**LMI-based $H_\infty$ Fuzzy Equalizer Design for Discrete-time Nonlinear Channels**

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Abstract

This paper investigates the problems of nonlinear channel equalization based on LMI-fuzzy methodology. According to Takagi-Sugeno (T-S) fuzzy modeling concept, the discrete-time nonlinear channel can be constructed by the piecewise linear subsystems. The FIR fuzzy equalizer design for nonlinear channel is transformed into a standard linear matrix inequality (LMI) optimization problem, and the coefficients of the fuzzy equalizer are obtained by solving LMIs. Thus, the result is very simple and the computation is numerically tractable. Finally, simulation result is given to demonstrate the effectiveness of the proposed methodology.

**Keywords:** $H_\infty$; FIR; Equalizer; LMI; T-S fuzzy model.

**I. Introduction**

In modern digital communication systems, there have been several methods proposed to treat the problem of the channel equalization. The equalizer design problem via two-block $H_\infty$ optimization technique is presented in [1]. The risk-sensitive FIR equalizer has been formulated as the constrained analytic centering problem, which is another type of convex problem [2,3]. The maximum likelihood Viterbi algorithm and the conventional decision feedback equalizer are used for the Bayesian decision feedback equalizer [4]. However, most of them mentioned above are firmly based on linear filter algorithms. In general, equalization is a nonlinear classification problem and it is more difficult to model and control the nonlinear systems.

Recently, fuzzy techniques have been applied in various fields such as control systems, communication systems, signal processing, and so on. Among various fuzzy modeling themes, Takagi-Sugeno (T-S) model based fuzzy control approach has been rapidly and successfully developing in nonlinear control frameworks [5]. The T-S fuzzy model offers an effective way to represent the nonlinear systems with a set of fuzzy If-Then rules, each of the rules can be represented as a local linear state equation and the overall fuzzy system is achieved by fuzzy “blending” of the these linear state equation.

We propose an LMI-fuzzy based approach to design the FIR fuzzy equalizer for nonlinear discrete-time channels from $H_\infty$ perspective, and employ the state-space description in conjunction with bounded real lemma [3,6] to calculate the optimal $\gamma$ value and the coefficients of the FIR fuzzy equalizer, where the $\gamma$ value, a tolerance level, can be regarded as an indication of the quality of the filter. Moreover, the effect of the $\gamma$ value for different equalizer lengths is discussed. The FIR fuzzy equalizer design presented in this paper involves only solving a set of LMIs. Thus, the result is very simple and the computation is numerically tractable.

Two primary contributions of this paper are (i) deal with the nonlinear system by T-S fuzzy model and (ii) eliminating the effect of the external disturbance as much as possibly.

The notation used in this paper is fairly standard. $M > 0$ ($M < 0$) means that the matrix $M$ is symmetric and positive (negative) definite, $M^T$ represents transpose of $M$. $I_k$ stands for the identity matrix with dimension $k$. $\| \cdot \|$ denotes the infinity norm of a discrete-time stable proper transfer function matrix. $\min$ stands for minimum operation.

The rest of this paper is organized as follows. In Section II, the system model is described and a T-S fuzzy model is used to construct the nonlinear channel. Furthermore, the state-space representation for the error transfer function is presented. Converting the problems to the task of finding an optimal $\gamma$ value by solving LMIs is described in Section III. In Section IV, a numerical example is proposed and simulation results are demonstrated as well. Finally, conclusion is drawn in Section V.

**II. System model description**

The general structure of the error transfer function corresponding to the nonlinear channel equalization problem is illustrated in Fig.1 [7], where $b_i$ is the transmitted digital information sequence, $v_i$ is the unknown noise, $e_i$ is the error between the equalizer output and the delay of the desired transmitted sequence, $H_{NL}(z)$ is the discrete equivalent of the nonlinear time-invariant communication channel, $K(z)$ is the
equalizer to be designed and \( L(z) = z^{-d} \) is the delay. The discrete data sequence \( b_i \) passes through the nonlinear time-invariant channel \( H_{NL}(z) \), the observation sequence \( y_i \) is then formed by the addition of an unknown measurement disturbance \( v_i \) with the output of the nonlinear communication channel \( H_{NL}(z) \).

