Spectral conditions for positive realness of single-input single-output LTI systems

Robert Shorten
The Hamilton Institute
NUI Maynooth
Ireland
email:robert.shorten@may.ie

Christopher King
Department of Mathematics
Northeastern University
Boston, MA 02115 USA
email: king@neu.edu

Abstract—In this note we derive necessary and sufficient conditions for a SISO system to be (strictly) positive real.

This paper is dedicated to the memory of Professor John T. Lewis

I. INTRODUCTION

In this note we consider the problem of determining whether the transfer function $H(j\omega)$ associated with the linear time invariant (LTI) system

$$
\Sigma : \dot{x} = Ax + bu \\
y = c^T x + du
$$

is positive real, where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{1 \times n}$, $d \in \mathbb{R}$, where $x \in \mathbb{R}^{n \times 1}$, $u, y \in \mathbb{R}$, and where $H(j\omega)$ is given by

$$
H(j\omega) = d^T (j\omega I - A)^{-1} b.
$$

Recently, several papers have appeared that give compact conditions to test whether a given transfer function is (strictly) positive real [1], [2], [3], [4]. In this paper we show that (strict) positive realness can be easily determined from: (i) the spectrum of the matrix $(A - \frac{1}{2} bc^T)A$ when $d \neq 0$; and (ii) the spectrum of the matrix $A(I - \frac{1}{c^T A^p} A^p bc^T)A$ for some odd integer $p$ when $d = 0$.

II. DEFINITIONS

Let $A$ be a real $n \times n$ matrix, and suppose the transfer function $H(s) = d + c^T (sI - A)^{-1} b$ has poles and zeros that lie in the closed left half of the complex plane. Any poles on the imaginary axis are assumed to be simple. It follows that $H(s)$ is real for all real $s$, and that $H(s)$ is analytic in $Re(s) > 0$. Then $H(s)$ is said to be positive real (strictly positive real) if the following conditions are satisfied [5], [6].

Definition 2.1: (i) $Re(H(j\omega)) \geq 0$ for all $\omega \in \mathbb{R}$ (excluding any poles on the imaginary axis); and (ii) all residues of $H(s)$ at poles on the imaginary axis are positive.

Definition 2.2: Define $H_\epsilon(s) = H(s - \epsilon)$. Then $H(s)$ is strictly positive real (SPR) if $H_\epsilon(s)$ is PR for some $\epsilon > 0$.

Comment: Note that Definition 2.1 implies that if $H(s)$ is PR then $Re(H(s)) > 0$ whenever $Re(s) > 0$. Also Definition 2.2 implies that if $H(s)$ is SPR then $H(s)$ is a stable transfer function (the matrix $A$ has all of its eigenvalues in the open left half of the complex plane and is said to be stable). Furthermore $Re(H(j\omega))$ cannot decay more rapidly than $\omega^{-2}$ as $|\omega| \to \infty$ [5].

III. MAIN RESULTS

We state results separately for the cases $d > 0$ and $d = 0$ as they require different conditions.

Theorem 3.1: Consider the transfer function $H(s) = d + c^T (sI - A)^{-1} b$ with $d > 0$. $H(s)$ is strictly positive real (SPR) if and only if (i) $A$ is stable and (ii) the matrix $(A - \frac{1}{2} bc^T)A$ has no eigenvalues on the closed negative real axis $(-\infty, 0]$. $H(s)$ is positive real (PR) if and only if (i) the matrix $(A - \frac{1}{2} bc^T)A$ has no eigenvalue of odd (algebraic) multiplicity on the open negative real axis $(\infty, 0)$, and (ii) all residues of $H(s)$ at poles on the imaginary axis are positive.

Theorem 3.2: Consider the transfer function $H(s) = c^T (sI - A)^{-1} b$. $H(s)$ is SPR if and only if: (i) $c^T Ab < 0$; (ii) $c^T A^{-1} b < 0$; (iii) $A$ is stable; and (iv) $A(I - \frac{1}{c^T A^p} A^p bc^T)A$ has no eigenvalues of odd (algebraic) multiplicity on the open negative real axis $(\infty, 0)$; and (iii) all residues of $H(s)$ at poles on the imaginary axis are positive.

Comment: The definition of strict positive realness given in [5], [6], [3] (Definition 2.2) is motivated by the desire that any proper system which is SPR should also satisfy the Kalman-Yacubovic-Popov lemma. Some authors relax this requirement and use the following definition of a strict positive real transfer function:

$H(s)$ is strictly positive real (SPR) if $Re(H(j\omega)) > 0$ for all $\omega \in \mathbb{R}$.

