = \alpha_0 + k \sigma_i^{(p)} = \sum_{j=1}^{m} \frac{\partial \alpha_{j-1}}{\partial y} \beta_j(y) - \mu_{n, p, 1}, \quad (3.28)$$

such that $v_j = \text{sign}[\beta_j(y)] b_j(y) v_0$, $j = 1, 2, \ldots, m$, and the adaptive laws for $\dot{k}$, $\dot{k}$, and $\dot{k}_3$ as

$$\dot{k} = -\gamma_1 (\dot{y} + \bar{\alpha}_1 - \gamma_1 \sigma_k(\bar{k}) \bar{k})$$

$$\dot{k}_3 = \Gamma_3 \bar{\alpha}_0 - \Gamma_3 \sigma_k(\bar{k}) \bar{k} \bar{k}.$$  \quad (3.29)

Consider a Lyapunov function

$$V = V_0 + \frac{1}{2} \bar{K} G^{-1} \bar{K} + \frac{1}{2} \bar{K} \sigma \bar{K} + \sum_{i=1}^{n} \frac{1}{2} d_i e^T P e_i,$$  \quad (3.30)

where $P = P^T > 0$ satisfying the Lyapunov equation

$$PA_o + A_o^T P = -I. \quad (3.31)$$

From (3.13) and Assumption (A1), it follows that

$$\dot{V} \leq -\sum_{i=1}^{\rho} c_i \zeta_i^2 - \sum_{i=1}^{\rho} d_i \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \zeta_i^2 - \sum_{i=1}^{\rho} \zeta_i \frac{\partial \alpha_{i-1}}{\partial y} \zeta_2$$

$$- \sum_{i=1}^{\rho} \lambda_i \left(\frac{\partial \alpha_{i-1}}{\partial y}\right)^2 \zeta_i^2 + \sum_{i=1}^{\rho} \left|z_i \frac{\partial \alpha_{i-1}}{\partial y}\right| \left|\zeta_2^2\right|$$

$$+ \zeta_i \frac{\partial \alpha_{i-1}}{\partial y} \left|\phi_1(x, t)\right| + k \sigma_k(\bar{k}) \bar{k} \bar{k}$$

$$+ \left|\sigma_k(\bar{k}) \bar{k} \sigma_k(\bar{k}) \bar{k} \bar{k} - \sigma_k(\bar{k}) \bar{k} \bar{k} \sigma_k(\bar{k}) \bar{k} \bar{k}\right. - \sum_{i=1}^{\rho} \frac{1}{2} d_i e^T e \left.$$  \quad (3.32)

where $\lambda = \sum_{i=1}^{\rho} \frac{L_i (\sigma_i^2 + 1)}{4k_i}$ is a positive constant.

### 3.3 Stability Analysis

The proposed adaptive scheme has the following properties:

**Theorem 3.1** The adaptive output feedback control scheme consisting of the controller (3.28) and the filters (3.7) along with the parameter update laws (3.29) applied to the system (2.1), based on Assumptions (A1), (A2) and (A3), ensures global boundedness of all closed-loop signals and desired output tracking performance: \( \int_0^t y_i(t) - y_i(t)^2 dt \leq \frac{1}{c_1} (t - t_1) + \gamma_0 \), for any $t_2 > t_1 \geq 0$ and some constant $\gamma_0 > 0$, and the tracking error can be made as small as desired by choosing sufficiently large $c_1$ and $\lambda_i$, $i = 1, 2, \ldots, \rho$.

**Proof:** For each time interval $(t_i, t_i + 1)$, $k = 0, 1, \ldots, q$, we have such a V defined by (3.30) whose time-derivative V satisfies (3.32). Starting from the first time interval, we conclude that $V(t) \in L^{\infty}$ for all $t \geq [0, t_1]$, so that $\zeta$, $\dot{\zeta}$, $\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3$, and $\varepsilon$ are bounded for $t \geq [0, t_1]$. As $z_i = \dot{y} - \gamma_i$, $y_i$ is bounded.

