An Improved Frequency-domain Method for the Fractional Order $PI^\lambda D^\mu$ Controller Optimal Design

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Abstract: An improved frequency-domain design method is proposed to design the fractional order $PI^\lambda D^\mu$ controller. Using this improved method, the parameters of the fractional order $PI^\lambda D^\mu$ controllers can be obtained immediately according to the model characteristics and design specifications. A proportional relation between the integral gain and derivative gain is built, while the derivative order is set to be equal to the integral order. The proportional coefficient between integral gain and derivative gain is studied and modeled based on priori knowledge and data fitting, and then the estimation model for the optimal proportional coefficient is built. The proposed tuning method is applied to design a fractional order $PI^\lambda D^\mu$ controller for a permanent magnet synchronous motor servo system. Motor speed control simulations are performed to verify the proposed method. Simulation results show that the obtained control system can achieve robustness and the optimized step response performance.

Keywords: PID control, fractional order controller, frequency-domain method, feedback control, data fitting

1. INTRODUCTION

The proportional integral derivative (PID) control is the most widely used control method in the industrial control area. In recent years, fractional calculus has aroused interest and attention of scholars (Podlubny (1999a), Monje et al. (2010), Luo et al. (2010), Luo and Chen (2009)). The fractional order proportional integral derivative ($PI^\lambda D^\mu$) controller has the potential to achieve better control performance over the traditional PID controller because the adjustable integral order $\lambda$ and derivative order $\mu$ are introduced, expanding the control scope of the controller (Podlubny (1999b)). However, on the other hand, the tuning of the $PI^\lambda D^\mu$ controller is more complicated.

The tuning methods of fractional order $PI^\lambda D^\mu$ controller can mainly be divided into two kinds, the frequency-domain design method (Luo et al. (2010), Luo and Chen (2009)) and the optimization methods (Biswa et al. (2009), Zheng and Pi (2016)). The frequency-domain method is often applied to design the fractional order $PI^\lambda$ or $PD^\mu$ controller. Based on the given gain crossover frequency and phase margin, the controller parameters are calculated according to the gain robustness specification. The obtained control system achieves reliable stability and the robustness to gain variations. However, as discussed in this paper, this method cannot be directly applied to tune the fractional order $PI^\lambda D^\mu$ controller.

A tuning method based on the differential evolution (DE) algorithm is proposed (Zheng et al. (2017)), satisfying the specifications in both frequency-domain and time-domain simultaneously. The obtained control system achieves the optimal dynamic performance, while the frequency-domain design requirements are also satisfied. However, applying this method, large amount of space and time are spent in the numerical optimization. Therefore, this method may not be suitable for engineering application.

An improved frequency-domain design method is proposed to design the fractional order $PI^\lambda D^\mu$ controller in this paper. In order to reduce the pending parameters of the controller, the proportional relation between the integral gain $K_i$ and derivative gain $K_d$ is built, while the derivative order $\mu$ is set to be equal to the integral order $\lambda$. Based on this modification, the number of the pending parameters is reduced from five to three. Therefore, the current frequency-domain design method can be applied to tune the fractional order $PI^\lambda D^\mu$ controller. The proportional coefficient between $K_i$ and $K_d$ is studied and modeled based on priori knowledge and data fitting, and then the estimation model for the optimal proportional coefficient
is built. The proposed tuning method is applied to design a fractional order PI\(^{\lambda}\)D\(^{\mu}\) controller for a permanent magnet synchronous motor (PMSM) servo system. Motor speed control simulations are performed and the advantage of the improved frequency-domain method is demonstrated.

2. IMPROVED FREQUENCY-DOMAIN DESIGN METHOD

The fractional order PI\(^{\lambda}\)D\(^{\mu}\) controller is described by (1),

\[
C(s) = K_p \left(1 + \frac{K_i}{s^\lambda} + K_d s^\mu\right),
\]

where \(K_p, K_i\) and \(K_d\) are proportional, integral and derivative gains, respectively; \(\lambda\) and \(\mu\) are the fractional orders.

