PID controller design with an $H_\infty$ criterion

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Abstract: This paper deals with the design of Proportional-Integral (PI) and Proportional-Integral-Derivative (PID) controllers. The main result is a constructive determination of the set $S_\gamma$ of stabilizing PI and PID controllers achieving an $H_\infty$ norm bound of $\gamma$ on the error transfer function. This result utilizes the computation of the complete stabilizing set $S$ recently obtained. We also point out connections between this $H_\infty$ design and Gain and Phase Margin designs. Illustrative examples are presented.

Keywords: PID, stabilizing set, $H_\infty$ norm, gain and phase margin, Nyquist plot

1. INTRODUCTION AND BACKGROUND

In servomechanisms the tracking error needs to be small for the class of reference signals and disturbance signals encountered. For step references and disturbances the integral controller yields zero steady state error provided the closed loop is stable. Integral control is usually implemented as a PI or PID controller in practice. In this paper we consider the additional design criterion of an $H_\infty$ norm specification on the error transfer function and show that the complete set of PI and PID controllers satisfying $H_\infty$ norm specification can be constrictively determined.

In the next section we develop a relationship between $H_\infty$ norm specification on the error transfer function and guaranteed classical gain and phase margins. Following this we present our constructive calculation of $S_\gamma$ for PI or PID controller sets satisfying the given $H_\infty$ norm specification of $\gamma$.

In Emami and Watkins (2009), the 2D regions of stabilizing PID controllers achieving the $H_\infty$ norm bound of $\gamma$ on the sensitivity and complementary sensitivity functions with weightings were found by using Neimark’s D-decomposition. The difference is that our approach explicitly uses the stabilizing set $S$, and thus can determine the limits of achievable performance.

Similar approach was adopted by Tantaris et al. (2006) for first order controllers and in this case the stability region was computed a priori. Krajewski and Viaro (2012) showed that at a fixed frequency (and for a fixed $k_p$, the derivative gain) the $L_2$ norm of the error transfer function being equal to $\gamma$ was represented by an ellipse in $(k_i,k_p)$ space where $k_p$ was the proportional gain and $k_i$ was the integral gain.

An $H_\infty$ optimal PID design using a frequency loop-shaping approach was reported in Tsakalis and Dash (2013); Ashfaque and Tsakalis (2012). PID gains were chosen by an optimization problem with a desired open loop transfer function. All PID gains were assumed to be positive in order for the constraint to be a convex set. However, the stabilizing set is not convex in general. See, for instance, Example 2.2 in Bhattacharyya et al. (2009).

The computation of all PID stabilizing controllers, the stabilizing set, and extensions were developed in Bhattacharyya et al. (2009). In Díaz-Rodríguez and Bhattacharyya (2016) the subset of the stabilizing set achieving prescribed gain and phase margin specifications were found.

2. $H_\infty$ CONTROL AND STABILITY MARGINS

Consider the unity feedback system (see Fig. 1)

$$r(t) + e(t) \rightarrow y(t)$$

Fig. 1. Unity feedback control loop.

with the error transfer function

$$\frac{e(s)}{r(s)} = \frac{1}{1 + G(s)}. \tag{1}$$

Suppose that $G(s)$ includes a controller designed to make the $H_\infty$ norm of (1) less than $\gamma$, a prescribed real positive number. Then

$$\frac{1}{|1 + G(j\omega)|} < \gamma, \quad \text{for all } \omega \geq 0 \tag{2}$$

and (2) is equivalent to

$$|1 + G(j\omega)| > \frac{1}{\gamma}, \quad \text{for all } \omega \in [0, \infty). \tag{3}$$

We will now establish that (3) implies guaranteed gain and phase margins at the loop breaking point ‘m’ in Fig. 1.

Remark 1. Let $\gamma^*$ denote the infimum value of $\gamma$ satisfying (3). When $G(s)$ is strictly proper, $\gamma^* \geq 1$. When $G(s)$ is proper, $\gamma^* > 1/|1 + G(j\infty)|$. 

