Nonlinear Stochastic $H_2/H_\infty$ Control With State-Dependent Noise

Weihai Zhang, Jun’e Feng, Bor-Sen Chen and Zhaolin Cheng

Abstract—For a system governed by Itô-type nonlinear stochastic differential equation with state-dependent noise, the $H_2/H_\infty$ control problem is considered, which combines the $H_2$ optimization with the robust $H_\infty$ performance. A cross-coupled Hamilton-Jacobi equations associated with the nonlinear stochastic $H_2/H_\infty$ control are obtained, based on which, sufficient conditions for designing the finite and infinite horizon nonlinear stochastic $H_2/H_\infty$ controllers are derived. Some results on linear stochastic $H_2/H_\infty$ control can be viewed as corollaries of this paper.

I. INTRODUCTION

One of the most important robust control approaches is the so-called $H_\infty$ control, which has made great progresses since the foundation work of [1]. $H_\infty$ control demands that one design a controller to eliminate the external disturbance below a given level, obviously, there may be more than one controller to $H_\infty$ control problem. In practice, we often need a control $u^*$ not only to restrain the exogenous disturbance, but also to minimize a cost function when the worst case disturbance $v$ is implemented, this is the so-called $H_2/H_\infty$ control problem. Up to now, most of the results on $H_\infty$ or mixed $H_2/H_\infty$ control are concentrated on deterministic systems, we refer the reader to [2], [3], [9], [10], [14]-[16] and the references therein.

It is fair to say that stochastic $H_\infty$ and mixed $H_2/H_\infty$ control problems have become attractive research areas in the recent years, we can only mention the following work here. In [4], linear stochastic $H_\infty$ control has been studied, and a stochastic bounded real lemma was also obtained. While [11] was on nonlinear stochastic $H\infty$ control problem, and an Hamilton-Jacobi equation (HJE) associated with nonlinear $H_\infty$ was derived, which can be viewed as an extension of [14] in some sense. A recent paper [8] generalized the mixed $H_2/H_\infty$ consequences of [2] to stochastic counterpart. [7] discussed the output feedback $H_\infty$ control for stochastic uncertain systems. Now, in this present paper, we will continue the work of [8] to nonlinear case, basically follow the line of [9] for the treatment of deterministic nonlinear $H_2/H_\infty$ control.

Weihai Zhang is with Information and Control Research Center, Shenzhen Graduate School of Harbin Institute of Technology, HIT Campus, Shenzhen University Town, Xili, Shenzhen 518055, P.R. China. Also with College of Information and Electrical Engineering, Shandong University of Science and Technology, Qingdao 266510, P.R. China. Email: wzhzhang@163.com

Jun’e Feng is with School of Mathematics and System Sciences, Shandong University, Jinan 250100, P.R. China. Email: fengjun@sdu.edu.cn

Bor-Sen Chen is with Department of Electrical Engineering National Tsing Hua University, Hsin Chu 30043, Taiwan. Email: bschen@moti.ee.nthu.edu.tw

Zhaolin Cheng is with School of Mathematics and System Sciences, Shandong University, Jinan 250100, P.R. China

Concretely speaking, the main contribution of this paper is as follows: A sufficient condition for finite/infinite horizon mixed $H_2/H_\infty$ control is given via the coupled differential/algebraic HJE’s, respectively, which are nonlinear second-order partial differential equations. Some further deserved study problems are also presented. This paper also extends the results of [9] to stochastic systems.

For convenience, we adopt the following notations.

$A'$ the transpose of the corresponding matrix $A$;

$I$: the identity matrix;

$L_{f,g}^2V(x) := \frac{\partial V(x)}{\partial x} f(x);
L_{g,h}^2([0,T],\mathbb{R}^l) := \{ l(t) \in \mathbb{R}^l \text{ with respect to an increasing } \sigma\text{-algebra } \mathcal{F}_t \text{ satisfying } E \int_0^\infty \|y(t)\|^2 dt < \infty \}$.

