On the Use of SoS Methods for Analysis of Connection-Level Stability in the Internet

Ashvin Lakshmikantha, C. L. Beck and R. Srikant

Abstract—We study connection-level models of file transfer requests in the Internet, where connection arrivals to each route occur according to Poisson processes and the file-sizes have phase-type distributions. We investigate the use of the Sum-of-Squares technique to construct Lyapunov functions that satisfy Foster’s condition for stochastic stability.

I. INTRODUCTION

Over the last few years, it has been observed that TCP-like protocols can be viewed as distributed mechanisms for resource allocation in the Internet. Much attention has been devoted to designing algorithms that are stable when the number of competing sources in the network is fixed; see [6] for a comprehensive survey. In this work, we study the stochastic stability of the network at the connection level, i.e., the dynamics of connection arrivals and departures. This problem was first considered in [8] and generalized in [2]; these papers assume that the connection arrival processes are Poisson and the file sizes are exponentially distributed. We relax the exponential file size distribution assumption and consider a class of file size distributions which are dense in the set of all possible distributions. Note that TCP-compliant streaming flows are not considered in our model (see [3]).

In order to numerically establish connection-level stability of a variety of network topologies, we apply the Sum-of-Squares (SoS) technique to establish the positive recurrence of a certain Markov chain; this Markov chain describes the evolution of the number of connections in a congestion-controlled network. In the systems we consider, the state variables are always non-negative. Therefore, to prove stability, it is sufficient to find Lyapunov functions which are copositive and whose time derivatives are non-negative. A sufficient condition for \( f(x) \) to be copositive is that the function \( g(z_1, z_2, \ldots, z_n) = f(z_1^2, z_2^2, \ldots, z_n^2) \) is positive definite. One can then apply the SoS technique directly to the polynomial \( g(z) \).

We now describe our connection-level model for the Internet. The SoS technique is then used to establish connection-level stability for linear and star network topologies.

II. MATHEMATICAL NETWORK MODEL

Consider a network consisting of a set of links \( L \) with each link \( l \in L \) having a capacity \( c_l > 0 \). A number of flows compete for access to these links and each type of flow is associated with a route consisting of a subset of \( L \). The set of all routes (all possible types of flows) is denoted by \( R \). Let \( n_r \) denote the number of flows and \( x_r \) denote the bandwidth associated with each flow, using route \( r \). All the flows that take a route \( r \) are considered type \( r \). The set of rates \( \{x_r\} \) is chosen as a function of the number of sources on each route, \( \{n_r\} \), by solving an optimization problem which will be described later. Flows arrive to and depart from the network and thus, \( \{n_r\} \) changes as a function of time. It is assumed that as the flows arrive and depart, the bandwidths associated with the flows \( \{x_r\} \) change instantaneously. Note that we further assume resource allocation in the Internet is accomplished using a congestion controller algorithm that, for our model, converges instantaneously.

Flows of type \( r \) arrive according to a Poisson process of rate \( \lambda_r \). Each file of type \( r \) is distributed according to a phase-type distribution (i.e., a Coxian distribution). The number of phases in file type \( r \)'s distribution is \( N_r \). Each phase consists of an exponential distribution with mean \( 1/\mu_r \). With the exception of the final phase, after a phase is completed, the file exits the system with probability \( (1-p_r) \) and begins a new exponential phase with probability \( p_r \). After the final phase, the file exits the system w. p. 1. Thus, the mean file-size distribution is given by

\[
\frac{1}{\mu_r} \left[ \frac{1-p_r^{N_r}}{1-p_r} \right].
\]

For any flow \( r \in R \), we denote the number of flows in the \( i^{th} \) stage by \( n_{r,i} \). The set of rates \( \{x_r\} \) at any time \( t \) when the number of flows in the network is \( \{n_r\} \) is given by the solution to the following optimization problem:

\[
\max_{\{x_r\}} \sum_r n_r w_r \log x_r
\]

subject to \( \sum_{r \in R} n_r x_r \leq c_l, \forall l \), and \( x_r \geq 0, \forall r \),

where \( n_r = (\sum_{j=1}^{N_r} n_{r,j}) \).