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Case 1: \( \gamma > 1 \)

The condition (3) implies that the Nyquist plot \( G(j\omega) \) stays out of the circle \( CEDB \) centered at \(-1 + j0\) and of radius \( 1/\gamma \). In Fig. 2, we have the limiting case in which \( G(j\omega) \) passes through \( B \), the phase margin is \( \phi \) and \( G(j\omega) = \overrightarrow{OB} \), \(-1 + j0 = \overrightarrow{OA} \), \(1 + G(j\omega) = \overrightarrow{AB} \).

Since \( \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \) and \( \overrightarrow{OAB} \) is an isosceles triangle,
\[
-1 + j0 + \frac{1}{\gamma} e^{-j\theta} = -1 e^{j\phi} \quad \text{and} \quad 2\theta + \phi = \pi. \tag{4}
\]

From (4),
\[
-1 + \frac{1}{\gamma} \sin \left( \frac{\phi}{2} \right) = -\cos \phi \tag{5}
\]
\[
\sin \phi = \frac{1}{\gamma} \cos \left( \frac{\phi}{2} \right). \tag{6}
\]

From (6),
\[
\phi = 2 \sin^{-1} \left( \frac{1}{2\gamma} \right) \tag{7}
\]
which is the guaranteed minimum phase margin for the \( H_\infty \) controller with the norm less than \( \gamma \).

The guaranteed gain margin is the interval:
\[
\frac{1}{\text{OD}} \frac{1}{\text{OC}} = \left[ \frac{\gamma}{\gamma + 1}, \frac{\gamma}{\gamma - 1} \right]. \tag{8}
\]

Case 2: \( \gamma = 1 \)

In this case, Fig. 2 is replaced by Fig. 3. It is easy to see that the guaranteed phase margin is \( \phi = \pi/3 \) and the guaranteed gain margin is \( \left[ \frac{1}{2}, \infty \right] \). These also follow from formulas (7) and (8) evaluated at \( \gamma = 1 \).

Case 3: \( \gamma < 1 \)

The geometry corresponding to this case is shown in Fig. 4 below.

Fig. 2. \( \gamma > 1 \)

In this case, it also follows that the guaranteed phase margin is
\[
\phi = 2 \sin^{-1} \left( \frac{1}{2\gamma} \right) \tag{9}
\]
and the guaranteed gain margin is
\[
\left\| \frac{1}{1 + G(s)} \right\|_\infty < \gamma, \tag{11}
\]
then the guaranteed phase margin at the loop breaking point ‘m’ is:
\[
\phi = 2 \sin^{-1} \left( \frac{1}{2\gamma} \right) \tag{12}
\]

The guaranteed gain margin is:
\[
g_m = \left\{ \frac{\gamma}{\gamma + 1}, \frac{\gamma}{\gamma - 1} \right\}, \quad \text{for} \quad \gamma > 1
\]
\[
\left\{ \frac{\gamma}{\gamma + 1}, \infty \right\}, \quad \text{for} \quad \gamma \leq 1. \tag{13}
\]

3. PROBLEM FORMULATION

Consider now the control system in Fig. 5.

Fig. 5. Unity feedback control loop.

\( r(t) \) is the reference signal, \( e(t) \) is the error signal, \( u(t) \) is the input signal (to the plant), \( y(t) \) is the output signal, \( P(s) \) is the plant transfer function and \( C(s) \) is the controller transfer function which we will consider to be either PI or PID.

The problem to be solved in this paper is: find the set \( S_\gamma \) of all stabilizing PI or PID controllers satisfying
\[
\left\| \frac{1}{1 + P(s)C(s)} \right\|_\infty < \gamma. \tag{14}
\]

4. MAIN RESULTS

In this section we develop the computation of \( S_\gamma \) for PI and PID controllers. Note that (14) is equivalent to
\[ |1 + P(j\omega)C(j\omega)| > \frac{1}{\gamma}, \quad \forall \omega \in [0, \infty). \] (15)

### 4.1 Computation of \( S_\gamma \) for PI controllers

PI controllers have the form:

\[ C(s) = k_p + \frac{k_i}{s}. \] (16)

Write

\[ P(j\omega) = P_r(\omega) + j\omega P_i(\omega), \] (17)

\[ C(j\omega) = k_p - \frac{k_i}{\omega}. \] (18)