II. FINITE HORIZON $H_2/H_\infty$ CONTROL

Consider the following stochastic nonlinear system governed by Itô-type differential equation

$$dx = (f(x) + g(x)u + h(x)v) dt + l(x)dW$$

$$f(0) = l(0) = 0$$

with controlled output

$$z = \begin{bmatrix} C(x) \\ u \end{bmatrix}$$

where $x(t) \in \mathbb{R}^n$ is called the system state, $z(t) \in \mathbb{R}^m$ is the penalty output, $u(t)$ and $v \in L_2^2(\mathbb{R}_+,\mathbb{R}^{n_x})$ stand for the control and exogenous disturbance signal, respectively. $f,g,h,l$ and $C$ are smooth functions with suitable dimensions. $W(\cdot)$ is a one-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ relative to an increasing family $(\mathcal{F}_t)_{t \geq 0}$ of $\sigma$-algebras $\mathcal{F}_t \subset \mathcal{F}$.

Now, we first define the finite horizon nonlinear stochastic $H_2/H_\infty$ control as follows:

Definition 1 (Finite horizon nonlinear stochastic $H_2/H_\infty$ control): Find, if possible, a state feedback control law $u = u^*(t,x)$ such that

(i) For any given $\gamma > 0$, $T > 0$, $v \in L_2^2([0,T],\mathbb{R}^{n_x})$, the trajectory of the closed-loop system (1) starting from $x(0) = x_0 = 0$ satisfies

$$E \int_0^T \|C(x)\|^2 + \|u^*\|^2 dt \leq \gamma E \int_0^T \|v\|^2 dt \quad (3)$$

(ii) When the worst case disturbance $v^*$ is implemented in (1), $u^*$ minimizes the quadratic performance

$$J_2^2(u^*, v^*) = \min_{u \in L_2^2([0,T],\mathbb{R}^{n_x})} J_2^2(u, v^*) = \min_{u \in L_2^2([0,T],\mathbb{R}^{n_x})} E \int_0^T \|C(x)\|^2 + \|u\|^2 dt.$$
If we define
\[ J_1^T(u, v) := E \int_0^T (\gamma^2 \|v\|^2 - \|z\|^2) \, dt \]
and
\[ J_2^T(u, v) := E \int_0^T \|z\|^2 \, dt \]
then it can be seen that the mixed \( H_2/H_\infty \) control problem is equivalent to finding the Nash equilibria \((u^*, v^*)\) defined as
\[ J_1^T(u^*, v^*) \leq J_1^T(u^*, v), \quad \forall v \in L_2^2([0, T], \mathbb{R}^m) \]  
(4)
\[ J_2^T(u^*, v^*) \leq J_2^T(u, v^*), \quad \forall u \in L_2^2([0, T], \mathbb{R}^n). \]  
(5)
The first Nash inequality is associated with the \( H_\infty \) performance, since \( J_1^T(u^*, v^*) \geq 0 \) implies (3), while the second one is related with the \( H_2 \) performance. Clearly, if the Nash equilibria \((u^*, v^*)\) exist, \( u^* \) is our desired \( H_2/H_\infty \) controller, and \( v^* \) is the worst case disturbance. In this case, we also say that nonlinear stochastic \( H_2/H_\infty \) control admits a pair of solutions \((u^*, v^*)\). The following theorem is a sufficient condition for the existence of a finite horizon \( H_2/H_\infty \) controller.