A function \( f \) is called copositive if \( f(x) > 0 \) \( \forall x \in \mathbb{R}_+^n \). If \( -f \) is copositive, then we will call \( f \) negative.
The following is clearly a necessary condition for the Markov chain \( \{n_{rj}(t)\} \) to be stable:

\[
\sum_{i \in \tau} \lambda_i \mu_i \left( \frac{1 - p_i^{N_r}}{1 - p_r} \right) < c_i \quad \forall i.
\]

That is, the load on each link should be less than its capacity. We use the term “stable” to indicate that the Markov chain is positive recurrent. To prove that (1) is also sufficient for stability, we first uniformize the continuous-time Markov chain and obtain a discrete-time Markov chain [4]. To obtain the discrete-time chain, we note that the largest rate at which transitions occur out of any state is bounded above by

\[
\sum_r \lambda_r \max_i c_i \sum_r \mu_r N_r.
\]

Without loss of generality, we normalize this rate to 1. The equivalent discrete-time Markov chain \( \{n_{rj}(k)\}, k = 0, 1, 2, \ldots \) then is obtained as follows:

- an arrival of type \( r \) w.p. \( \lambda_r \),
- a departure from the network from a phase that is not the last phase, i.e., a transition from \( (n_{rj}, n_{rj+1}) \) to \( (n_{rj} - 1, n_{rj+1} + 1) \) w.p. \( \mu_r r_{rj} x_{rj} \mu_r \), if \( n_{rj} > 0 \) and \( j < N_r \),
- a departure from the network from the last phase, i.e., a transition from \( (n_{rj}, N_r, x_{rj}, \mu_r) \) if \( n_{rj} > 0 \) and \( j < N_r \),
- a fictitious event, i.e., no change in the state of the system, w.p. \( 1 - (\sum_r \lambda_r + \sum_{rj} n_{rj} x_{rj} \mu_r) \).

We use the following lemma, which presents a more stringent sufficient condition than the original Foster-Lyapunov theorem [1, Prop. 5.3], to verify stability of the Markov chain \( \{n_{rj}(k)\} \).

**Lemma 1:** If there exist positive constants \( \{\kappa_{rj}\}, \{a_{rj}\} \) and \( C \) such that

\[
V(n) := \sum_{rj} \kappa_{rj} n_{rj}^2 \text{ satisfies } V(n(k + 1)) - V(n(k)) | n(k) = n | \leq -\epsilon \left( \sum_{rj} \sum_{j=1}^{N_r} a_{rj} n_{rj}^2 \right) + C,
\]

\( \forall n \geq 0 \), then the Markov chain is positive recurrent.

**III. STABILITY ANALYSIS**

In this section, we make use of SoS procedures to find Lyapunov functions that satisfy Lemma 1. We consider two types of network topologies, linear and star networks.

**A. Linear Network Topologies**

**Example 1:** Consider a linear network consisting of two links with unit link capacities as shown in Figure 1. There are three routes in the network: 0, 1 and 2. Route 0 consists of both the links while route 1 uses link 1 exclusively and route 2 uses link 2 exclusively. The number of phases associated with each route is given by: \( N_0 = 2, N_1 = 1, N_2 = 1 \). From (1), a necessary condition for stability is

\[
\frac{\kappa_0}{\mu_0} (1 + p_0) + \frac{\kappa_1}{\mu_1} (1 + p_1) < 1,
\]

\[
\frac{\kappa_2}{\mu_2} (1 + p_2) < 1.
\]