It follows from (3.7) that $x_1$ and $x_2$, $j = 0, 1, \ldots, n^*$, are bounded. As in [4], it can be concluded that

$$\mu_{n, j} = e^T (sI - A_o)^{-1} e_{n, j} K(s) \left[ d^{\infty} \frac{d}{dt} - \sum_{i=1}^{n} \frac{d^i}{dt^i} \phi_i(x, t) \right]$$

which results in the boundedness of $\mu_{n, j}$, $j = 0, 1, \ldots, n^*$, due to the smoothness of $K$ and $\phi_i$. The boundedness of $\mu_{n, j}$, $j = 0, 1, \ldots, n^*$, $\zeta_1$, $\zeta_2$, and $\zeta_3$, $\mu_{n, j}$, $j = 0, 1, \ldots, n^*$, are bounded too because of the smoothness. It follows from (3.6) and the boundedness of $\varepsilon$ and $\zeta_1$ that $x_1$ is bounded. According to (3.28), it can also be seen that $v_0$ is a bounded signal. Since $\beta_j(y) \neq 0$ for any $y \in R$, the boundedness of $v_1$ is guaranteed too for $j = 1, 2, \ldots, m$. Therefore, all closed-loop signals are bounded for $t \in [0, t_1]$.

At time $t = t_1$, there occur $p_1$ actuator failures, which result in the abrupt change of $\kappa$, $k_1, k_2$, and $k_3$. Since the change of values of these parameters is finite, and $\zeta$, $\dot{\zeta}$, $\varpi_i$, $\dot{\varpi}_i$, $\kappa$, $\dot{k}$, and $k_3$ are continuous, $V(t_{i+1}) = V(t_i) + \dot{V}_1$. Therefore it can be concluded from (3.32) that $V(t) \in L^{\infty}$ for $t \in (t_i, t_{i+1})$. Therefore, repeating the argument above, all the signals are bounded for the time interval $(t_i, t_{i+1})$. Continuing in the same way, finally we have that $V(t) \in L^{\infty}$ for $t \in (t_q, \infty)$ with $V(t_q) = V(t_q) + \dot{V}_q$ for a finite $V_q$. Due to the finite times of actuator failures, it can be concluded that $V(t)$ is bounded, $\forall t > t_0$, and so are all the closed-loop signals.

To show the tracking performance, we consider the last time interval $(t_q, \infty)$ with a positive finite initial $V(t_q)$. Because $\sigma_k(\bar{k}) \bar{k} \bar{k} \bar{k} \leq 0$, $\sigma(\bar{k}) \bar{k} \bar{k} \bar{k} \leq 0$, and $\sigma_3(\bar{k}) \bar{k} \bar{k} \bar{k} \bar{k} \leq 0$, it follows from (3.32) that $\int_{t_1}^{t_2} \varepsilon^2 dt \leq \frac{1}{c_1} (t_2 - t_1) + \gamma_0$, for any $t_2 > t_1 \geq 0$ and some constant $\gamma_0 > 0$, and the tracking error can be made as small as desired by choosing sufficiently large $c_1$ and $\lambda_i$, $i = 1, 2, \ldots, \rho$. \( \nabla \)
4 An Application to Aircraft Control

The nonlinear longitudinal dynamics of the twin otter aircraft [5] is used for our actuator failure compensation study. Choosing the velocity, angle of attack, pitch angle, and pitch rate as the states $x_1, x_2, x_3$ and $x_4$, and the elevator angles of an augmented two-piece elevator as the inputs $u_1$ and $u_2$, we write the nonlinear aircraft into the state-space form:

$$\begin{align*}
\dot{x}_1 &= (c_1^T \phi_0(x_2) x_1 + \phi_1(x)) \cos(x_2) \\
&+ (c_2^T \phi_0(x_2) x_1 + \phi_2(x)) \sin(x_2) + \Delta g_1(x) u_1 + \Delta g_2(x) u_2 \\
\dot{x}_2 &= x_4 - \left( c_1^T \phi_0(x_2) x_1 + \phi_1(x) \frac{1}{x_1} \right) \sin(x_2) \\
&+ (c_2^T \phi_0(x_2) x_1 + \phi_2(x) \frac{1}{x_1} \cos(x_2)) + \Delta g_2(x) u_1 + \Delta g_2(x) u_2 \\
\dot{x}_3 &= x_4, \quad \dot{x}_4 = \theta^T \phi(x) + b_1 x_4^2 u_1 + b_2 x_4^2 u_2,
\end{align*}$$

where $c_1 \in R^3, c_2 \in R^3, d_1 \in R, d_2 \in R, b_1 \in R, b_2 \in R, \theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T \in R^4$ is constant, and