**Theorem 1:** Suppose there exist a non-positive definite function \( V_1 \in C^{1,2}([0, T], \mathbb{R}^n) \), \( V_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) with \( V_1(0,0) = 0 \), and a non-negative definite function \( V_2 \in C^{1,2}([0, T], \mathbb{R}^n) \), \( V_2 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) with \( V_2(0,0) = 0 \), such that they solve a pair of cross-coupled HJEs
\[ \mathcal{L}_{u=v^*} V_1(t,x) - \|C(x)\|^2 - \gamma^2 \|v^*\|^2 - \|u^*\|^2 = 0, \quad V_1(T,x(T)) = 0 \]  
(6)
\[ \mathcal{L}_{u=u^*,v=v^*} V_2(t,x) + \|C(x)\|^2 + \|u^*\|^2 = 0, \quad V_2(T,x(T)) = 0 \]  
(7)
with
\[ u^*(t,x) = -\frac{1}{2} \gamma g(x) V_2(t,x), \]  
(8)
\[ v^*(t,x) = -\frac{1}{2\gamma^2} h(x) V_1(t,x) \]  
(9)
and \( \mathcal{L}_{u,v} \) being the infinitesimal operator of (1), then the mixed \( H_2/H_\infty \) control problem admits a pair of solutions \((u^*, v^*)\). Moreover,
\[ J_2^T(u^*, v^*) = V_2(0, x_0). \]
**Proof:** We follow the line of [8] and [9]. By use of the completion technique of square argument and (6), it follows
\[ J_2^T(u, v^*) = V_1(0, x_0) - EV_1(T,x(T)) + E \int_0^T (\gamma^2 \|v\|^2 - \|z\|^2) \, dt + dV_1(t,x(t)) = V_1(0, x_0) + E \int_0^T (\gamma^2 \|v\|^2 - \|z\|^2) \, dt + \mathcal{L}_{u,v} V_1(t,x(t)) \, dt = V_1(0, x_0) + E \int_0^T (\gamma^2 \|v\|^2 - \|u\|^2 + \|u^*\|^2 + (\gamma^2 - 2) \|v^*\|^2) \, dt + L_g(x) V_1(t,x) \, dt \]
\[ = V_1(0, x_0) + E \int_0^T (\gamma^2 \|v\|^2 - \|u\|^2 + \|u^*\|^2 + (\gamma^2 - 2) \|v^*\|^2) \, dt + L_h(x) V_1(t,x) \, dt \]
\[ \geq J_2^T(u^*, v^*) = V_2(0, x_0) \]
which verifies the requirement (ii), the proof of Theorem 1 is complete.

For linear system
\[
\begin{align*}
\begin{cases}
dx(t) = (A(t)x(t) + B_2(t)u(t) + B_1(t)v(t)) \, dt + A_1(t)x(t) \, dw_1 \\
x(0) = x_0 \\
z(t) = \begin{bmatrix} C(t)x(t) \\ D(t)u(t) \end{bmatrix}, \quad D'(t)D(t) = I
\end{cases}
\end{align*}
\]  
(10)
if we take \( V_1(t, x) = x'P_1(t)x, V_2(t, x) = x'P_2(t)x \) with \( P_1 \leq 0, P_2 \geq 0 \), then Theorem 1 yields the sufficiency part of Theorem 5 of [8]. In particular, the coupled HJEs (6) and (7) down to a pair of coupled Riccati equations

\[
\begin{align*}
\dot{P}_1 &= A'P_1 + P_1A + A'_1P_1A_1 - C'C \\
\dot{P}_2 &= A'P_2 + P_2A + A'_1P_2A_1 + C'C
\end{align*}
\]

\( P_1(T) = 0 \) \( P_2(T) = 0 \).

(11) (12)

III. A UNIFIED TREATMENT FOR \( H_2, H_\infty \) AND MIXED \( H_2/H_\infty \) CONTROL

As done in [2] and [9], under the framework of a nonzero sum, two player Nash differential game, we can give a unified treatment for \( H_2, H_\infty \) and mixed \( H_2/H_\infty \) control problems. Consider system (1) with the penalty output (2), associated with the following two performance

\[
\begin{align*}
J_f^T(u, v) &= E \int_0^T \langle \gamma^2 \|v\|^2 - \|z\|^2 \rangle \, dt \\
J_g^T(u, v) &= E \int_0^T \langle \|z\|^2 - \rho \|v\|^2 \rangle \, dt
\end{align*}
\]

Similar to the discussion of Theorem 1, it can be shown that if the following cross-coupled HJEs

\[
\begin{align*}
\mathcal{L}_{u=0, v=0} \tilde{V}_1(t, x) - \|C(x)\|^2 - \gamma^2 \|u^*\|^2 - \|u^*\|^2 &= 0, \\
\tilde{V}_1(T, x(T)) &= 0 \\
\mathcal{L}_{u=u^*, v=v^*} \tilde{V}_2(t, x) + \|C(x)\|^2 + \|u^*\|^2 - \rho\|v^*\|^2 &= 0, \\
\tilde{V}_2(T, x(T)) &= 0
\end{align*}
\]

