Robust $H_\infty$ Control for Uncertain Takagi-Sugeno Fuzzy Systems with Interval Time-Varying Delay

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Abstract—This paper is concerned with the problem of delay-dependent robust $H_\infty$ control for uncertain time-delay fuzzy systems with norm-bounded uncertainty. The time-delay is assumed to be a time-varying continuous function belonging to a given interval, which means that the lower and upper bounds for the delay are available. No restriction on the derivative of the time-varying delay is needed, which allows the time-delay to be a fast time-varying function. The state-space Takagi-Sugeno (T-S) uncertain fuzzy model with interval time-varying delay is adopted. Delay-dependent conditions for the existence of robust $H_\infty$ controller are presented in the form of linear matrix inequalities (LMIs). A numerical example is given to demonstrate the effectiveness of the proposed method.

I. INTRODUCTION

Fuzzy systems in the form of the Takagi-Sugeno model have attracted great interests in the past decade. It has shown that the T-S model method can give an effective way to represent complex nonlinear systems by some simple local linear dynamic systems with their linguistic description. And some nonlinear dynamic systems can be approximated by the overall fuzzy linear T-S models for the purposes of stability analysis and controller design.

Delayed fuzzy systems were introduced and studied in [5] by developing the T-S fuzzy model based on the Lyapunov-Krasovskii approach. After then, the T-S model of fuzzy systems with time-delay have been widely studied, and many relevant results have been reported ([6], [7] and references therein). It is well known that delays appear in many dynamic systems. Fuzzy delayed systems of T-S models provide a method of using local linear delayed systems combined with fuzzy linguistic descriptions to achieve nonlinearity.

It is clear that the stability analysis and stabilization are important issues in analysis and design of control systems. In the time-domain, the direct Lyapunov method is a powerful tool. There are two different ideas to pursue this method: the Lyapunov-Krasovskii approach and the Lyapunov-Razumikhin approach. Both approaches can be used to handle systems with time-varying delay. The former usually requires both the upper bound of the time-varying delay and additional information on its derivative [5], [8], [9], while the latter has no restriction on the derivative of the time-varying delay, which allows a fast time-varying delay [10]. However, the obtained results using the Lyapunov-Krasovskii approach are usually less conservative than those using the Lyapunov-Razumikhin approach since the former takes advantage of the additional information of the delay.

It is well known that there exist some systems which are stable with some nonzero delay, but are unstable without delay [11], [12]. For such case, if there is a time-varying perturbation on the nonzero delay, it is of great significance to consider the stability analysis and controller design of the systems with interval time-varying delay. Other typical examples of the systems with interval time-varying delay are networked control systems [13], [14]. The stability of such kinds of systems was investigated in [15] using the Lyapunov-Krasovskii approach, where the derivative bound of the interval time-varying delay, i.e. $\dot{\tau}(t) \leq \tau_d < 1$, was needed. However, in most of practical applications, it is not easy to estimate the bound of the derivative of time-varying delay in advance. Sometimes, such derivative bound can not satisfy $\dot{\tau}(t) \leq \tau_d < 1$, or the time-varying delay is not even differentiable at all, such as networked control systems. To the best of our knowledge, for the case where only the upper and lower bounds of the interval time-varying delay are precisely known, there is no result available for the delay-dependent robust $H_\infty$ controller design for such kinds of systems, especially for fuzzy systems by employing the Lyapunov-Krasovskii approach.

In this paper, we will consider the problem of delay-dependent robust $H_\infty$ control design for uncertain fuzzy systems with interval time-varying delay. The restriction on the derivative of the interval time-varying delay is removed, which means that a fast interval time-varying delay is allowed. Based on the Lyapunov-Krasovskii functional approach, the existence condition of a delay-dependent robust $H_\infty$ controller will be derived by introducing some relaxation matrices which can be used to reduce the conservatism of the obtained criterion which will be formulated in the form of linear matrix inequalities (LMIs). A numerical example will be given to show the effectiveness of the method.