Substituting (17) and (18) in (15) we get

\[ |1 + \frac{k_p P_r(\omega) + k_i P_i(\omega) + j(\omega k_p P_i(\omega) - k_i P_r(\omega))}{L_0(\omega)}| > \frac{1}{\gamma}, \] (19)

which can be rewritten as

\[ (1 + L_0(\omega))^2 + L_1^2(\omega) > \frac{1}{\gamma^2} \] (20)

\[ \begin{bmatrix} P_r(\omega) & P_i(\omega) \\ \omega P_r(\omega) - P_i(\omega) \end{bmatrix} \begin{bmatrix} k_p \\ k_i \end{bmatrix} = \begin{bmatrix} L_0(\omega) \\ L_1(\omega) \end{bmatrix}. \] (21)

(21) has a unique solution if

\[ |P(j\omega)| \neq 0, \] (22)

that is the plant has no \( j\omega \) axis zeros.

Assuming (22), (21) can be solved:

\[ \begin{bmatrix} k_p \\ k_i \end{bmatrix} = \frac{1}{|P(j\omega)|^2} \begin{bmatrix} P_r(\omega) & \omega P_i(\omega) \\ -\omega^2 P_i(\omega) - \omega P_r(\omega) \end{bmatrix} \begin{bmatrix} L_0(\omega) \\ L_1(\omega) \end{bmatrix} \] (23)

(20) represents the outside of a circle \( C_\gamma \) of radius \( \frac{1}{\gamma} \) in the \((L_0, L_1)\) plane centered at \((-1, 0)\):

**Lemma 1.** Condition (15) at a fixed \( \omega \) is equivalent to \( k_p, k_i \) lying in the complement of the interior of the axis parallel ellipse \( E_\gamma(\omega) \) with center \( o' \) at \((\frac{-\omega^2 P_i(\omega)}{|P(j\omega)|^2}, \frac{-P_r(\omega)}{|P(j\omega)|^2})\), principal axes \( \frac{2\omega}{|P(j\omega)|} \cdot \frac{\omega}{|P(j\omega)|} \).

**Proof.** For each \( \omega \geq 0 \), (19) is

\[ \begin{aligned}
&\left|1 + \left(P_r(j\omega) + j\omega P_i(j\omega)\right)(k_p - \frac{k_i}{\omega})\right| > \frac{1}{\gamma} \\
\iff &\left(1 + P_r(j\omega)k_p + P_i(j\omega)k_i\right)^2 + \left(\omega P_i(j\omega)k_p - P_r(j\omega)\frac{k_i}{\omega}\right)^2 > \frac{1}{\gamma^2};
\end{aligned} \] (24)

After some algebra we obtain

\[ \frac{(k_i - c_1)^2}{a^2} + \frac{(k_p - c_2)^2}{b^2} > 1 \] (25)

where

\[
\begin{aligned}
c_1 &= -\frac{\omega^2 P_i(\omega)}{|P(j\omega)|^2}, & c_2 &= \frac{-P_r(\omega)}{|P(j\omega)|^2}, \\
a &= \frac{\omega}{|P(j\omega)|}, & b &= \frac{1}{\gamma}. 
\end{aligned} \] (26)

\[ \]
\[ S_\gamma(\omega) = S \setminus E_\gamma(\omega) \quad \forall \omega \in [0, \infty). \quad (27) \]

Since (15) must hold for all \( \omega \),
\[ S_\gamma = \bigcap_{\omega=0}^{\infty} S_\gamma(\omega) \quad (28) \]
as shown in Fig. 9.

We state this result as the following theorem.