admit solutions \( \tilde{V}_1, \tilde{V}_2 \in C^{1,2}([0, T], \mathbb{R}^n), \tilde{V}_1 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \tilde{V}_2 : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \tilde{V}_1(0, 0) = \tilde{V}_2(0, 0) = 0 \), then \((u^*, v^*)\) is the so-called Nash equilibrium point, which satisfies (4) and (5), where

\[
\begin{align*}
u^*(t, x) &= -\frac{1}{2} \mathcal{L}_g \tilde{V}_2(t, x) \\
v^*(t, x) &= \frac{1}{2 \gamma^2} \mathcal{L}_h \tilde{V}_1(t, x)
\end{align*}
\]

i) The nonlinear quadratic optimal control problem

\[
\min_{u \in \mathcal{L}_2^2([0, T], \mathbb{R}^n)} \{ \mathcal{J}_2^T(u, 0) = E \int_0^T \|z\|^2 \, dt \}
\]

subject to

\[
\begin{align*}
dx = (f(x) + g(x)u) \, dt + l(x) \, dW \\
x(0) = x_0 \in \mathbb{R}^n \end{align*}
\]

can be solved by setting \( \rho = 0, \gamma \rightarrow \infty \). It can be shown that the solutions \( \tilde{V}_1 \) and \( \tilde{V}_2 \) of the coupled HJEs (13) and (14) are \( \tilde{V}_1(t, x) \rightarrow -\tilde{V}(t, x) \) and \( \tilde{V}_2(t, x) \rightarrow \tilde{V}(t, x) \) respectively, where \( \tilde{V}(t, x) \) solves the following HJE

\[
\mathcal{L}_{u=u^*, v=0} \tilde{V}(t, x) + \|C(x)\|^2 + \|u^*\|^2 = 0, \quad \tilde{V}(T, x(T)) = 0
\]

with \( u^*(t, x) = -\frac{1}{2} \mathcal{L}_g \tilde{V}(t, x) \), or equivalently,

\[
\begin{align*}
\mathcal{L}_{u=u^*, v=0} \tilde{V}(t, x) + \|C(x)\|^2 \\
+ \frac{1}{4} \mathcal{L}_h \tilde{V}(t, x) = 0
\end{align*}
\]

Moreover,

\[
\min_{u \in \mathcal{L}_2^2([0, T], \mathbb{R}^n)} \mathcal{J}_2^T(u, 0) = \mathcal{J}_2^T(u^*, 0) = \tilde{V}(0, x_0)
\]

ii) If we set \( \rho = \gamma \), then \( \tilde{V}_\infty(t, x) = \tilde{V}_2(t, x) = -\tilde{V}_1(t, x) \), where \( \tilde{V}_\infty \) is a solution to HJE

\[
\begin{align*}
\mathcal{L}_{u=u^*, v=0} \tilde{V}_\infty(t, x) + \|C(x)\|^2 + \|u^*\|^2 + \gamma^2 \|v^*\|^2 &= 0, \\
\tilde{V}_\infty(T, x(T)) &= 0
\end{align*}
\]

or

\[
\begin{align*}
\mathcal{L}_{u=u^*, v=0} \tilde{V}_\infty(t, x) + \|C(x)\|^2 \\
+ \frac{1}{2 \gamma^2} \mathcal{L}_h \tilde{V}_\infty(t, x) = 0
\end{align*}
\]

with

\[
u^*(t, x) = -\frac{1}{2} \mathcal{L}_g \tilde{V}_\infty(t, x), \quad \mathcal{J}_2^T(u^*, 0) = \frac{1}{2 \gamma^2} \mathcal{L}_h \tilde{V}_\infty(t, x)
\]

More specifically, the above \( u^*(t, x) \) is our desired \( H_\infty \) control law, which makes

\[
E \int_0^T \|z(t)\|^2 \, dt \leq \gamma^2 E \int_0^T \|v(t)\|^2 \, dt
\]

hold for any nonzero \( v \in \mathcal{L}_2^2([0, T], \mathbb{R}^n) \).

iii) By taking \( \rho = 0 \), the mixed \( H_2/H_\infty \) control is retrieved. In this case, \( \tilde{V}_1 = \tilde{V}_1 \), \( \tilde{V}_2 = \tilde{V}_2 \).