Notation. For a symmetric matrix $X$, the notation $X \geq 0$ ($X > 0$) means that the matrix $X$ is positive semi-definite.
continuous time-varying function satisfying corresponding to \( \Delta F \) real constant matrices of appropriate dimensions. \( z \) where \( A \) and \( B \) be 

\[
\begin{align*}
\text{Plant Rule} \quad i \quad &: \quad \text{IF } z_i(t) \text{ is } M_{i1}, \ z_2(t) \text{ is } M_{i2}, \ldots, \ z_g(t) \text{ is } M_{ig} \\
\quad &\quad \text{THEN } \left\{ 
\quad \dot{x}(t) = [A_{i0} + \Delta A_{i0}(t)]x(t) \\
\quad &+ [A_{i1} + \Delta A_{i1}(t)](x(t - \tau(t)) \\
\quad &+ B_x D_i u(t)] + B_{wi}w(t), \\
\quad \ddot{z}(t) = C_i x(t) + D_i u(t), \\
\quad x(t) = \phi(t), \ t \in [-\tau_M, 0] \right\}
\end{align*}
\]

where \( i = 1, 2, \ldots, r \) is the number of IF-THEN rules; \( z_i(t), z_2(t), \ldots, z_g(t) \) are the premise variables of \( i \) and \( M_{ij} \) \( (i = 1, 2, \ldots, r; j = 1, 2, \ldots, g) \) are the fuzzy sets corresponding to \( z_j(t) \) and plant rules \( r \); \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the input, \( w(t) \in L_2[0, \infty) \) is the exogenous disturbance, and \( \dot{z}(t) \in \mathbb{R}^p \) is the controlled output; \( A_{i0}, A_{i1}, B_x, w_i, C_i \) and \( D_i \) \( (i = 1, 2, \ldots, r) \) are known parameter matrices of appropriate dimensions; \( \Delta A_{i0}(t), \Delta A_{i1}(t) \) and \( \Delta B_i(t) \) \( (i = 1, 2, \ldots, r) \) are real-valued unknown matrices representing time-varying parameter uncertainties of \( i \), and are to be formed as the form 

\[
\Delta A_{i0}(t), \Delta A_{i1}(t), \Delta B_i(t) = H_i F_i(t) [E_{i0}, E_{i1}, E_{ib}]
\]

where \( H_i, E_{i0}, E_{i1} \) and \( E_{ib} \) \( (i = 1, 2, \ldots, r) \) are known real constant matrices of appropriate dimensions. \( F_i(t) \in \mathbb{R}^{l_{i1} \times l_{i2}} \) \( (i = 1, 2, \ldots, r) \) are unknown time-varying matrix functions with Lebesgue measurable elements satisfying 

\[
F_i(t)^T F_i(t) \leq I, \ i = 1, 2, \ldots, r.
\]

\( \phi(t) \) is the initial condition of system \( (1) \); \( \tau(t) \) is a uniformly continuous time-varying function satisfying 

\[
0 < \tau_m \leq \tau(t) \leq \tau_M,
\]

where \( \tau_M \) and \( \tau_m \) are two known constants. Let \( \mu_i(z(t)) \) be the normalized membership function of the inferred fuzzy set \( \mu_i(z(t)) \), i.e. 

\[
\mu_i(z(t)) = \frac{\mu_i(z(t))}{\sum_{i=1}^{g} \mu_i(z(t))},
\]

\[
\mu_i(z(t)) = \prod_{j=1}^{g} M_{ij}(z_j(t)),
\]

\( M_{ij}(z_j(t)) \) is the grade of membership of \( z_j(t) \) in \( M_{ij} \). It is assumed that 

\[
\mu_i(z(t)) \geq 0, \quad i = 1, 2, \ldots, r, \quad \sum_{i=1}^{r} \mu_i(z(t)) > 0, \quad \forall t \geq 0.
\]

Then 

\[
h_i(z(t)) \geq 0, \quad i = 1, 2, \ldots, r, \quad \sum_{i=1}^{r} h_i(z(t)) = 1, \quad \forall t \geq 0.
\]

Using a center average defuzzifier, product inference, and singleton fuzzifier, the fuzzy model \( (1) \) can be expressed by the following global model 

\[
\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))[A_{i0}(t)x(t) + A_{i1}(t)x(t - \tau(t)) + B_x D_i u(t)] + B_{wi}w(t),
\]

\[
\ddot{z}(t) = \sum_{i=1}^{r} h_i(z(t))[C_i x(t) + D_i u(t)],
\]

\[
x(t) = \phi(t), \quad t \in [-\tau_M, 0].
\]

where \( A_{i0}(t) \triangleq A_{i0} + \Delta A_{i0}(t), A_{i1}(t) \triangleq A_{i1} + \Delta A_{i1}(t), B_x(t) \triangleq B_i + \Delta B_i(t), i = 1, 2, \ldots, r. \)