Based on (1), the amplitude and phase of the controller are obtained as described by (2) and (3),

\[
\begin{align*}
|C(j\omega)| &= K_p \sqrt{P(\omega)^2 + Q(\omega)^2}, \\
\angle C(j\omega) &= \arctan \left(\frac{Q(\omega)}{P(\omega)}\right),
\end{align*}
\]

where

\[
\begin{align*}
P(\omega) &= 1 + K_i \omega^{-\lambda} \cos\left(\frac{\pi}{2}\lambda\right) + K_d \omega^{\mu} \cos\left(\frac{\pi}{2}\mu\right), \\
Q(\omega) &= K_d \omega^{\mu} \sin\left(\frac{\pi}{2}\mu\right) - K_i \omega^{-\lambda} \sin\left(\frac{\pi}{2}\lambda\right).
\end{align*}
\]

The plant model for controller design has the form represented by (6),

\[
G(s) = \frac{K}{s^3 + \tau_1 s^2 + \tau_2 s}.
\]

Based on (6), the amplitude and phase of the plant model are obtained as described by (7) and (8),

\[
\begin{align*}
|G(j\omega)| &= \frac{K}{\sqrt{A(\omega)^2 + B(\omega)^2}}, \\
\angle G(j\omega) &= -\arctan \left(\frac{B(\omega)}{A(\omega)}\right),
\end{align*}
\]

where

\[
A(\omega) = -\tau_1 \omega^2, B(\omega) = \tau_2 \omega - \omega^3.
\]

Based on the frequency-domain design method, the control system should be robust to the loop-gain variations. According to the robustness specification, the derivative of the phase-frequency curve is zero, namely, the phase Bode plot is flat at the gain crossover frequency. For the plant model \(G(s)\) and controller \(C(s)\), the loop-gain robustness equation is then described by (10),

\[
\frac{d|\angle G(j\omega)C(j\omega)|}{d\omega}\bigg|_{\omega=\omega_c} = 0,
\]

where \(\omega_c\) is the given gain crossover frequency, satisfying

\[
|G(j\omega_c)C(j\omega_c)| = 1.
\]

To ensure the stability of the control system, the phase margin \(\varphi_m\) is also given, satisfying

\[
\angle G(j\omega_c) + \angle C(j\omega_c) = -\pi + \varphi_m.
\]

The fractional order PI\(^{\lambda}\)D\(^{\mu}\) controller has five parameters to be tuned, \(K_p, K_i, K_d, \lambda\) and \(\mu\). However, only three equations are derived from the design specifications. Therefore, the current frequency-domain design method cannot be applied to tune the fractional order PI\(^{\lambda}\)D\(^{\mu}\) controller directly.

A modification on the current design method is proposed to solve this problem. The proportional relation between the integral gain \(K_i\) and derivative gain \(K_d\) is built, as described by (13),

\[
K_d = a K_i,
\]

where \(a\) is the proportional coefficient. Besides, the derivative order \(\mu\) is set to be equal to the integral order \(\lambda\). Therefore, the modified fractional order PI\(^{\lambda}\)D\(^{\mu}\) controller is described by (14),

\[
C(s) = K_p \left(1 + \frac{K_i}{s^\lambda} + a K_i s^\lambda\right).
\]

Substituting (14) into (12), \(K_i\) can be represented as (15),

\[
K_i = \frac{-aM}{M \omega_c^{-\lambda} \cos(\frac{\pi}{2}\lambda) + aM \omega_c^{\lambda} \cos(\frac{\pi}{2}\mu) + a M \omega_c^{-\lambda} \cos(\frac{\pi}{2}\mu) - N \omega_c^{-\lambda} \sin(\frac{\pi}{2}\lambda)},
\]

where

\[
M = A(\omega_c) \tan(-\pi + \varphi_m) + B(\omega_c),
\]

\[
N = B(\omega_c) \tan(-\pi + \varphi_m) - A(\omega_c).
\]