To show (3) is also sufficient, we consider a quadratic Lyapunov function of the form:

\[
V(n(k)) = \frac{\kappa_0}{\mu_0} n_{00}(k)^2 + \frac{\kappa_1}{\mu_1} n_{02}(k)^2 + \frac{\kappa_2}{\mu_2} n_{11}(k)^2 + \frac{\kappa_2}{\mu_2} n_{22}(k).
\]

Denoting \( \frac{1}{\mu_r} \) by \( \rho_r \), we have

\[
E[V(n(k + 1))] - V(n(k)) | n(k) = n | \leq \sum_{r=0}^{2} \kappa_r \rho_r (1 + 2n_r)
\]

\[
+ \kappa_0 \rho_0 n_{00}(k) + \kappa_2 \rho_2 n_{02}(k) + \kappa_0 \rho_0 n_{00} + \kappa_2 \rho_2 n_{02} + C.
\]

where \( C \) is an appropriately defined constant. Given the state of the system \( n \), the rates allocated to each flow can be calculated by solving the optimization problem:

\[
\max_{x_{rj} \geq 0} n_{00} \log x_{00} + n_{11} \log x_{11} + n_{22} \log x_{22}
\]

subject to

\[
n_{00} x_{00} + n_{11} x_{11} + n_{22} x_{22} = 1,
\]

\[
n_{00} x_{00} + n_{11} x_{11} + n_{22} x_{22} = 1.
\]

The optimal rates are given by

\[
n_{r} x_{r} = \begin{cases} \frac{n_{00} x_{00} + n_{11} x_{11} + n_{22} x_{22}}{n_{00} x_{00} + n_{11} x_{11} + n_{22} x_{22}} & \text{if } r = 0, \\ \frac{n_{00} x_{00} + n_{11} x_{11} + n_{22} x_{22}}{n_{00} x_{00} + n_{11} x_{11} + n_{22} x_{22}} & \text{if } r = 1, 2. \end{cases}
\]

Substituting for \( \{x_r\} \) in (4) we get,

\[
E(V(n(k + 1)) - V(n(k)) | n(k)) \leq \sum_{r=0}^{2} \kappa_r \rho_r n_r \left( \sum_{r} n_{rj} x_{rj} \right)
\]

\[
+ \kappa_0 \rho_0 n_{00}(k) + \kappa_2 \rho_2 n_{02}(k) + \kappa_0 \rho_0 n_{00} + \kappa_2 \rho_2 n_{02} + C + C.
\]

To apply Lemma 1, we need to show that there exists \( \epsilon > 0 \) and \( \{\kappa_i > 0\} \) such that

\[
\left( \sum_r \kappa_r \rho_r n_r \left( \sum_r n_{rj} x_{rj} \right) + \kappa_0 \rho_0 n_{00}(k) + \kappa_2 \rho_2 n_{02}(k) + \kappa_0 \rho_0 n_{00} + \kappa_2 \rho_2 n_{02} \right) \leq -\epsilon \sum_{r} n_{rj}^2.
\]
In other words, we need to show the polynomial $F(n)$ is conegative, where

$$F(n) = (\sum_{r=01,2} \kappa_r \rho_r \nu_r)(\sum_r n_r w_r) + \kappa_{02} \rho_0 \nu_0 n_0 \nu_0 + \sum_{j=1,2} \kappa_j \nu_j^2 w_j + \epsilon \sum_r n_r^2)$$

By substituting $n_i = z_i^2$, it can be seen that $F(n)$ is conegative when the following inequalities are satisfied:

$$\begin{bmatrix}
  b_1 - \kappa_n - \kappa_0 - y_1 - y_2 - y_3 \\
  -b_0 - y_1 - b_2 - \epsilon - b_4 - y_4 - y_5 \\
  -b_4 - y_2 - b_6 - y_4 - b_3 - \epsilon \\
  -b_4 - y_3 - b_6 - y_5 - b_4 - \epsilon
\end{bmatrix} > 0,
$$