Throughout this paper, delay-dependent state feedback T-S fuzzy-model-based \( H_\infty \) control laws are utilized for the robust stabilization of the T-S fuzzy system \( (1) \) as follows 

\[
R^i : \quad \text{IF } z_i(t) \text{ is } M_{i1}, z_2(t) \text{ is } M_{i2}, \ldots, z_g(t) \text{ is } M_{ig} \quad \text{THEN } u(t) = K_i x(t),
\]

where \( K_i \) \( (i = 1, 2, \ldots, r) \) are the controller gains of \( (7) \) to be determined. The defuzzified output of the controller rules is given by 

\[
u(t) = \sum_{i=1}^{r} h_i(z(t)) K_i x(t).
\]

\( (6) \) with controller \( (8) \) can be represented as 

\[
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t))[(A_{i0}(t) + B_i(t) K_j)x(t) + A_{i1}(t)x(t - \tau(t)) + B_{wi}w(t)],
\]

\[
\ddot{z}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t))[C_i + D_i K_j] x(t),
\]

\[
x(t) = \phi(t), \quad t \in [-\tau_M, 0].
\]

For a prescribed scalar \( \gamma > 0 \), we define the performance index 

\[
J(w) = \int_{0}^{\infty} [\ddot{z}(\theta) - \gamma^2 w(\theta) w(\theta)] d\theta.
\]

The purpose of this paper is to design a delay-dependent robust \( H_\infty \) controller \( (8) \) for the T-S global model \( (6) \) such that for all admissible uncertainties satisfying \( (2), (3), (4) \), and any \( \tau(t) \) satisfying \( (4) \) for a prescribed scalar \( \gamma > 0 \).

\( (1), (9) \) with \( w(t) = 0 \) is asymptotically stable; \( (2) \) under the zero initial condition, \( (9) \) satisfies 

\[
\|\ddot{z}(\theta)\|_2 < \gamma \|w(\theta)\|_2
\]

for all nonzero \( w(t) \in L_2[0, \infty) \).
III. MAIN RESULT

In this section some delay-dependent sufficient conditions for the existence of robust $H_\infty$ controller (8) for T-S fuzzy system (9) will be presented.

Defining $\tau_{av} = \frac{1}{2} (M + M_0)$ and $\delta = \frac{1}{2} (\tau_M - \tau_m)$, \(\tau(t)\) satisfying (4) can be expressed as

$$\tau(t) = \tau_{av} + \delta q(t),$$

where

$$q(t) = \begin{cases} \frac{2r(t) - (\tau_M + \tau_m)}{\tau_M - \tau_m}, & \tau_M > \tau_m \\ 0, & \tau_M = \tau_m \end{cases}$$

It is clear that $|q(t)| \leq 1$. For this case, $\tau(t)$ is a function belonging to the interval $[\tau_{av} - \delta, \tau_{av} + \delta]$, where $\delta$ can be taken as the range of variation of the time-varying delay $\tau(t)$.

Based on the Lyapunov-Krasovskii functional approach, the delay-dependent $H_\infty$ fuzzy controller (8) for the following nominal closed-loop system is first investigated

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \left[ (A_{i0} + B_i K_j) x(t) + A_{i1} (x(t - \tau(t)) + B_{wi} w(t)), \right] \\ \dot{z}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \left[ C_i + D_{ij} K_j \right] x(t), \\ x(t) = \phi(t), \quad t \in [-\tau_{av}, 0]. \end{cases}$$

Using the fact

$$x(t - \tau_{av} - x(t - \tau(t)) = \int_{t-\tau(t)}^{t-\tau_{av}} \dot{x}(s) ds,$$

system (13) can be rewritten as

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \left[ (A_{i0} + B_i K_j) x(t) + A_{i1} x(t - \tau_{av}) + B_{wi} w(t), \right] \\ -A_{i1} \int_{t-\tau_{av}}^{t-\tau(t)} \dot{x}(s) ds, \\ \dot{z}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) \left[ C_i + D_{ij} K_j \right] x(t), \\ x(t) = \phi(t), \quad t \in [-\tau_{av}, 0]. \end{cases}$$

For the delay-dependent $H_\infty$ controller design for fuzzy system (13), we now state and establish the following result.