The nonlinear channel \( H_{NL}(z) \), contaminated with the noise at the output, is described by the following piecewise linear T-S fuzzy model:

Model Rule \( j \):

\[
\text{IF } z_1 \text{ is } M_{j1} \text{ and } \ldots \text{ and } z_p \text{ is } M_{jp},
\]

\[\text{THEN } s_{i+1} = A_{cj} s_i + B_{cj} u_i, \quad (1)\]

\[y_i = C_{cj} s_i + D_{cj} u_i, \quad j = 1, 2, \ldots, L.\]

Where \( M_{jp} \) are the fuzzy sets and \( L \) is the number of model rules; \( s_i \) is the state vector, \( u_i = [b_i \ v_i]^T \) is the input vector and \( y_i \) is the measured output; \( A_{cj}, B_{cj}, C_{cj}, D_{cj} \) are constant matrices; \( z_1, \ldots, z_p \) are premise variables that may be functions of the state variables, external disturbances, and/or time.

The final outputs of the fuzzy systems (1) and (2) are inferred as follows:

\[
s_{i+1} = \frac{\sum_{j=1}^{L} w_j(z) (A_{cj} s_i + B_{cj} u_i)}{\sum_{j=1}^{L} w_j(z)},
\]

\[
y_i = \frac{\sum_{j=1}^{L} w_j(z) (C_{cj} s_i + D_{cj} u_i)}{\sum_{j=1}^{L} w_j(z)}, \quad j = 1, 2, \ldots, L.
\]

The term \( M_{jk}(z_k) \) is the grade of membership function of \( z_k \) in \( M_{jk} \). It is easy to find that

\[w_j(z) \geq 0, \quad j = 1, 2, \ldots, L\]

and \( \sum_{j=1}^{L} w_j(z) > 0 \).

Therefore, \( h_j(z) \geq 0 \) for \( j = 1, 2, \ldots, L \) and \( \sum_{j=1}^{L} h_j(z) = 1 \).

The delay operator \( L(z)=z^{-d} \) is represented by

\[\ell_{i+1} = A_d \ell_i + B_d u_i, \quad (8)\]

\[z_i = C_d \ell_i, \quad (9)\]

where \( d > 0 \), \( \ell \) is the state vector, \( u_i = [b_i \ v_i]^T \) is the input vector and \( A_d = \begin{bmatrix} 0 & 0 \\ I_{(d-1)(d-1)} & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0 \end{bmatrix} \)

The state-space model for the FIR equalizer with order \( R-1 \) has the following state-space structure:

Model Rule \( j \):

\[
\text{IF } z_1 \text{ is } M_{j1} \text{ and } \ldots \text{ and } z_p \text{ is } M_{jp},
\]

\[\text{THEN } \alpha_{i+1} = A_{cj} \alpha_i + B_{cj} y_i, \quad (10)\]

\[\hat{z}_j = C_{cj} \alpha_i + D_{cj} y_i, \quad (11)\]

where \( z = [z_1 \ z_2 \ \ldots \ z_p]^T \), \( w_j(z) = \prod_{k=1}^{p} M_{jk}(z_k) \), and \( h_j(z) = \frac{w_j(z)}{\sum_{j=1}^{L} w_j(z)} \).