$$\begin{align*}
\phi_0(x_2) &= [x_2, x_2^2, 1]^T, \phi(x) = [x_2, x_2^2, x_1, x_2]^T, \\
\phi_1(x) &= \rho_1 + p_1 x_1 x_2^2 - \rho_0 \sin(x_1), \\
\phi_2(x) &= p_2 + p_2 x_1 x_2^2 + \rho_0 \cos(x_1), \\
g_1(x) &= (a_1 \sin^2(x_1) + a_2 x_1^2)^2, \\
g_2(x) &= -a_1_1 x_2^2 \sin^2(x_1) + a_2_2 x_1^2 \cos(x_1).
\end{align*}$$

Assume that the velocity $x_1$ is measured, while $x_2$ and $x_4$ are not measured. The control objective is to design an adaptive scheme to control the elevator angles such that the pitch angle $\gamma = x_3$ tracks a reference signal as close as desired even if any one of the elevator segments may be stuck at an unknown angle, i.e., $u_1(t) = \bar{u}_1, t \geq t_j, j = 1$ or 2 with unknown $\bar{u}_1, t_j$.

To design an adaptive compensation scheme using output feedback for the twin otter aircraft model, two issues need to be handled: zero dynamics and unmeasured states $x_2$ and $x_4$.

**Zero Dynamics.** The aircraft system does not have the form of the system (2.1) so that we need to obtain the inherent zero dynamics to investigate the minimum-phase property. According to the control objective, the states $x_1$ and $x_2$ in fact construct the zero dynamics. Apply a change of coordinates

$$\begin{align*}
\eta_1 &= f(x_1, x_2) + q_1 x_4, \quad \eta_2 = x_1 \sin(x_2) - q_2 x_4,
\end{align*}$$

where $q_1 = \frac{a_1}{r}, q_2 = \frac{a_2}{r},$ and $k = b_1 x_4, b_2 x_4$, and obtain the inherent zero dynamics as

$$\begin{align*}
\dot{\eta}_1 &= \omega_1(\eta_1, \bar{\xi}) = c_1^T \phi_0(x_2) x_1 + \phi_1(x) - x_1 \sin(x_2) - q_1 \theta \phi(x), \\
\dot{\eta}_2 &= \omega_2(\eta_1, \bar{\xi}) = c_1^T \phi_0(x_2) x_1 + \phi_1(x) - x_1 \sin(x_2) - q_2 \theta \phi(x),
\end{align*}$$

where $\eta = [\eta_1, \eta_2]^T$, $\bar{\xi} = [x_3, x_4]^T$. With the aircraft parameters obtained in a certain design condition [5], we investigate the unforced zero dynamics $\dot{\eta} = f(\eta, 0)$ with $\bar{\xi} = 0$ and $f(\eta, \xi) = \{ f_1(\eta, x_3, x_4), f_2(\eta, x_3, x_4) \}^T$. Investigate the stability of the unforced zero dynamics at the equilibrium point $\eta_0 = [71.24, 1.701]^T$ by checking the matrix $\frac{\partial f(\eta_0, 0)}{\partial \eta}$ at $\eta_0$. The eigenvalues of the matrix are $-1.950$ and $-0.05897$, which indicates that the unforced zero dynamics are locally asymptotically stable. It in turn implies that the zero dynamics are locally input-to-state stable in a neighborhood of $\eta_0$ with $\bar{\xi}$ in a neighborhood of 0 as the input [3].

**Dynamic Bounding.** For the unmeasured state $x_2$, the zero dynamics are input-to-state stable so that $x_2$ can be dynamically bounded by $y$. The zero dynamics $\dot{\eta} = f(\eta, \bar{\xi})$ are however highly nonlinear, which can hardly be handled. To apply the dynamic bounding design, we investigate the local property of the zero dynamics by linearizing the zero dynamics around a nominal trajectory. Since $y = x_3$ is desired to track a reference signal $y_r(t)$ and $x_4 = \bar{x}_4$, it is reasonable to take $\bar{x}_r(t) = [y_r(t), y_r(t)]^T$ as the nominal input. We further assume that the nominal reference signal $y_r(t)$ and its time derivative $\dot{y}_r(t)$ are close to some constant $\bar{x}_r = [x_3, x_4]^T$ with which we consider $y_r(t) - x_3, \dot{y}_r(t) - x_4$ as some additional small perturbation around $\bar{x}_r$.