Similarly, substituting (14) into (10), the equation about \(K_1\) is obtained, as described by (18),

\[
Q_2 K_1^2 + Q_1 K_1 + Q_0 = 0,
\]

where

\[
Q_2 = \frac{2aK_i \sin(\pi\lambda) + 2aQ_0 \cos(\pi\lambda) + a^2 Q_0 \omega_c^{-2\lambda} + \frac{Q_0}{\omega_c^2}},
\]

\[
Q_1 = \frac{a^2 K_i \sin(\frac{\pi}{2}\lambda) + \frac{a^2 K_i}{\omega_c^2} \sin(\frac{\pi}{2}\mu) + 2a Q_0 \omega_c^{-\lambda} \cos(\frac{\pi}{2}\mu) + 2a^2 K_i \cos(\frac{\pi}{2}\mu)}{\omega_c^2},
\]

\[
Q_0 = \frac{d|\angle G(j\omega)|}{d\omega}\bigg|_{\omega=\omega_c}.
\]

According to the frequency-domain method, the gain crossover frequency \(\omega_c\) and the phase margin \(\varphi_m\) are given in advance. Therefore, for a specific plant model \(G(s)\), once the coefficient \(a\) between \(K_i\) and \(K_d\) is determined, \(K_i\) and \(\lambda\) can be calculated based on (15) and (18). Then \(K_d\) and \(\mu\) can also be obtained. Finally, \(K_p\) can be calculated according to (11). Thus, all the parameters of the fractional order PI\(^{\lambda}\)D\(^{\mu}\) controller are obtained.

3. ESTIMATION MODEL OF THE PROPORTIONAL COEFFICIENT

According to the improved frequency-domain method, the proportional coefficient \(a\) between \(K_i\) and \(K_d\) should be determined before the controller parameters are calculated. Therefore, the dynamic performance of the control system is significantly affected by the value of \(a\). To ensure good dynamic performance of the obtained control system, the estimation model of \(a\) is needed to be built.

The estimation model of \(a\) is built based on the following assumption. For a plant model \(G(s)\) and a specific \((\omega_c, \varphi_m)\) setting, an optimal fractional order PI\(^{\lambda}\)D\(^{\mu}\) controller can be derived from the optimal value of \(a\). The distributions

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of the optimal $a$ corresponding to each $(\omega_c, \phi_m)$ setting can be approximated by a model related to the $(\omega_c, \phi_m)$ setting and the plant model characteristics. Furthermore, the approximated model of the optimal $a$ is continuously defined in the hyperspace of $\omega_c$, $\phi_m$ and the related model characteristic parameters.

The estimation model of the optimal $a$ is built according to the following steps. First, several test models are built based on the interested model parameter ranges. Similarly, several $(\omega_c, \phi_m)$ pairs are selected according to the design requirements. Second, the optimal values of $a$ corresponding to different $(\omega_c, \phi_m)$ pairs and test models are collected. Third, based on the collected data, the distribution rules of the optimal $a$ for different $(\omega_c, \phi_m)$ settings and model characteristics are studied. Finally, the estimation model of $a$ is built according to the summarized distribution rules.

3.1 Optimal Data Collection

In this paper, the estimation model of the optimal $a$ is built for the PMSM servo system having the form described by (6). According to the commonly used PMSM models, the range of parameter $\tau_1$ is set to be 100 to 140, while that of $\tau_2$ is set to be 8000 to 11000. Besides, in order to satisfy the general design requirement, the range of the given gain crossover frequency $\omega_c$ is set to be $35\text{rad/s}$ to $60\text{rad/s}$, while that of the given phase margin $\phi_m$ is set to be $45^\circ$ to $60^\circ$ (Ruan et al. (2016)).