Proposition 1: For a prescribed scalar $\gamma > 0$ and some given scalars $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\tau_m$ and $\tau_M$, system (13) is stable and satisfies $\|\dot{z}(t)\|_2 < \gamma \|w(t)\|_2$ for all nonzero $w(t) \in L_2(0, \infty)$ and any $\tau(t)$ satisfying (4), if there exist $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $X_j$ ($j = 1, 2, \cdots, r$), and $M_i$ ($i = 1, 2, 3$) of appropriate dimensions such that the following LMI’s simultaneously hold for $i, j = 1, 2, \cdots, r$,

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_1 \\ \ast & \Theta_{22} & \Theta_{23} & \Theta_2 \\ \ast & \ast & \Theta_{33} & \Theta_3 \\ \ast & \ast & \ast & \Theta_4 \end{bmatrix} < 0,$$

where

$$\begin{align*} \Theta_{11} &= \tilde{Q} + A_{i0} X + B_i Y_j + (A_{i0} X + B_i Y_j)^T + M_i^T + M_j \\ \Theta_{12} &= A_{i1} X + \varepsilon_2 (A_{i0} X + B_i Y_j)^T - M_i^T + M_j, \\ \Theta_{13} &= \tilde{P} - X + \varepsilon_3 (A_{i0} X + B_i Y_j)^T + M_i, \\ \Theta_{22} &= -\tilde{Q} + \varepsilon_2 (A_{i1} X + X^T A_i^T) - M_i^T - M_j, \\ \Theta_{23} &= -\varepsilon_2 X + \varepsilon_3 X^T A_i - M_j, \\ \Theta_{33} &= \tau_{av} R + 2 \delta S - \varepsilon_3 X^T + X, \\ \Theta_1 &= [B_{wi}, -\tau_{av} M_i^T, \delta A_{i1} X, (C_i X + D_{ij} Y_j)^T], \\ \Theta_2 &= [\varepsilon_2 B_{wi}, -\tau_{av} M_i^T, \delta \varepsilon_2 A_{i1} X, 0], \\ \Theta_3 &= [\varepsilon_3 B_{wi}, -\tau_{av} M_i^T, \delta \varepsilon_3 A_{i1} X, 0], \\ \Theta_4 &= diag\{-\gamma^2, I, -\tau_{av}, \tilde{R}, -\delta S, -I\}. \end{align*}$$

Moreover, the state feedback controller gains of (8) are given by $K_j = Y_j X^{-1}$ for $j = 1, 2, \cdots, r$.

In order to prove the above proposition, the following result is needed.

Lemma 1: [16] There exists a symmetric matrix $X$ such that

$$\begin{bmatrix} P_1 + X & Q_1 & 0 \\ Q_1^T & R_1 & 0 \\ 0 & 0 & R_2 \end{bmatrix} > 0,$$

if and only if

$$\begin{bmatrix} P_1 + X & Q_1 & 0 \\ Q_1^T & R_1 & 0 \\ 0 & 0 & R_2 \end{bmatrix} > 0.$$
where \( \Delta \tau(t) \equiv \tau(t) - \tau_{av} \).