The final outputs of the fuzzy equalizer are inferred as follows:

\[A_c = \frac{\sum_{j=1}^{L} w_j(z) A_{cj}}{\sum_{j=1}^{L} w_j(z)}, \quad B_c = \frac{\sum_{j=1}^{L} w_j(z) B_{cj}}{\sum_{j=1}^{L} w_j(z)},
\]

\[C_c = \frac{\sum_{j=1}^{L} w_j(z) C_{cj}}{\sum_{j=1}^{L} w_j(z)}, \quad D_c = \frac{\sum_{j=1}^{L} w_j(z) D_{cj}}{\sum_{j=1}^{L} w_j(z)}. \quad (5)\]

The state variable is the grade of membership function of \( z_k \) in \( M_{jk} \). It is easy to find that

\[w_j(z) \geq 0, \quad j = 1, 2, \ldots, L\]

and \( \sum_{j=1}^{L} w_j(z) > 0 \).

Therefore, \( h_j(z) \geq 0 \) for \( j = 1, 2, \ldots, L \) and \( \sum_{j=1}^{L} h_j(z) = 1 \).

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\[\hat{z}_j = C_{cj} \alpha_i + D_{cj} y_i, \quad (11)\]

where \( z = [z_1 \ z_2 \ \ldots \ z_p]^T \), \( w_j(z) = \prod_{k=1}^{p} M_{jk}(z_k) \), and \( \hat{z}_j \) is the output of the fuzzy equalizer, \( z_1, \ldots, z_p \) are premise variables that may be functions of the state variables, external disturbances, and/or time.

\[A_{cj} = \begin{bmatrix} 0 & 0 \\ I_{(R-2)(R-2)} & 0 \end{bmatrix}, \quad B_{cj} = \begin{bmatrix} I \\ 0_{(R-2)1} \end{bmatrix}, \quad C_{cj} = [k_{1j} \ldots k_{R-1j} k_{R-1j}], \quad D_{cj} = k_{0j} \]

and \( k_{0j}, k_{1j}, \ldots, k_{R-1j} \) are the coefficients of the equalizer which to be designed.

The final outputs of the fuzzy equalizer are inferred as follows:
\[ \omega_{i+1} = \frac{1}{\sum_{j=1}^{L} w_j(z)} \sum_{j=1}^{L} w_j(z) \left[ A_{ij} \omega_i + B_{ij} y_i \right] \]

\[ = \frac{1}{\sum_{j=1}^{L} w_j(z)} \sum_{j=1}^{L} w_j(z) \left[ h_j(z) \{ A_{ij} \omega_i + B_{ij} y_i \} = A_e \omega_i + B_e y_i \right], \quad (12) \]

\[ \hat{z}_i = \frac{1}{\sum_{j=1}^{L} w_j(z)} \sum_{j=1}^{L} w_j(z) \left[ C_{ij} \omega_i + D_{ij} y_i \right] \]

\[ = \frac{1}{\sum_{j=1}^{L} w_j(z)} \sum_{j=1}^{L} w_j(z) \left[ h_j(z) \{ C_{ij} \omega_i + D_{ij} y_i \} = C_e \omega_i + D_e y_i \right]. \quad (13) \]

where \( z = [z_1 \ z_2 \ \cdots \ z_p] \), \( w_j(z) = \prod_{k=1}^{p} M_{jk}(z_k) \),

\[ h_j(z) = \frac{w_j(z)}{\sum_{j=1}^{L} w_j(z)}, \quad A_e = \frac{\sum_{j=1}^{L} w_j(z) A_{ij}}{\sum_{j=1}^{L} w_j(z)}, \]

\[ B_e = \frac{\sum_{j=1}^{L} w_j(z) B_{ij}}{\sum_{j=1}^{L} w_j(z)}, \quad C_e = \frac{\sum_{j=1}^{L} w_j(z) C_{ij}}{\sum_{j=1}^{L} w_j(z)} \]

\[ D_e = \frac{\sum_{j=1}^{L} w_j(z) D_{ij}}{\sum_{j=1}^{L} w_j(z)}. \quad (14) \]