Based on the range of $\tau_1$, three values of $\tau_1$: 100, 120 and 140 are selected to build the test models. Similarly, three values of $\tau_2$: 8000, 9500 and 11000 are also selected. The gain $K$ has no influence on the value of $a$, it is fixed to be 30000. Therefore, nine test models are built by combining the values of $\tau_1$ and $\tau_2$, as described from (22) to (30),

$$G_1(s) = \frac{30000}{s^3 + 100s^2 + 8000s},$$  \hspace{1cm} (22)

$$G_2(s) = \frac{30000}{s^3 + 120s^2 + 8000s},$$  \hspace{1cm} (23)

$$G_3(s) = \frac{30000}{s^3 + 140s^2 + 8000s},$$  \hspace{1cm} (24)

$$G_4(s) = \frac{30000}{s^3 + 100s^2 + 11000s},$$  \hspace{1cm} (25)

$$G_5(s) = \frac{30000}{s^3 + 120s^2 + 11000s},$$  \hspace{1cm} (26)

$$G_6(s) = \frac{30000}{s^3 + 140s^2 + 11000s},$$  \hspace{1cm} (27)

$$G_7(s) = \frac{30000}{s^3 + 100s^2 + 9500s},$$  \hspace{1cm} (28)

$$G_8(s) = \frac{30000}{s^3 + 120s^2 + 9500s},$$  \hspace{1cm} (29)

$$G_9(s) = \frac{30000}{s^3 + 140s^2 + 9500s}. $$  \hspace{1cm} (30)

Based on the range of $\omega_c$, seven values of $\omega_c$: $35\text{rad/s}$, $37\text{rad/s}$, $40\text{rad/s}$, $45\text{rad/s}$, $50\text{rad/s}$, $55\text{rad/s}$, $60\text{rad/s}$ are selected to tune the controllers. Similarly, four values of $\phi_m$: $45^\circ$, $50^\circ$, $55^\circ$, $60^\circ$ are also selected.

![Fig. 1. The distributions of optimal $a$ with regard to $\phi_m$](image)

The integrated time absolute error (ITAE) is selected to be the dynamic performance index of the control system. The ITAE index is described by (31),

$$J = \int_0^\infty t |e(t)| \, dt, $$  \hspace{1cm} (31)

where $e(t)$ represents the deviation between the expected output and the actual output.

For each test model $G_j(s)$, the optimal values of $a$ are collected following the steps below.

1. Combine the values of given $\omega_c$ and $\phi_m$ and then generate the $(\omega_c, \phi_m)$ groups.
2. Select 25 values of $a$ uniformly in the range of $a$, obtaining $a_1, a_2, ..., a_{25}$.
3. Calculate the controller parameters based on each $a_j$ ($j = 1, 2, ..., 25$).
4. Step response simulation is performed and the ITAE of the control system output is calculated.
5. The value of $a$ with the least ITAE is selected to be the optimal one.

Following these steps, the optimal values of $a$ for nine test models are collected.

3.2 Estimation Model Study

The optimal value of $a$ should be related to the given crossover frequency $\omega_c$, phase margin $\phi_m$ and the phase-frequency characteristics of the plant model. First, the relation between the optimal $a$ and the given $\phi_m$ is studied, under the condition that the given $\omega_c$ is fixed. Taking $G_1(s)$ as an example, the distributions of the optimal $a$ with regard to $\phi_m$ are plotted in Fig. 1. Based on Fig. 1, the optimal $a$ satisfies the linear relationship with $\phi_m$ when $\omega_c$ is fixed. Therefore, the optimal $a$ can be described by (32),

$$a = A\phi_m + B,$$  \hspace{1cm} (32)

where $A$ and $B$ are related to the given gain crossover frequency $\omega_c$ and the phase-frequency characteristics of the plant model.

The distributions of the optimal $a$ with regard to $\phi_m$ are fitted applying the least square method. Thus, the values of $A$ and $B$ corresponding to different given $\omega_c$ for model $G_1(s)$ are obtained. The same process is performed on the
other eight test models and then the values of $A$ and $B$
for $G_1(s)$ to $G_9(s)$ are obtained. The distributions of $A$
with regard to $\omega_c$ for different test models are plotted in
Fig. 2. Based on Fig. 2, $A$ approximately satisfies the linear
relationship with $\omega_c$. Therefore, $A$ can be described by
(33),

$$A = M_1 \omega_c + N_1,$$

where $M_1$ and $N_1$ are also related to the phase-frequency
characteristics of the plant model. The values of $M_1$ and
$N_1$ are obtained by fitting the distributions of $A$ with regard to $\omega_c$ using the least square method. Thus, the
values of $M_1$ and $N_1$ corresponding to different test models
are obtained.

