**Abstract**—This note presents the stability analysis for missile guidance systems with linear time-invariant autopilot systems. The concept of \( L_2 \) stability is employed in order to analyze missile systems. The well-known PN guidance law is considered as a missile guidance law and the nonlinear planar pursuit situation is fully considered in the analysis. It is shown first that the missile systems can be \( L_2 \) stable for target acceleration inputs which belong to \( L_2 \) space. A sufficient condition for a missile autopilot transfer function to guarantee \( L_2 \) stability of the missile systems is also provided as a function of total miss distance, which offers a criterion for missile autopilot design.

I. INTRODUCTION

In this paper, we attempt to analyze missile systems considering autopilot dynamics and nonlinear dynamics of pursuit geometry.

In fact, missile subsystems such as autopilot and guidance systems are usually designed separately. When they are combined, the performance analysis is very difficult because of their own nonlinearity and coupling effects. Thus, the analysis of whole missile systems still poses severe challenges to control engineers [1]-[4].

In case of planar pursuit situation and linear autopilot, the performance of PN guidance systems were analyzed under the assumptions for the linearization of nonlinear pursuit dynamics [5]. In that analysis, the target acceleration was assumed to be zero and the dynamics including linear autopilot and nonlinear planar pursuit dynamics were linearized and combined under the assumptions that the closing velocity is constant.

In this note, we propose an \( L_2 \)-space approach to performance analysis of missile systems. In the analysis, target acceleration functions are assumed to belong to \( L_2 \) space and nonlinear planar pursuit dynamics are fully considered. This note presents a sufficient condition that guarantees \( L_2 \) stability of missile guidance systems with linear autopilot and objective performances such as a miss distance. This sufficient condition is a guideline to the design of linear autopilot which guarantees the objective miss distance.

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II. PRELIMINARIES

In this note, a planar pursuit situation is considered as in [2] and [5]. The planar pursuit equation is described as follows.

\[
\frac{d}{dt}(r\lambda) = -\dot{r}\lambda - a_M\cos(\theta_M) + a_T\cos(\theta_T)
\]

\[
\dot{r} = -V_M\cos(\theta_M) + V_T\cos(\theta_T)
\]

\[
r\dot{\lambda} = -V_M\sin(\theta_M) + V_T\sin(\theta_T)
\]

where \( r, \dot{r} \) are the relative distance between target and missile, its derivative, \( \dot{\lambda} \) is Line of sight (LOS) angular rate, \( a_M, a_T \) are missile and target acceleration, \( V_M, V_T \) are missile and target velocity, \( \theta_M \) is the angle between missile velocity vector and LOS, and \( \theta_T \) is the angle between target velocity vector and LOS.

The missile guidance system with linear autopilot can be written by the following equations similar to [5].

\[
\dot{x} = Ax + bu, u = NV_M\dot{\lambda}
\]

\[
a_M = cx + du
\]

where \( A \) is stable, \((A, b)\) is controllable, \((c, A)\) is observable, \( V_M \) is a missile velocity, and \( N \) is a navigation constant.

Using the result of lemma 2 in [2], the following lemma can be easily obtained.

**Lemma 1**: The following assumptions are satisfied.

(A2.1) \( 0 \leq \rho < 1 \).

(A2.2) The target acceleration \( a_T \) is a function in \( L_2 \) space and For some \( \alpha > 0 \),

\[
\alpha_T(t) \leq \alpha, \forall t \geq 0.
\]

(A2.3) \( |\theta_m(t)| \leq \frac{\pi}{2} \forall t \geq 0 \).

(A2.4) The navigation constant \( N \) satisfies the following inequality for \( \beta \leq 1 - \rho \).

\[
N > 1 + \frac{\rho + \alpha}{\beta V_M} \frac{(\beta|V_M|^2)}{\sqrt{1 - (\rho + \beta)^2}}.
\]

Then, the following inequalities holds.

\[
\frac{d}{dt}|r\lambda| \leq -\frac{\alpha}{\beta V_M}|r\dot{\lambda}| + \{|a_T| + |a_M - NV_M\dot{\lambda}|, \forall t \geq 0.
\]
The inequality (6) implies by the triangle inequality that
\[
\|r\dot{\lambda}\|_2 \leq \frac{\beta V_M}{\alpha}\{\|a_T\|_2 + \|a_M - NV_M\dot{\lambda}\|_2\}. \tag{7}
\]
Thus, if we redefine the equation (1) by
\[
\frac{d}{dt}(r\lambda) = -\{\dot{r} + NV_M\sin(\theta_M)\}\lambda + v(t)
\]
\[
\dot{r} = -V_M\cos(\theta_M) + V_T\cos(\theta_T)\tag{8}
\]
and the input to this dynamics is defined as \(v(t) \triangleq a_T\cos(\theta_T) - \{a_M - NV_M\dot{\lambda}\}\cos(\theta_M)\), then the system defined by (8) is \(L_2\) stable.