The zero dynamics $\dot{\eta} = f(\eta, \bar{\xi})$ are thus linearized at $\eta = f(\eta_0, \bar{x}_r)$, where $\eta_0$ is the equilibrium point of $\eta = f(\eta, \bar{x}_r)$. Therefore we can have $x_2$ dynamically bounded by $y$ via the linear dynamics $\eta_3 = A \eta_3 + B \bar{x}_r$, where $A = \frac{\partial f(\eta, \bar{x}_r)}{\partial \eta}$ and $B = \frac{\partial f(\eta, \bar{x}_r)}{\partial \bar{x}_r}$, which locally approximate the nonlinear zero dynamics. Noticing that $x_4 = x_3$, we have $\eta_3 = G(s)[\bar{x}r] = [G_1(s), G_2(s)]^T [x_3] = [G_1(s), G_2(s)]^T [y_r(t)]$, where $G_1(s) = z_{12} + \frac{z_{12} z_{22} + \frac{z_{12}}{z_{22}}}{z_{22}}$ and $G_2(s) = \frac{z_{12} z_{22} + \frac{z_{12}}{z_{22}}}{z_{22}}$ with $D(s) = \text{det}(sI - A)$. Let $G^*(s) = \frac{\rho(t)}{1 + \rho(t)}$ and $G^*(s) = \frac{\rho(t)}{1 + \rho(t)}$, and define a signal $\rho(t) = F(s)\left(\frac{z_{12}}{z_{22}} + \frac{z_{12}}{z_{22}}\right)$ with $F(s) = \frac{z_{12}}{z_{22}}$, where $\delta > 0$ is a constant such that $D(s - \delta \bar{s})$ is stable. Define $H_i(s) = G_i(s)$ ($s + \delta \bar{s}$).

Consider that in normal operation $x_1 > 0$ and $|x_2| \leq \frac{r}{2}$ with which we have $|x_2| \leq \frac{r}{2} |\sin(x_2)|$. It follows from (4.3) that

$$\begin{align*}
|x_2| &\leq 2 \frac{|q_1|}{x_1} + 2 \frac{|q_2|}{x_1} |x_1| + 2 \frac{|q_2|}{|x_1|} x_1.
\end{align*}$$

Defining $\delta(x) = [x_2^T x_2^2 x_2^2 x_2^2]^T$, we thus have Assumption (A1) satisfied by showing that

$$\begin{align*}
|\hat{\eta}^T \phi(x)| &\leq |\theta_1 |x_1^2 |x_2| + |\theta_2 |x_2^2 |x_2| + |\theta_3 |x_2|
\leq \Theta_1 + \Theta_2 x_2 + \Theta_3 x_2^2 + \Theta_4 x_1^2 + \Theta_5 x_1^2 + \Theta_6 \rho(t)
+ \Theta_7 x_1 \rho(t) + \Theta_8 \rho^2(t) + \Theta_9 \rho^2(t),
\end{align*}$$

where $\Theta_k > 0, i = 1, 2, \ldots, 9$, are unknown constants. Notice that in Assumption (A1) there is no a such term as $\rho^2(t)$. Since $p_{k_1}(t) < p_{k_2}(t)$ for any $|p_{k_1}(t)| < |p_{k_2}(t)|$, our dynamic bounding design is however applicable to this case.

**Observer.** The unmeasured state $x_4$ exhibits the appearance of a linear form in $\phi(x)$. Taking advantage of this property, next we design a time-varying observer to estimate $x_4$. Assuming that $\theta_4$ is known, we introduce a signal
\[ \beta(t) = -\theta_4 x_1^2(t). \] Since \( \theta_4 < 0 \), we have \( \beta(t) \geq 0, \forall t \geq 0. \) We now give a nominal state observer for \( \dot{x}' = [x_3', x_4']^T \) as \( \dot{x}' = \dot{x} + k_2 \xi + k_1 \mu \) with the filters given by

\[
\begin{align*}
\dot{\xi} &= A_0 \xi - \beta(t) \xi + l y + [\beta(t), 0]^T, \\
\dot{\zeta} &= A_0 \zeta - \beta(t) \zeta + [0, 1]^T \bar{q}(x(t)) \Phi, \\
\dot{\zeta} &= A_0 \zeta - \beta(t) \zeta + [0, x_2^2]^T, \quad \mu = A_0 \mu - \beta(t) \mu + [0, v_0]^T, \quad (4.7)
\end{align*}
\]