In order to study the distributions of $M_1$ and $N_1$ with
regard to the phase-frequency characteristics of the plant
model, seven frequency points: $35 \text{ rad/s}$, $37 \text{ rad/s}$, $40 \text{ rad/s}$, $45 \text{ rad/s}$, $50 \text{ rad/s}$, $55 \text{ rad/s}$, $60 \text{ rad/s}$ are selected in the
gain crossover frequency range. The phases at these frequency
points are calculated and the mean phase $\varphi_{m0}$
within the frequency range is obtained by calculating the
average of seven values. Performing such calculations on
nine test models, the mean phases corresponding to nine models, $\varphi_{m0}(G_1(s))$, $\varphi_{m0}(G_2(s))$, ..., $\varphi_{m0}(G_9(s))$ are ob-
tained.

The distributions of $M_1$ with regard to mean phase $\varphi_{m0}$
are plotted as data points in Fig. 3. Based on Fig. 3, $M_1$
satisfies the linear relationship with $\varphi_{m0}$, as described by
(34),

$$M_1 = P_1 \varphi_{m0} + Q_1,$$

where $P_1$ and $Q_1$ are pending constants. Applying the
least square method to fit the distributions of $M_1$ with
regard to $\varphi_{m0}$, the values of $P_1$ and $Q_1$ are obtained,
$P_1 = 5.033 \times 10^{-8}$, $Q_1 = 5.627 \times 10^{-6}$, the fitting line
is plotted in red in Fig. 3.

The distributions of $N_1$ with regard to mean phase $\varphi_{m0}$
are plotted as data points in Fig. 4. Based on Fig. 4, $N_1$
satisfies the linear relationship with $\varphi_{m0}$, as described by
(35),

$$N_1 = P_2 \varphi_{m0} + Q_2,$$

where $P_2$ and $Q_2$ are pending constants. Applying the
least square method to fit the distributions of $N_1$ with
regard to $\varphi_{m0}$, the values of $P_2$ and $Q_2$ are obtained,
$P_2 = -2.838 \times 10^{-6}$, $Q_2 = -3.044 \times 10^{-4}$, the fitting line
is plotted in red in Fig. 4.

In order to study the relations between $B$ in (32) and
the given $\omega_c$, the distributions of $B$ with regard to $\omega_c$
for different test models are plotted in Fig. 5. Based on Fig. 5,
$B$ approximately satisfies the linear relationship with $\omega_c$, as described by (36),

$$B = M_2 \omega_c + N_2,$$

where $M_2$ and $N_2$ are related to the phase-frequency
characteristics of the plant model. The values of $M_2$ and
$N_2$ are obtained by fitting the distributions of $B$ with
regard to $\omega_c$ using the least square method. Thus, the
values of $M_2$ and $N_2$ corresponding to different test models
are obtained.
In order to check the robustness of the obtained control system, the loop-gain of \( C_1(s) \) is set to be 100%, 120% and 80% of its nominal value to simulate the plant model uncertainty. Setting the reference speed \( n_r \) to be 1000rpm, the motor speed step response simulations are performed.
Based on the simulation results, the proposed tuning method is valid and the obtained control system can achieve robustness and the optimized step response performance.

5. CONCLUSION

An improved frequency-domain design method for fractional order PI\(^3\)D\(^\mu\) controllers is proposed. The proportional relation between \(K_i\) and \(K_d\) is built, while \(\mu\) is set to be equal to \(\lambda\). The estimation model of the proportional coefficient between \(K_i\) and \(K_d\) is built for the commonly used PMSM servo systems. Motor speed step response simulations are performed to verify the estimation model and the proposed tuning method. Simulation results show that the improved frequency-domain design method is valid and the obtained control system can achieve robustness and the optimized step response performance.

REFERENCES