Remark 1: It can be seen that all results obtained in this section still hold for the time-varying stochastic system

\[
\begin{align*}
dx = (f(t, x) + g(t, x)u + h(t, x)v) \, dt + l(t, x) \, dW \\
n(0, t) = n(0, t) = 0, \quad \forall t \geq 0
\end{align*}
\]

with penalty output

\[
z = \begin{bmatrix} C(t, x) \\ u \end{bmatrix}
\]

Remark 2: A more general HJE than (15) was derived in [12], while (17) is also a special case of the corresponding one of [11].
IV. INFINITE HORIZON $H_2/H_\infty$ CONTROL

To discuss the infinite horizon nonlinear stochastic $H_2/H_\infty$ control problem, the internal stability requirement is needed, so we should introduce the following definition on stochastic stability.

Definition 2 [5]: Consider the following uncontrolled stochastic system
\[ dx = f(x)dt + l(x)dW, \quad x(0) = x_0, \quad f(0) = l(0) = 0, \] (18)

1) $x \equiv 0$ of (18) is said to be stable in probability if for any $\epsilon > 0$
\[ \lim_{x_0 \to 0} P(\sup_{t \geq 0} |x| > \epsilon) = 0. \] (19)

2) $x \equiv 0$ of (18) is said to be locally asymptotically stable in probability if (19) holds and there exists a neighborhood $U_0$ of the origin, such that
\[ P(\lim_{t \to \infty} |x(t)| = 0, \forall x_0 \in U_0) = 1 \] (20)

Remark 3: In the previous references, we can find another definition form on locally asymptotic stability (e.g. [17]), which said that $x \equiv 0$ of (18) is locally asymptotically stable in probability if (19) holds and
\[ \lim_{x_0 \to 0} P(\lim_{t \to \infty} |x(t)| = 0) = 1 \]

Here, we adopt Definition 2 in order to be consistent with the deterministic one [9]. The following lemma is well known for stability in probability.

Lemma 1: If there exists a neighborhood $U_0$ of 0, a Lyapunov function $V(x) \in C^2(U)$, $V(x) > 0$ for $x \neq 0$ in the domain $U_0$, such that
\[ \mathcal{L}_{u,v=0}V(x) = \frac{\partial V(x)}{\partial x}f(x) + \frac{1}{2}\frac{\partial^2 V(x)}{\partial x^2}l(x) \leq 0 \] (21)

for $x \neq 0$, then $x \equiv 0$ of system (18) is stable in probability.

Below, we state the infinite horizon nonlinear stochastic $H_2/H_\infty$ control as follows.

Definition 3 (Infinite horizon nonlinear stochastic $H_2/H_\infty$ control): Find, if possible, a static state feedback control law $u = u^*(x) \in \mathcal{L}_2^2(\mathbb{R}^+, \mathbb{R}^{n_u})$ such that

(i) For any given $\gamma > 0$ and any nonzero $v \in \mathcal{L}_2^2(\mathbb{R}^+, \mathbb{R}^{n_v})$, the trajectory
\[ dx = (f(x) + g(x)u^*(x) + h(x)v)^dt + l(x)dW \] (22)

starting from $x_0 = 0$ satisfies
\[ E\int_0^\infty (||C(x)||^2 + ||u^*(x)||^2)dt \leq \gamma^2 E\int_0^\infty ||v||^2dt \] (23)

(ii) When the worst case disturbance $v^*$ is implemented in (1), $u^*$ minimizes the quadratic performance
\[ J_2^\infty(u^*, v^*) = \min_{u \in \mathcal{L}_2^2(\mathbb{R}^+, \mathbb{R}^{n_u})} \mathcal{J}_2^\infty(u) \]
\[ = \min_{u \in \mathcal{L}_2^2(\mathbb{R}^+, \mathbb{R}^{n_u})} E\int_0^\infty (||C(x)||^2 + ||u^*(x)||^2)dt \]

where $\mathcal{L}_2^2$ consists of all measurable, adaptive process $u(x)$ (with respect to $\mathcal{F}_t$), which makes the following trajectory
\[ dx = (f(x) + g(x)u + h(x)v^*)dt + l(x)dW \] (24)

\[ \text{to be locally asymptotically stable in probability.} \]

(iii) The system
\[ dx = (f(x) + g(x)u^*(x))dt + l(x)dW \] (25)

is locally asymptotically stable in probability.