Note that
\[
2 \xi^T(t) N_T \left\{ \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) ((A_{i0} + B_i K_j)x(t) + A_{i1} x(t - \tau_{av}) + B_{ui} u(t) - A_{i1} \int_{t - \tau(t)}^{t} \dot{x}(s) ds - \dot{x}(t) = 0, \right.
\]
and
\[
2 \xi^T(t) M_T [x(t) - x(t - \tau_{av}) - \int_{t - \tau(t)}^{t} \dot{x}(s) ds] = 0.
\]
where \( \xi^T(t) = [x^T(t) \ x^T(t - \tau_{av}) \ \dot{x}^T(t)] \), \( N = [N_1 \ N_2 \ N_3] \), \( M = [M_1 \ M_2 \ M_3] \), \( N_i, M_i \) \( (i = 1, 2, 3) \) are some matrices of appropriate dimensions. Then according to (5) and (17)-(20), we have
\[
\dot{V}(x_t) \leq \begin{cases} \sum_{i=1}^r \sum_{j=1}^r h_i h_j \Sigma_{ij}, & \text{if} \ \tau(t) \neq \tau_{av}, \\ \sum_{i=1}^r \sum_{j=1}^r h_i h_j \tilde{\Sigma}_{ij}, & \text{if} \ \tau(t) = \tau_{av}, \end{cases}
\]
where
\[
\Sigma_{ij} = \frac{1}{\tau_{av}} \int_{t - \tau_{av}}^{t} \xi^T(t, s) \Phi_1 \xi(t, s) ds + \frac{1}{\tau(t) - \tau_{av}} \int_{t - \tau(t)}^{t - \tau_{av}} \xi^T(t, s) \Phi_2 \xi(t, s) ds, \\
\tilde{\Sigma}_{ij} = \frac{1}{\tau_{av}} \int_{0}^{t} \xi^T(t, s) \tilde{\Phi}_1 \xi(t, s) ds,
\]
\[
\xi^T(t, s) = [x^T(t) \ x^T(t - \tau_{av}) \ w^T(t) \ \dot{x}^T(s)], \\
\Phi_1 = \tilde{\Phi}_1 + Z, \\
\Phi_2 = \begin{bmatrix} 0 & 0 & 0 & -\Delta \tau(t) N_T A_{i1} \\
* & 0 & 0 & -\Delta \tau(t) N_T^2 A_{i1} \\
* & * & 0 & -\Delta \tau(t) N_T^3 A_{i1} \\
* & * & * & -\text{sgn}(\Delta \tau(t)) \cdot \Delta \tau(t) S \end{bmatrix}, \\
\tilde{Z} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & 0 \\
* & Z_{22} & Z_{23} & 0 \\
* & * & Z_{33} & 0 \\
* & * & * & 0 \end{bmatrix}, \\
\Phi_1 = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & -\tau_{av} M_{i1}^T \\
* & \Phi_{22} & \Phi_{23} & -\tau_{av} M_{i2}^T \\
* & * & \Phi_{33} & -\tau_{av} M_{i3}^T \\
* & * & * & -\tau_{av} R \end{bmatrix},
\]
\[
\begin{bmatrix} -Z_{11} & -Z_{12} & -Z_{13} & \delta N_T A_{i1} \\
* & -Z_{22} & -Z_{23} & \delta N_T^2 A_{i1} \\
* & * & -Z_{33} & \delta N_T^3 A_{i1} \\
* & * & * & -\delta S \end{bmatrix} < 0
\]
\[
\begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & -\tau_{av} M_{i1}^T \\
* & \Phi_{22} & \Phi_{23} & -\tau_{av} M_{i2}^T \\
* & * & \Phi_{33} & -\tau_{av} M_{i3}^T \\
* & * & * & -\tau_{av} R \end{bmatrix} < 0.
\]

From (22) we can see that if \( \Omega_1 < 0, \Omega_2 < 0 \) or \( \Omega_1 < 0 \), then \( \dot{V}(t) \leq -\lambda x^T(t)x(t) \) for some scalar \( \lambda > 0 \). Noting that \( |\Delta \tau(t)| \leq \delta, \forall t \), pre- and post-multiplying the both sides of \( \Omega_2 < 0 \) by diag\([I, I, I, \text{sgn}(\Delta \tau(t))I]\), it is easy to see that matrix inequalities
\[
\begin{bmatrix} -Z_{11} & -Z_{12} & -Z_{13} & \delta N_T A_{i1} \\
* & -Z_{22} & -Z_{23} & \delta N_T^2 A_{i1} \\
* & * & -Z_{33} & \delta N_T^3 A_{i1} \\
* & * & * & -\delta S \end{bmatrix} < 0
\]
implies \( \Omega_2 < 0, \forall t \geq 0 \) for \( i = 1, 2, \ldots, r \). From Lemma 1 and (23), the following matrix inequalities imply \( \Omega_1 < 0, \Omega_2 < 0 \) or \( \Omega_1 < 0 \) for \( i, j = 1, 2, \ldots, r \).