We briefly note that our conditions can be easily extended to account for this definition of strict positive realness. In particular, this definition allows $A$ to have eigenvalues on the imaginary axis and $H(s)$ is SPR if and only if

$[\text{when } d > 0 :]$ (i) $H(s)$ is positive real, (ii) $d + c^T A^{-1} b > 0$, and (iii) $A - \frac{1}{c^T A^p} A^p bc^T A$ has no nonzero eigenvalue $-\omega^2$ with (algebraic) multiplicity $m$, then $m = 2$ and $-\omega^2$ is also an eigenvalue of $A^2$ with (algebraic) multiplicity 2.

$[\text{when } d = 0 :]$ (i) $H(s)$ is positive real, (ii) $c^T A^{-1} b < 0$, and (iii) if $A(I - \frac{1}{c^T A^p} A^p bc^T)A$ has a non-zero eigenvalue $-\omega^2$ with (algebraic) multiplicity $m$, then $m = 2$ and $-\omega^2$ is also an eigenvalue of $A^2$ with (algebraic) multiplicity 2.

Proof of Theorem 3.1:
The proof is based on the determinant representation of the transfer function that was first derived in [7]. Assume that \( j \omega \) is not in the spectrum of \( A \) so that \((j\omega I - A)^{-1}\) is well-defined. Using the identity

\[
(j\omega I - A)^{-1} + (-j\omega I - A)^{-1} = -2A(\omega^2 I + A^2)^{-1}
\]

we can rewrite \( \text{Re}(H) \) as

\[
\text{Re}\{H(j\omega)\} = d - c^T A(\omega^2 I + A^2)^{-1}b = d \left[ 1 - \frac{1}{d} c^T A(\omega^2 I + A^2)^{-1}b \right]
\]

We use the following observation: for any pair of vectors \( u \) and \( v \) in \( \mathbb{R}^n \),

\[
det[I_n + uv^T] = 1 + v^T u
\]

Applying (6) with \( v^T = -\frac{1}{d} c^T A \) and \( u = (\omega^2 I + A^2)^{-1}b \) gives

\[
\text{Re}\{H(j\omega)\} = d \det[I_n - \frac{1}{d} (\omega^2 I + A^2)^{-1}b c^T A] = d \frac{\det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A]}{\det[\omega^2 I_n + A^2]} \tag{7}
\]

Conditions for SPR: Suppose first that \( H(s) \) is SPR. Then for some \( \varepsilon > 0 \) the matrix \( A + \varepsilon I \) has spectrum in the left half of the complex plane. Hence \( A \) has no spectrum on the imaginary axis, and therefore \( \det[\omega^2 I_n + A^2] = |\det[j\omega I_n + A]|^2 > 0 \) for all real \( \omega \). Also the comment after Definition 2.1 implies that \( \text{Re}\{H(j\omega)\} = \text{Re}\{H_1(j\omega + \varepsilon)\} > 0 \) since \( H_1(s) \) is PR. It follows from (7) that \( \det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A] > 0 \) for all real \( \omega \), and this establishes conditions (i) and (ii).

Conversely, suppose that \( \det[\omega^2 I_n + A^2] > 0 \) and \( \det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A] > 0 \) for all real \( \omega \). Then by continuity the same is true if \( A \) is replaced by \( A + \varepsilon I \) for sufficiently small \( \varepsilon \), which implies that \( H(s) \) is PR. Hence \( H(s) \) is SPR.

Conditions for PR: Suppose first that \( H \) is PR. Since \( \det[\omega^2 I_n + A^2] = |\det[j\omega I_n + A]|^2 \geq 0 \), (7) implies that \( \text{Re}\{H(j\omega)\} \) can change sign if and only if \( \det[\omega^2 I_n + A^2 - \frac{1}{d} bc^T A] \) has a zero of odd multiplicity for some \( \omega \neq 0 \). In this case \(-\omega^2\) would be an eigenvalue of \( A^2 - \frac{1}{d} bc^T A \) with odd algebraic multiplicity. This establishes condition (i). Condition (ii) is required by the definition of PR.

Conversely, suppose that conditions (i) and (ii) are satisfied. Then (7) implies that \( \text{Re}\{H(j\omega)\} \) does not change sign along the imaginary axis, and since it is positive for large \( \omega \) this establishes (1) in Definition 2.1. Hence \( H(s) \) is PR.