where \( \ell = [l_1, l_2]^T \) is chosen such that \( A_0 = A - l c^T \) is stable with \( A \in \mathbb{R}^{2 \times 2} \) in the canonical form, and \( k_1 \) and \( k_2 \) are defined as \( k_1 = b_1 + b_2 \) and \( k_2 = 0 \), if both elevator segments work, and \( k_1 = b_1 \) and \( k_2 = b_2 \), if the \( i \)th elevator segment works and the \( j \)th elevator segment fails, \( i \neq j \in \{1, 2\} \). The nominal observation error \( \varepsilon = \bar{x} - \dot{x}' \) satisfies \( \dot{A}_2 \dot{\varepsilon} = -\dot{\beta}(t) \varepsilon \), which implies \( \varepsilon \) converges to zero exponentially.

For unknown \( \xi^*_0 \), we introduce dynamic bounding signals

\[
\dot{\chi}_0(t) = -\delta_0 \chi_0(t) + \frac{\gamma^2 + 1}{2}, \quad \dot{\zeta}(t) = -\delta \chi(t) + \Phi, \quad (4.8)
\]

where \( \chi_0 \in \mathbb{R} \) and \( \chi \in \mathbb{R}^9 \), \( \delta > 0 \) is chosen such that \( A_0 + \delta I \) is stable, \( \delta_0 > 0 \) is such that \( F(s - \delta_0) \) is stable, and \( \Phi = [1, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9]^T \). It follows from (4.6) that \( |\delta \phi(s)| \leq \Theta \Phi \), where \( \Theta = [\Theta_1, \Theta_2, \ldots, \Theta_9]^T \). Although \( \xi^*_0 \) is governed by a linear time-varying system, it is proved that Lemma 2.6 in [8] can still be applied to this case.

**Adaptive Design.** Define \( \kappa = [\Theta_1, \Theta_2, \ldots, \Theta_9]^T \) and \( \kappa = \frac{1}{\kappa_1} \), and signals \( \zeta = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_9]^T \) and \( \zeta = [x_1^2 + 1, x_2^2 + 1, x_3^2 + 1, x_4^2 + 1, x_5^2 + x_6^2, x_7^2 + x_8^2 + x_9^2]^T \). With Assumption (A3) that the sign of \( k_1 \) is known, the adaptive controller is derived as

\[
v_j = 1, \quad j = 1, 2, 0 = \alpha_2 + \bar{\kappa}_0 + \frac{\partial \alpha_1}{\partial \xi^*_0} \chi^2, \quad (4.9)
\]

and the adaptive laws are \( \dot{k}_1 = -\gamma \text{sign}[k_1] (\bar{y}_1 + \alpha_1 - \gamma \sigma_k(k)) \), \( \dot{k}_2 = -\Gamma_2 - \gamma \sigma_k(k) \), \( \dot{k}_3 = -\Gamma_3 - \gamma \sigma_k(k) \), \( \dot{k}_4 = -\Gamma_4 - \gamma \sigma_k(k) \), via a backstepping procedure as shown in Section 3.2, where \( \gamma > 0, \Gamma = \Gamma^T > 0, \) and \( \Gamma_3 = \Gamma_4 > 0 \) are the adaptive gains, \( \sigma(k), \sigma_1(k), \) and \( \sigma_3(k) \) are the switching signals in the form of (3.20) and (3.25).

For simulation, the initial states are chosen as \( x(0) = [62.5, -0.04, 0, 0]^T \) and \( l = [2, 1], \delta_0 = 0.05, \delta = 0.5, c_1 = c_2 = d_1 = d_2 = 0.01, \lambda_1 = \lambda_2 = 0.000001, \gamma = 0.0005, c_1 = 0.1, \theta_2 = 0.001, \Gamma_3 = I \). Figure 1 shows the results when \( u_1 \) fails at 150 second with an unknown failure value 0.04 (rad).

5 **Concluding Remarks**

This paper addresses actuator failure compensation using output feedback for a class of nonlinear systems with unknown nonlinearities bounded by output-dependent functions and dynamic signals. An adaptive controller using dynamic bounding design is proposed for desired objective and an application to a nonlinear aircraft dynamics is also presented.

**References**


Figure 1: Output and tracking error.