**Lemma 2:** Suppose that the assumptions in Lemma 1 are satisfied. Then, the system described by (8) with an input \(v(t)\) and an output \(y \triangleq r\lambda\) is \(L_2\) stable. That is,
\[
\|r\dot{\lambda}\|_2 \leq \gamma_1\{\|a_T\|_2 + \|a_M - NV_M\dot{\lambda}\|_2\} \tag{9}
\]
where \(\gamma_1 \triangleq \frac{\beta V_M}{\alpha}\). \(\square\)

### III. Main Results

The systems considered in this note can be considered as an interconnected systems composed of the following subsystems.

\[
\Sigma_1:\begin{align*}
\dot{x}_1 &= -\frac{\dot{r}}{r}x_1 + bu_1 \\
y_1 &= x_1 \tag{10}
\end{align*}
\]
\[
\Sigma_2:\begin{align*}
\dot{x}_2 &= Ax_2 + bu_2 \\
y_2 &= cx_2 + (d-1)u_2 \tag{11}
\end{align*}
\]

where
\[
\begin{align*}
u_1 &\triangleq a_T\cos(\theta_T) - y_2\cos(\theta_M), y_1 \triangleq r\dot{\lambda} \\
u_2 &\triangleq NV_M\dot{\lambda} - \frac{\dot{r}}{r}y_1 \tag{12}
\end{align*}
\]
Define the transfer function of the linear autopilot by
\[
G(s) \triangleq c(sI - A)^{-1}b + d. \tag{13}
\]
Since the system matrix \(A\) is stable, note that the system \(\Sigma_2\) is \(L_2\) stable for any \(u_2 \in L_2\). Thus, the following inequality holds from the well-known input-output properties[6].
\[
\|y_2\|_2 \leq \gamma_2\|u_2\|_2 \tag{14}
\]
where \(\gamma_2 \triangleq \sup\{|G(jw) - 1| : w \in [0,\infty)\}\).

In order to prove the main result, we need the following lemma.

**Lemma 3:** Assume that (A2.1)-(A2.4) in Lemma 1 are satisfied. If
\[
|r\dot{\lambda}| \leq \beta V_M, \forall t > t_0 \tag{15}
\]
then
\[
\dot{r} < -V_M\sqrt{(1-\rho)^2 - \beta^2}, \forall t > t_0. \tag{16}
\]
Note that Lemma 3 can be obtained from Lemma 1 in [2]. Now, the main result is given in Theorem 1.

**Theorem 1:** Suppose that the assumptions in Lemma 1 are satisfied and the transfer function \(G(s)\) of linear autopilot satisfies the following inequality for some \(r_M > 0\).
\[
\gamma_1\gamma_2\frac{NV_M}{r_M} < 1 \tag{17}
\]
Then, for any \(a_T \in L_2\), the missile system is \(L_2\) stable until the final miss distance is achieved and the final miss distance \(r_m \triangleq \min(r(t))\) is less than \(r_M\). \(\square\)

**Proof:** We prove by contradiction. Suppose \(r(t) > r_M, \forall t\). From (14) and the definition of \(u_2\), it follows that
\[
\|y_2\|_2 \leq \gamma_2\frac{NV_M}{r_M}\|a_T\|_2 \tag{19}
\]
Thus, it follows that there exists a \(T > 0\) such that for a given positive constant \(\epsilon < \beta V_M\)
\[
|r\dot{\lambda}| < \epsilon, \forall t > T. \tag{20}
\]
From Lemma 3, it follows that
\[
\dot{r} < -V_M\{(1-\rho)^2 - \beta^2\}^{1/2}, \forall t > T \tag{21}
\]
Therefore, there exists a \(T_1 > T\) such that \(r(T_1) \leq r_M\). This contradicts to the assumption \(r_m > r_M\). \(\Box\)

### IV. Conclusions

In this note, stability and performance analysis of missile systems has been considered in terms of \(L_2\) stability. This approach facilitates the performance analysis of missile systems composed of complicate subsystems. A sufficient condition was provided which could offer a criterion for missile systems to guarantee a guidance performance such as miss distance. The analysis has taken into account the exact nonlinear pursuit geometry without any assumptions for linearization. This approach can be easily extended to 3-dimensional pursuit situations.

**References**