where

$$
\begin{align*}
  b_1 &= \kappa_{01} \nu_0 (1 - \rho_0) \\
  b_2 &= \kappa_{02} \nu_0 \\
  b_3 &= \kappa_{11} \nu_1 (1 - \rho_1) \\
  b_4 &= \kappa_{22} \nu_2 (1 - \rho_2) \\
  b_5 &= (\kappa_{02} \rho_0 + \kappa_{11} \rho_1) \\
  b_6 &= (\kappa_{01} \rho_0 \nu_1 + \kappa_{11} \rho_1 \nu_0) \\
  b_7 &= (\kappa_{01} \rho_0 \nu_2 + \kappa_{22} \rho_2 \nu_0) \\
  b_8 &= \kappa_{12} \rho_2 \nu_0
\end{align*}
$$

and $b_{01} = \kappa_{01} \nu_1 (1 - \rho_2) + \kappa_{11} \nu_2 (1 - \rho_1)$.

Therefore to show the system is stable, we have to find parameters $\{\kappa_i\}, \{y_i\}$ such that the above linear matrix inequalities (LMIs) are satisfied (see, for example [7]).

A set of numerical experiments have been completed for a large variety of system parameter values. A small subset of these parameter values are given in Table I. In all experiments described herein, $\mu_r = 1 \forall r$; the arrival rates $\{\lambda_r\}$ range from 0 to 0.99; the probability of re-entry ranges from 0.1 to 0.99; and $\epsilon$ is set to 0.001. The link utilization is set to 0.99\(^2\) i.e.,

$$\sum_{t \in T} \lambda_r \mu_r \left(1 - p_r^{\lambda_t} \right) = 0.99 \epsilon_i \forall i.$$ 

The resulting coefficients for the Lyapunov functions obtained via solution of the above LMI feasibility problem are given in Table II.

**Example 2:** The same network topology is now considered, however, the number of phases associated with each route is as follows: $N_0 = 1, N_1 = 2, N_2 = 2$. Again from (1), a necessary condition for stability is

$$\frac{\lambda_0}{\mu_0} + \frac{\lambda_1}{\mu_1} (1 + p) < 1,$$

$$\frac{\lambda_2}{\mu_2} + \frac{\lambda_3}{\mu_3} (1 + p) < 1.$$ 

\(^2\)It is easy to show that if the utilization is less than 0.99 then one can always find $\{\lambda_r\}$ with utilization equal to 0.99 such that the Lyapunov function associated with these parameters is also a Lyapunov function for the original problem.

The optimal resource allocation does not change with the number of phases and hence the rates $\{x_r\}$ are given by (5). To verify that (6) is sufficient for stability, we consider the following Lyapunov function.

$$V(k) = \frac{\kappa_0}{\mu_0} \nu_0^2 + \frac{\kappa_{11}}{\mu_1} \nu_1^2 + \frac{\kappa_{12}}{\mu_2} \nu_2^2 + \frac{\kappa_{21}}{\mu_1} \nu_1^2 + \frac{\kappa_{22}}{\mu_2} \nu_2^2.$$

As in Example 1, it is possible to convert the Lyapunov function search into an LMI (solved, for example, using [7]). A small subset of the system parameter values considered are given in Table III, and corresponding Lyapunov function coefficients are given in Table IV.

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**TABLE I**

<table>
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<tr>
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**TABLE II**

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<td>$b_2$</td>
<td>31.33</td>
</tr>
<tr>
<td>$b_3$</td>
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</table>

**TABLE III**

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</tr>
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<tr>
<td>$\lambda_2$</td>
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<tr>
<td>$\mu$</td>
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</table>

**TABLE IV**

<table>
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</tr>
<tr>
<td>$\kappa_{22}$</td>
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</tbody>
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**Fig. 1. Linear Network**