**Proof of Theorem 3.2:**

The proof is based on a representation similar to (7). First, by considering its behavior for large \( \omega \), and using the definition of \( p \), we see that the leading part of \( \text{Re}\{H\} \) is

\[
\text{Re}\{H(j\omega)\} \sim -\omega^{(p+1)/2} \frac{1}{\omega^{p+1}} c^T A^p b
\]

Define

\[
\nu_p = -\omega^{(p+1)/2} \frac{1}{\omega^{p+1}}
\]

Following (8) we write

\[
\text{Re}\{H(j\omega)\} = \nu_p c^T A^p b - c^T A \left( \omega^2 + A^2 \right)^{-1} \nu_p A^{p-1} b
\]

\[
= \nu_p c^T A^p b \left[ 1 - \nu_p c^T A b \right] \left( \omega^2 + A^2 \right)^{-1} \nu_p A^{p-1} b
\]

We now repeat the argument that leads from (5) to (7), with \( d \) replaced by \( \nu_p c^T A^p b \) and \( A^2 \) replaced by \((\omega^2 + A^2)^{-1} \). The result is \( \text{Re}\{H(j\omega)\} \)

\[
= \nu_p c^T A^p b \frac{\det[\omega^2 I_n + A^2 - \frac{1}{c^T A b} A^{p+1} b c^T A]}{\det[\omega^2 I_n + A^2]} \tag{10}
\]

where

\[
R_p = \frac{1}{\nu_p c^T A b} (I_n + \nu_p A^{p-1}) b c^T A
\]

Notice that \( R_t = 0 \), since \( 1 + \nu_p A^2 = 0 \).

We claim that for \( p \geq 3 \)

\[
\det[\omega^2 I_n + A^2 - \frac{1}{c^T A b} A^{p+1} b c^T A] = \det[\omega^2 I_n + A^2 - \frac{1}{c^T A b} A^{p+1} b c^T A] \tag{12}
\]

To see this, first define

\[
M = \omega^2 I_n + A^2 - \frac{1}{c^T A b} A^{p+1} b c^T A
\]

and note that for \( p \geq 3 \)

\[
R_p = \frac{1}{\nu_p c^T A b} \sum_{j=0}^{(p-3)/2} \frac{(-1)^j}{\omega^{2j+2}} A^{2j+1} b c^T A
\]

Since \( R_p \) is rank 1, it follows from (6), (14) and the definition of \( p \) that \( \text{det}[M - R_p] = \det[M] \det[I - M^{-1} R_p] = \det[M] \left( 1 - \frac{1}{\nu_p c^T A b} \sum_{j=0}^{(p-3)/2} \frac{(-1)^j}{\omega^{2j+2}} A^{2j+1} b c^T A \right) \]

\[
= \det[M] \tag{15}
\]

which is just (12). Using (9) we get the that \( \text{Re}\{H(j\omega)\} \) is given by

\[
\left( -1 \right)^{(p+1)/2} k(\omega) \frac{\det[\omega^2 I_n + A^2 - \frac{1}{c^T A b} A^{p+1} b c^T A]}{\det[\omega^2 I_n + A^2]} \tag{16}
\]

for all \( \omega \neq 0 \) where \( k(\omega) = \frac{1}{\nu_p c^T A b} \).

Conditions for SPR: Suppose first that \( H \) is SPR. Then as noted after Definition 2.2, \( \text{Re}\{H(j\omega)\} \) can decay no faster than \( \omega^{-2} \) as \( \omega^2 \to \infty \), which implies that \( p = 1 \). From (8) it then follows that \( c^T A b < 0 \). The SPR condition requires that \( A \) is stable, and that \( \text{Re}\{H(j\omega)\} > 0 \) for all real \( \omega \). For \( \omega \neq 0 \), (16) implies that \( \det[\omega^2 I_n + A^2 - \frac{1}{c^T A b} A^{p+1} b c^T A] > 0 \). For \( \omega = 0 \), direct calculation yields the condition \( \text{Re}\{H(0)\} = -c^T A b > 0 \).

Conversely suppose that conditions (i) – (iv) are satisfied. These conditions remain true if \( A \) is replaced by \( A + \varepsilon I \) for \( \varepsilon \) sufficiently small. Therefore \( H \) is PR for some \( \varepsilon > 0 \), and hence \( H \) is SPR.

Conditions for PR: Comparing (16) and (7), the only difference is that the overall multiplicative factor \( d \) is replaced by \( (\omega^2 + A^2)^{-1} \). Verification that \( \text{Re}\{H(j\omega)\} \) is SPR requires that \( (\omega^2 + A^2)^{-1} \) \( c^T A b > 0 \). The other conditions for PR follow by repeating the arguments from the proof of Theorem 3.1, except at the point \( \omega = 0 \) where (16) does not apply. If \( H(s) \) has a pole at \( \omega = 0 \), then positivity is guaranteed by the positivity of the residue. Otherwise the continuity of \( \text{Re}\{H(j\omega)\} \) guarantees non-negativity at \( \omega = 0 \).
IV. Conclusions

In this paper we have presented simple algebraic conditions for checking (strict) positive realness of a single-input single-output transfer function.

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References