From the above preliminary, we can obtain the state-space model for the error transfer function with mapping input disturbances \( u_i \) to the error \( e_i = z_i - \hat{z}_i \) as follows:

\[ x_{i+1} = Ax_i + Bu_i, \quad (15) \]

\[ e_i = Cx_i + Du_i. \quad (16) \]

where \( d > 0 \) and

\[ A = \begin{bmatrix} A_e & 0 & 0 \\ 0 & A_d & 0 \\ B_e C & 0 & A_e \end{bmatrix}, \quad B = \begin{bmatrix} B_e & B_d & B_e D_e \end{bmatrix}^T, \]

\[ C = \begin{bmatrix} -D_e C_e & C_d & -C_e \end{bmatrix}, \quad D = \begin{bmatrix} -D_e D_e \end{bmatrix}, \]

\[ x_i = [s_i \ \ell_i \ \omega_i]^T, u_i = [h_i \ v_i]^T. \]

From eqns. (15) and (16), the equalizer coefficients are only included in the matrices \( C \) and \( D \). Therefore, the representation of the error transfer function can be specified as

\[ T(z) = C(zI - A)^{-1} B + D. \quad (17) \]

Denoting the disturbance attenuation value \( \gamma \), the obtained solution guarantees a disturbance rejection capability, which is optimal in the sense of \( H_\infty \) norm:

\[ \| T(z) \|_\infty < \gamma \quad (18) \]

Consequently, we will show how to cast the \( H_\infty \) control problem into LMI framework in the next section.

### III. Problem formulation

Almost all practical systems are subject to external disturbances that can in some situations degrade system performance if their effects are not considered during the design phase. There are many ways to eliminate the effects of the external disturbances in the current literature. Recently, interest has been devoted to the filtering problem with an \( H_\infty \) performance criterion. This approach consists of designing a controller that minimizes the \( H_\infty \) norm of the transfer function between the controlled output and the external disturbance, or at least guarantees that the \( H_\infty \) norm will not exceed a given level \( \gamma > 0 \).

In the previous section, the nonlinear channel has been represented as T-S fuzzy model, based on the fuzzy model; the state equation of the overall system can be obtained. Moreover, the \( H_\infty \) equalizer design problem can be formulated as a standard LMI optimization problem via the bounded real lemma. This lemma establishes the equivalence between the following statements:

**Theorem 3.1.** \([3,6]\) Consider a discrete-time transfer function \( T(z) \) of realization \( T(z) = C(zI - A)^{-1} B + D \).

The following statements are equivalent:

(i). \( \| C(zI - A)^{-1} B + D \|_\infty < \gamma \) and \( A \) is stable in the discrete-time sense.

(ii). There exists a solution to the LMI:

\[ \begin{bmatrix} A^T P A - P & A^T P B \ \\
B^T P A & B^T P B - \gamma^2 I \end{bmatrix} \begin{bmatrix} C^T \ \\
D^T \end{bmatrix} < 0. \quad (19) \]

**Proof:** Omitted due to space limit.

Based on Theorem 3.1, both of the internal stability and the \( H_\infty \)-norm constraint are equivalent to the feasibility of the above matrix inequality (19) for some symmetric matrix \( P > 0 \). Because of the unknown matrix is \( P \) and the fuzzy equalizer parameters entering in matrices \( C \) and \( D \), the inequality (19) is a standard LMI and the \( H_\infty \) constraints can be expressed as a single matrix inequality via the bounded real lemma. To obtain a better performance, the \( \gamma \) value can be reduced to the minimum.
possible value such that LMI (19) is satisfied. Therefore, the design procedure is summarized as follows.

**Design Procedures:**
Step 1) Identify the premise variables relevant to the model.
Step 2) Choose the structure of the membership functions.
Step 3) Determine the number of fuzzy rules.
Step 4) Construct the channel fuzzy model.
Step 5) Defuzzification of the channel fuzzy model.
Step 6) Construct the equalizer fuzzy model.
Step 7) Defuzzification of the equalizer fuzzy model.
Step 8) Decide the lengths of the equalizer and delay.
Step 9) Derive the error transfer function of the overall system.
Step 10) Solve the LMI in (19) to obtain the optimal value and the coefficients of the fuzzy equalizer.