If we define
\[ J_1^\infty(u, v) := \int_0^\infty (\gamma^2||v||^2 - ||z||^2)dt \]
and
\[ J_2^\infty(u, v) := \int_0^\infty ||z||^2dt \]
then the nonlinear stochastic $H_2/H_\infty$ control problem can be converted into solving the following two persons, nonzero sum Nash game associated with the $H_\infty$ and $H_2$ performance:

\[ J_1^\infty(u^*, v^*) \leq J_2^\infty(u, v), \forall v \in \mathcal{L}_2^2(\mathbb{R}^+, \mathbb{R}^{n_v}) \]
\[ J_2^\infty(u^*, v^*) \leq J_2^\infty(u, v^*), \forall u \in \mathcal{L}_2^2(\mathbb{R}^+, \mathbb{R}^{n_u}) \]

The following definition generalizes the zero-state detectability to the stochastic system
\[ \left\{ \begin{array}{l}
  dx = f(x)dt + l(x)dW, \\
  y = C(x).
\end{array} \right. \] (26)

Definition 4: System (26) is said to be locally zero-state detectable, if there exists a neighborhood $U_0$ of 0, that for all $x_0 \in U_0$, we have
\[ y(t) \equiv 0, \quad \forall t \geq 0 \Rightarrow P(\lim_{t \to \infty} x(t) = 0, x(0) = x_0) = 1. \] (27)

If $U_0 = \mathbb{R}^n$, then (26) is called zero-state detectable. In the sequel, when (26) is locally zero-state detectable (zero-state detectable), we also call $[f, l, C]$ locally zero-state detectable (zero-state detectable).

Theorem 2: Suppose the following assumptions hold.
1) $[f, l, C]$ is locally zero-state detectable.
2) there exists a locally negative definite function $V_1 : \Omega_0 \mapsto \mathbb{R}^-$, defined on a neighborhood $\Omega_0$ of the origin, and a locally positive definite function $V_2 : \Omega_0 \mapsto \mathbb{R}^+$, such that they satisfy a pair of cross-coupled HJE as follows:
\[ \mathcal{L}_{u=0, v=0}V_1(x) - ||C(x)||^2 - \gamma^2||u^*||^2 - ||u^*||^2 = 0 \] (28)
\[ \mathcal{L}_{u=0, v=0}V_2(x) + ||C(x)||^2 + ||u^*||^2 = 0 \] (29)

where $u^*$ and $v^*$ take the same form as in (8) and (9), respectively.

3) the pair $(f(x) + h(x)v^*, l(x)|C(x))$ is locally zero-state detectable.

Then the state feedback control law (8) and (9) solve the infinite time horizon $H_2/H_\infty$ control problem.

Proof: We first show (iii) of Definition 3 holds. For system (25) with $u^*$ given by (8), we apply Itô’s formula to Lyapunov function $-V_1(x)$, and consider equation (28), it concludes
\[ \mathcal{L}_{u=u^*, v=0}(-V_1(x)) = -||C(x)||^2 - \gamma^2||u^*||^2 - ||u^*||^2 \leq 0 \]

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which concludes

\[
J_T^1(u^*, v^*) = V_1(x_0) - EV_1(x(T)) + E \int_0^T \gamma^2\|v - v^*\|^2 dt \\
\geq V_1(x_0) - EV_1(x(T)) = J_T^1(u^*, v^*).
\]

Especially, for \( x_0 = 0 \), \( J_T^1(u^*, v^*) = -EV_1(x(T)) \geq 0 \), which follows

\[
E \int_0^\infty \|z\|^2 dt \leq \gamma^2 E \int_0^\infty \|v\|^2 dt, \\
x_0 = 0, \forall v \in L_r^2(\mathbb{R}^+, \mathbb{R}^n_v).
\]
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