Next, assuming that \( \phi(t) = 0, t \in [-\tau_M, 0] \), we consider the performance index (10) of system (13). From (21) we
have

\[ J(w) = \int_0^\infty [\dot{z}^T(\theta) \dot{z}(\theta) - \gamma^2 w^T(\theta) w(\theta)] d\theta \]
\[ \leq \int_0^\infty [\dot{z}^T(\theta) \dot{z}(\theta) - \gamma^2 w^T(\theta) w(\theta) + \dot{V}(x_0)] d\theta \]
\[ = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \tilde{\Sigma}_{ij} \text{ if } \tau(t) \neq \tau_{av}, \]
\[ = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \tilde{\Sigma}_{ij} \text{ if } \tau(t) = \tau_{av}, \quad (25) \]

where

\[ \tilde{\Sigma}_{ij} = \frac{1}{\tau_{av}} \int_0^{\tau_{av}} d\theta \int_0^\theta \xi^T(\theta,s) \Psi_1 \xi(\theta,s) ds \]
\[ + \frac{1}{\tau(t) - \tau_{av}} \int_0^{\tau(t) - \tau_{av}} d\theta \int_\theta^{\tau(t)} \xi^T(\theta,s) \Psi_2 \xi(\theta,s) ds, \]
\[ \Psi_1 = \Phi_1 + \Gamma, \]
\[ \Psi_2 = \Phi_2 + \Gamma, \]
\[ C^T \tilde{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]
\[ \Gamma = \begin{bmatrix} -\gamma I \\ \end{bmatrix}, \]
\[ \tilde{C} = C_i + D_i K_j. \]

From (25) one can see that if \( \Psi_1 < 0, \Phi_2 < 0 \) or \( \Psi_1 < 0 \) simultaneously hold for \( i, j = 1, 2, \ldots, r \), then \( J(w) < 0 \) for all nonzero \( w(t) \in L_2[0,\infty) \). Pre- and post-multiplying both sides of \( \Phi_2 < 0 \) by \( \text{diag}\{I, I, I, I, -\text{sgn}(\Delta(t) I)\} \), it is easy to see that

\[ \begin{bmatrix} -Z_{11} & -Z_{12} & -Z_{13} & -Z_{14} & \delta N_1^T A_{11} \\ * & -Z_{22} & -Z_{23} & -Z_{24} & \delta N_2^T A_{11} \\ * & * & -Z_{33} & -Z_{34} & \delta N_3^T A_{11} \\ * & * & * & -Z_{44} & \delta N_4^T A_{11} \\ * & * & * & * & -\delta S \end{bmatrix} < 0 \quad (26) \]

implies \( \Phi_2 < 0, \forall t \geq 0 \) for \( i = 1, 2, \ldots, r \).

By Lemma 1,

\[ \begin{bmatrix} \Phi_1 + \Gamma (1,2) \\ * -\delta S \end{bmatrix} < 0, \quad (27) \]

\[ (1,2) = \begin{bmatrix} \delta A_1^T N_1 & \delta A_1^T N_2 & \delta A_1^T N_3 & 0 & 0 \end{bmatrix}^T, \]

simultaneously hold for \( i, j = 1, 2, \ldots, r \) imply \( \Psi_1 < 0, \)
\( (26) \) or \( \Psi_1 < 0 \) for \( i = 1, 2, \ldots, r \). In addition, (27) imply (24) for \( i, j = 1, 2, \ldots, r \).

Noting that (27) are not LMIs, one can not solve it directly using MATLAB LMI Toolbox. In order to solve the matrix inequalities (27) efficiently, we define \( N_1 = N_0, N_2 = \varepsilon_2 N_0, \) and \( N_3 = \varepsilon_3 N_0, \) where \( \varepsilon_i > 0 \) \( (i = 2, 3) \). It is clear to see that (27) implies that \( N_0 \) is a nonsingular matrix. Furthermore, defining \( X = N_0^{-1}, Y_j = K_j X \) \((j = 1, 2, \ldots, r) \), \( P = X^T P X, \) \( \tilde{M}_t = X^T M_i X \) \((i = 1, 2, 3) \), \( Q = X^T Q X, \) \( \tilde{R} = X^T R X \) and \( \tilde{S} = X^T S X \) then pre- and post-multiplying both sides of (27) with \( \text{diag}\{X^T \ X^T \ X^T \ X^T \ X^T \ X^T \} \) and its transpose, respectively, we can obtain (15) by Schur complement. This completes the proof.