**IV. An example**

In this section, we will illustrate an example to demonstrate the effectiveness of the proposed methodology. It is assumed that the channel output is observed in additive white Gaussian noise with zero mean and power spectral density $\sigma^2 = 1$. This channel has two non-minimum phase zeros, the delay of this system is chosen as $d = 2$. The following discrete-time nonlinear channel model is considered:

$$y(k) = s(k) + 0.33562s(k-1) + 0.1s^2(k-1) + 4.6276s(k-2) + 0.2s(k-1)s^2(k-2) - 0.14487s(k-3) + 1.6837s(k-4) \quad (20)$$

It is assumed that $s_1, s_2 \in [-1, 1]$, for the nonlinear terms, define $z_1(k) = s_1$ and $z_2(k) = s_1 \times s_2$. To minimize the design effort and complexity, four rules are used to construct the nonlinear system (20), and the membership functions of $z_1(k)$ and $z_2(k)$ are simply defined using triangular types in Fig.2 and Fig.3. By using these fuzzy sets and rules, the nonlinear system is represented by the following piecewise linear T-S fuzzy model.

**Rule 1:**
If $z_1(k)$ is “Positive” and $z_2(k)$ is “Big”
Then $s_{i+1} = A_{i1}s_i + B_{i1}u_i$
$$y_i = C_{i1}s_i + D_{i1}u_i \quad (21)$$

**Rule 2:**
If $z_1(k)$ is “Positive” and $z_2(k)$ is “Small”
Then $s_{i+1} = A_{i2}s_i + B_{i2}u_i$
$$y_i = C_{i2}s_i + D_{i2}u_i \quad (22)$$

**Rule 3:**
If $z_1(k)$ is “Negative” and $z_2(k)$ is “Big”
Then $s_{i+1} = A_{i3}s_i + B_{i3}u_i$
$$y_i = C_{i3}s_i + D_{i3}u_i \quad (23)$$

**Rule 4:**
If $z_1(k)$ is “Negative” and $z_2(k)$ is “Small”
Then $s_{i+1} = A_{i4}s_i + B_{i4}u_i$
$$y_i = C_{i4}s_i + D_{i4}u_i \quad (24)$$

Where
$$A_{i1} = A_{i2} = A_{i3} = A_{i4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \, , \quad B_{i1} = B_{i2} = B_{i3} = B_{i4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \, ,$$

$$C_{i1} = [0.43562 \, \quad 4.8276 \, \quad -0.14487 \, \quad 1.6837] \, , \quad C_{i2} = [0.43562 \, \quad 4.4276 \, \quad -0.14487 \, \quad 1.6837] \, ,$$

$$C_{i3} = [0.23562 \, \quad 4.8276 \, \quad -0.14487 \, \quad 1.6837] \, , \quad C_{i4} = [0.23562 \, \quad 4.4276 \, \quad -0.14487 \, \quad 1.6837] \, ,$$

$$D_{i1} = D_{i2} = D_{i3} = D_{i4} = [1 \, \quad 1] \, .$$

Consider the operate point at $s_1 = -0.2$ and $s_2 = 0.2$, the defuzzification of the channel is carried out as

$$s_{i+1} = \frac{\sum_{j=1}^{4} w_j(z(k)) \{A_{j1}s_i + B_{j1}u_i\}}{\sum_{j=1}^{4} w_j(z(k))} = A_c s_i + B_c u_i \, , \quad (25)$$

Where
$$A_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \, , \quad B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \, .$$

$$y_i = \frac{\sum_{j=1}^{4} w_j(z(k)) \{C_{j1}s_i + D_{j1}u_i\}}{\sum_{j=1}^{4} w_j(z(k))} = C_c s_i + D_c u_i \quad (26)$$