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**TABLE V**

<table>
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<td>$\lambda_2$</td>
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<td>$\mu$</td>
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<td>$\kappa_{01}$</td>
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<td>$\kappa_{02}$</td>
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<td>$\kappa_{01}$</td>
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</tr>
<tr>
<td>$\kappa_{22}$</td>
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TABLE IV
EXAMPLE 2-LYAPUNOV FUNCTION COEFFICIENTS

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<th>$\kappa_{12}$</th>
<th>$\kappa_{21}$</th>
<th>$\kappa_{22}$</th>
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<td>18.77</td>
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<td>7.37</td>
<td>6.30</td>
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<tr>
<td>3.31</td>
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<td>675.14</td>
<td>485.28</td>
</tr>
<tr>
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<td>45.38</td>
<td>24.23</td>
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TABLE V
STAR NETWORK : SYSTEM PARAMETER VALUES

<table>
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<tr>
<th>$\lambda_0$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
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<td>0.90</td>
<td>0.37</td>
<td>0.45</td>
<td>0.05</td>
<td>0.99</td>
</tr>
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</table>

Fig. 2. Star Network Topology

B. Star Network Topologies

We now consider the star network topology\(^3\) which is comprised of 4 links, with unit capacities, as shown in Figure (2). There are 4 routes in the network : 0, 1, 2 and 3. The number of phases associated with each route is: $N_0 = 1, N_1 = 2, N_2 = 2, N_3 = 1$. From (1) a necessary condition for stability is

\[
\begin{align*}
\frac{\lambda_0}{\mu_0} + \frac{\lambda_1}{\mu_1} &< 1, \\
\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} &< 1, \\
\frac{\lambda_2}{\mu_2} (1 + p) + \frac{\lambda_3}{\mu_3} (1 + p) &< 1, \\
\frac{\lambda_3}{\mu_3} (1 + p) &< 1.
\end{align*}
\] (7)

Given the number of flows in the system, the optimal allocation of the rates $\{x_r\}$ can be found by solving the problem:

\[
\max\ n_0 w_0 \log x_0 + n_1 w_1 \log x_1 + n_2 w_2 \log x_2 + n_3 w_3 \log x_3 \quad (x_r \geq 0)
\]

subject to:

\[
\begin{align*}
n_0 x_0 + n_1 x_1 &\leq 1, \\
n_1 x_1 + n_2 x_2 &\leq 1, \\
n_2 x_2 + n_3 x_3 &\leq 1, \\
n_3 x_3 &\leq 1.
\end{align*}
\]

Solving the above problem we get

\[
\begin{align*}
n_0 x_0 &= \frac{n_0 w_0 + n_2 w_2}{n_1 w_1 + n_3 w_3}, \\
n_1 x_1 &= \frac{n_1 w_1 + n_3 w_3}{n_2 w_2 + n_3 w_3}.
\end{align*}
\]

\(^3\)This topology is motivated by the fact that the current Internet can be modelled as a star network since the core network links are relatively uncongested and the congestion occurs mainly on the access links.

To show that (7) is sufficient for stability, using Lemma 1, we consider the Lyapunov function:

\[
V(k) = \frac{\kappa_0}{\mu_0} n_0^2 + \frac{\kappa_1}{\mu_1} n_1^2 + \frac{\kappa_2}{\mu_2} n_2^2 + \frac{\kappa_3}{\mu_3} n_3^2 + \frac{\kappa_{21}}{\mu_2} n_2 n_1 + \frac{\kappa_{22}}{\mu_2} n_2^2 + \frac{\kappa_{31}}{\mu_3} n_3 n_1 + \frac{\kappa_{32}}{\mu_3} n_3^2
\]

As in Example 1, the search for such Lyapunov functions can be converted to a series of LMI feasibility problems. A selection of parameters for which the system has been studied are given in Table V and the corresponding Lyapunov function coefficients are given in Table VI.

TABLE VI
STAR NETWORK : LYAPUNOV FUNCTION COEFFICIENTS

<table>
<thead>
<tr>
<th>$\kappa_0$</th>
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REFERENCES