\textbf{Remark 1:} From the process of proof of Proposition 1, one can see that the system (14) with \( w(t) \equiv 0 \) is asymptotically stable if (24) hold for \( i, j = 1, 2, \ldots, r \).

In [15], Han and Gu studied the stability of the following system with time-varying interval delay using a generalized discretized Lyapunov functional approach.

\[ \dot{x}(t) = A x(t) + A_1 x(t - \tau(t)). \]

But additional information regarding the derivative of the time-varying delay, i.e. \( \dot{\tau}(t) \leq \tau(t), \) is needed. From Proposition 1, one can see that this restriction is removed, which means that a fast time-varying delay is allowed.

Concerning uncertainty appeared in system (9), by Proposition 1, one can easily obtain the following result.

\textbf{Corollary 1:} For a prescribed scalar \( \gamma > 0 \) and some given scalars \( \varepsilon_2 > 0, \varepsilon_3 > 0, \tau_m \) and \( \tau_M \), system (6) is robustly stabilizable under controller (8) and satisfies

\[ \|z(0)\|_2 < \gamma \|w(0)\|_2 \] for all nonzero \( w(t) \in L_2[0,\infty) \), any \( \tau(t) \) satisfying (4) and all admissible uncertainties satisfying (2) and (3), if there exist some scalars \( \chi_i > 0 \) \((i = 1, 2, \ldots, r) \), some matrices \( P > 0, Q > 0, R > 0, S > 0, X, Y_j \) \((j = 1, 2, \ldots, r) \), \( M_i \) \((i = 1, 2, 3) \) of appropriate dimensions such that the following LMIs simultaneously hold for \( i, j = 1, 2, \ldots, r \).

\[ \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2 & \Theta_1 - \chi_i I \end{bmatrix} < 0, \quad (28) \]

where

\[ \Theta_1 = [H_i^T \chi_i, H_i^T \varepsilon_2 \chi_i, H_i^T \varepsilon_3 \chi_i, 0, 0, 0]^T, \]
\[ \Theta_2 = [E_{i0} X + E_{i1} Y_j, E_{i1} X, 0, 0, 0, \varepsilon E_{i1} X, 0]^T. \]

Moreover, the state feedback controller gains of (8) are given by \( K_j = Y_j X^{-1} \) for \( j = 1, 2, \ldots, r \).

\textbf{Proof:} Replacing \( A_0, A_1 \) and \( B_i \) with \( A_0 + H_i F_i(t) E_{i0}, A_{i1} + H_i F_i(t) E_{i1} \) and \( B_i + H_i F_i(t) E_{i1, b} \), respectively, in (15) yields

\[ \dot{\theta}_{i1} = \dot{\theta}_{i2} = 0, \quad (29) \]
\[ \dot{\theta}_{i1} = [H_i^T \varepsilon_2 H_i, \varepsilon_3 H_i, 0, 0, 0, 0]^T, \]
\[ \dot{\theta}_{i2} = [E_{i0} X + E_{i1} Y_j, E_{i1} X, 0, 0, 0, \varepsilon E_{i1} X, 0]^T. \]

It is obvious that (29) is equivalent to

\[ \Theta + \chi_i \dot{\theta}_{i1}, \quad \chi_i = 1, \dot{\theta}_{i2} < 0, \quad (30) \]

for any \( \chi_i > 0 \) \((i = 1, 2, \ldots, r) \). By Schur complement, (30) is equivalent to (28). This completes the proof of this Corollary.

\textbf{IV. A NUMERICAL EXAMPLE}

In this section, we will apply the proposed method to design a delay-dependent robust \( H_\infty \) controller for an uncertain nonlinear delay system. The uncertain nonlinear time-delay system is described as follows