Where $C_c = [0.31562 \, \quad 4.6276 \, \quad -0.14487 \, \quad 1.6837] \, , \quad D_c = [1 \, \quad 1] \, .$

Furthermore, the FIR fuzzy equalizer with order five is represented by the following piecewise linear T-S fuzzy model:
Rule 1: 
If $z_1(k)$ is “Positive” and $z_2(k)$ is “Big”

Then $a_{01} = A_1 a_0 + B_1 y_i$

$\hat{z}_i = C_1 a_0 + D_1 y_i$ (27)

Rule 2: 
If $z_1(k)$ is “Positive” and $z_2(k)$ is “Small”

Then $a_{02} = A_2 a_0 + B_2 y_i$

$\hat{z}_i = C_2 a_0 + D_2 y_i$ (28)

Rule 3: 
If $z_1(k)$ is “Negative” and $z_2(k)$ is “Big”

Then $a_{03} = A_3 a_0 + B_3 y_i$

$\hat{z}_i = C_3 a_0 + D_3 y_i$ (29)

Rule 4: 
If $z_1(k)$ is “Negative” and $z_2(k)$ is “Small”

Then $a_{04} = A_4 a_0 + B_4 y_i$

$\hat{z}_i = C_4 a_0 + D_4 y_i$ (30)

Where $A_{e1} = A_{e2} = A_{e3} = A_{e4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

$B_{e1} = B_{e2} = B_{e3} = B_{e4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$C_{e1} = [k_{11} \  k_{21} \  k_{31} \  k_{41}]$, $C_{e2} = [k_{12} \  k_{22} \  k_{32} \  k_{42}]$, $C_{e3} = [k_{13} \  k_{23} \  k_{33} \  k_{43}]$, $C_{e4} = [k_{14} \  k_{24} \  k_{34} \  k_{44}]$,

$D_{e1} = k_{01}$, $D_{e2} = k_{02}$, $D_{e3} = k_{03}$ and $D_{e4} = k_{04}$.

By applying the procedures in the previous section, the optimal $\gamma$ value can be derived with 0.4548 and the coefficients of the fuzzy equalizer are obtained in Table 1. For compare the performance with different length of the fuzzy equalizer, the length with three and four of fuzzy equalizer are demonstrated, the optimal $\gamma$ values are obtained with 0.4919 and 0.4916, respectively. The coefficients of the defuzzification are indicated in Table 2 as well. The bit error rate (BER) simulation is also introduced to provide the true picture of performance of the system, where the BER is defined as the bit error probability with respective to SNR, and herein is relative to the transmitted signal $b_i$. The BER comparison for the length of fuzzy equalizer with five can be seen in Fig. 4.

From the demonstrated example, better performance will be obtained with increasing the length of the equalizer. The result in Fig.4 shows the better BER performance of the proposed method for T-S fuzzy model. This implies that the T-S fuzzy model can represent a highly nonlinear function relation and the designed equalizer can improve the BER performance.

V. Conclusion

In this paper, the LMI-based nonlinear channel equalization has been studied. The FIR fuzzy equalizer coefficients can be easily obtained by solving a set of LMIs only, it is simple and numerically tractable. A given numerical example has demonstrated the effectiveness of the proposed methodology. The results show that the longer equalizer outperforms the shorter equalizer and the nonlinear channel can be constructed as T-S fuzzy linear model by using a reasonable number of fuzzy rules. In general, the designed equalizer LMI-based fuzzy approach shows the facts of improving BER performance.

References


Table 1: The coefficients of fuzzy equalizer

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<th>Coefficients</th>
<th>$k_0$</th>
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<th>$k_2$</th>
<th>$k_3$</th>
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Table 2: $r$ values and equalizer coefficients

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<th>Equalizer length</th>
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<td>optimal $r$</td>
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