\[ \dot{x}_1(t) = -x_1(t)(2 + \sin^2 x_2(t)) + x_2(t) \]
\[ + 0.1 x_1(t - \tau(t)) + 0.2 x_2(t - \tau(t)) \cos^2 x_2(t) \]
\[ + e(t) + 0.1 x_2(t) \sin^2 x_2(t) + e(t) + c(t) x_1(t) \cos^2 x_2(t) \]
\[ + u_1(t) + (1 + \sin^2 x_2(t)) u(t) \]
\[ \dot{x}_2(t) = x_1(t) - x_2(t)(1 - \cos^2 x_2(t)) \]
\[ + 0.2 x_1(t - \tau(t)) \sin x_2(t) - 0.5 x_2(t - \tau(t)) \]
\[ + 0.5 u_2(t) + 0.1 c(t) x_2(t) \quad (31) \]
where \( c(t) \) is an uncertain parameter satisfying 
\[
\frac{c(t)}{c(t)} \in [-0.2, 0.2].
\tag{32}
\]
If we select the membership functions as
\[
M_1(x_2(t)) = \sin^2(x_2(t)) \quad \text{and} \quad M_2(x_2(t)) = \cos^2(x_2(t)),
\]
then, the nonlinear time-delay system (31) can be represented by the following uncertain time-delay T-S model

**Plant Rule 1:**

IF \( x_2(t) \) is \( M_1 \)

THEN
\[
\begin{align*}
\dot{x}(t) &= [A_{10} + \Delta A_{10}(t)]x(t) + A_{11}x(t - \tau(t)) + B_1 \Delta u(t) + B_{w1}w(t), \\
\dot{z}(t) &= C_1 x(t) + D_1 u(t),
\end{align*}
\]

and

**Plant Rule 2:**

IF \( x_2(t) \) is \( M_2 \)

THEN
\[
\begin{align*}
\dot{x}(t) &= [A_{20} + \Delta A_{20}(t)]x(t) + A_{21}x(t - \tau(t)) + B_2 \Delta u(t) + B_{w2}w(t), \\
\dot{z}(t) &= C_2 x(t) + D_2 u(t),
\end{align*}
\]

where
\[
\tau(t) = 1 + 0.2q(t), \quad \|q(t)\| \leq 1,
\tag{33}
\]

\[
A_{10} = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0.1 & 0 \\ 0.2 & -0.5 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{w1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix}, \quad D_1 = I;
\]

\[
A_{20} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0.1 & 0.2 \\ 0 & -0.5 \end{bmatrix},
\]

\[
B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{w2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix}, \quad D_2 = I,
\]

and \( \Delta A_{10}(t) \) and \( \Delta A_{20}(t) \) can be represented in the from of (2) and (3) with
\[
\begin{align*}
H_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{10} = \begin{bmatrix} 0 & 0.2 \\ 0.2 & 0 \end{bmatrix}, \\
H_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{10} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{align*}
\]

For this example, the \( H_{\infty} \) performance level is chosen as \( \gamma = 1 \). In order to design a robust \( H_{\infty} \) feedback controller (8) for the given T-S model, choosing \( \varepsilon_2 = 0.02, \varepsilon_3 = 0.54 \) and using MATLAB LMI Toolbox to solve the LMIs (28), a desired robust \( H_{\infty} \) fuzzy feedback controller can be constructed as (8) with
\[
K_1 = \begin{bmatrix} -1.1583 & -0.9259 \\ -0.8872 & -0.6348 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.7461 & -0.5782 \\ -0.5130 & -1.1010 \end{bmatrix}.
\tag{34}
\]

It is to say that the given T-S model is robustly stabilizable under controller (8) with controller gain (34) and satisfies \( \|z(t)\|_2, \|w(t)\|_2 \leq 2 \) for all nonzero \( w(t) \in \mathcal{L}_2[0, \infty) \), any time-varying delay \( \tau(t) \) satisfying (33) and all admissible uncertainties satisfying (32). In fact, setting \( \tau_{av} = 1 \), we can obtain the maximum allowed bound \( \delta_{\max} = 0.2358 \) to guarantee the robust \( H_{\infty} \) fuzzy stabilizable of (31) for any \( \tau(t) \in [0.7642, 1.2358] \).

**V. CONCLUSION**

The problem on the delay-dependent robust \( H_{\infty} \) controller design has been studied for a class of Takagi-Sugeno fuzzy-model-based systems with interval time-varying delay and norm-bounded parameter uncertainty. Based on the Lyapunov-Krasovskii Functional approach, a sufficient condition for the existence of the robust \( H_{\infty} \) controller, which robustly stabilizes the T-S fuzzy-model-based uncertain systems and guarantees a prescribed level on disturbance attenuation, has been obtained in an LMI form. No restriction on the derivative of the time-varying delay is needed, which has allowed a fast time-varying delay. The given numerical example has shown effectiveness of the proposed method.

**REFERENCES**