**Theorem 2.** In the unity feedback control loop, suppose that the plant \( P(s) \) has no \( j\omega \) axis zeros. All stabilizing PI controllers \( C(s) \) satisfying the \( H_\infty \) norm bound of \( \gamma \) on the error transfer function is the set \( S_\gamma \):
\[ S_\gamma = \bigcap_{\omega=0}^{\infty} S_\gamma(\omega). \quad (29) \]

**Proof.** \( S_\gamma(\omega) \) is the admissible set for each \( \omega \) and the controller must satisfy the \( H_\infty \) norm for all frequencies. Hence we have the set \( S_\gamma \) by intersecting the admissible sets \( S_\gamma(\omega) \) for all \( \omega \). \( \square \)

Note that \( S \) can be determined using the concept of signature developed in Bhattacharyya et al. (2009). If \( E_\gamma(\omega) \) is outside of \( S \) then \( S_\gamma(\omega) = S \). If \( S \subseteq E_\gamma(\omega) \) then \( S_\gamma \) is empty.

4.2 Computation of \( S_\gamma \) for PID controllers

PID controllers are of form:
\[ C(s) = k_p + \frac{k_i}{s} + k_ds. \quad (30) \]
Substituting \( s = j\omega \), we have
\[ C(j\omega) = k_p - \frac{1}{\omega} \left( k_i - \omega^2 k_d \right). \quad (31) \]

![Fig. 10. The \( E_\gamma(\omega) \) elliptic cylinder.](image)

Replace \( k_i \) in (19) with \( k_i^* = k_i - \omega^2 k_d \). By analysis similar to the PI case, it is easy to show that (15) implies that the controller parameters \( k_p, k_i, k_d \) must lie in the exterior of \( E_\gamma(\omega) \) described by:
\[ \frac{(k_i - \omega^2 k_d - c_1)^2}{a^2} + \frac{(k_p - c_2)^2}{b^2} > 1 \quad (32) \]
which is an elliptic cylinder with the center lying on the line
\[ \begin{cases} 
  k_i - \omega^2 k_d = \frac{-\omega^2 P(\omega)}{|P(j\omega)|^2}, \\
  k_p = \frac{-P(\omega)}{|P(j\omega)|^2}, 
\end{cases} \quad (33) \]
and principal axes
\[ \frac{2}{\sqrt{|P(j\omega)|}} \left( \frac{2}{\sqrt{|P(j\omega)|}} \right). \quad (34) \]

As before,
\[ S_\gamma(\omega) = S \setminus E_\gamma(\omega) \quad \forall \omega \in [0, \infty) \quad (35) \]
and
\[ S_\gamma = \bigcap_{\omega=0}^{\infty} S_\gamma(\omega). \quad (36) \]

The details of the computation are omitted.

**Remark 2.** We can consider the \( H_\infty \) norm with a weighting function \( W(s) \) multiplied by the error transfer function in (14). In this case, replace \( \gamma \) in (15) by \( \gamma' \) where \( \gamma' = \frac{W(j\omega)}{\gamma j\omega} \). Then, the principal axes of the axis parallel ellipse \( E_\gamma(\omega) \) are subject to change by \( W(j\omega) \). However, the derivation of the equations in this section remains the same.

**Remark 3.** If \( C(s) \) is replaced by \( C_\tau(s) = \frac{k_p + k_i + k_ds^2}{s(\tau + s + 1)} \), then \( C_\tau(s)P(s) = C(s) \frac{1}{\tau + s} P(s) \). Since \( \tau \) can be fixed a priori, we can replace \( P\gamma(j\omega) \) and \( P\gamma(j\omega) \) in (17) by
\[ P\gamma'(j\omega) = P\gamma(j\omega) + \frac{\tau \omega^2}{1 + \tau^2 \omega^2} \quad (37) \]
\[ \quad P\gamma'(j\omega) = \frac{P\gamma(j\omega) - \tau P\gamma(j\omega)}{1 + \tau^2 \omega^2}. \quad (38) \]
Then, the controller design can be carried out in a similar manner.

5. EXAMPLES

We present two examples to illustrate the steps to find the set \( S_\gamma \).

**Example 1.** Consider the second order plant and the PI controller:
\[ P(s) = \frac{s - 2}{s^2 + 4s + 3}, \quad C(s) = k_p + \frac{k_i}{s} \quad (36) \]

![Fig. 11. \( S_\gamma \) for \( \gamma = 1.6, 2, 4, 8 \) with the stabilizing set.](image)

The stabilizing set was first computed for the plant and the PI controller given in (36). Family of ellipses \( E_\gamma(\omega) \) were drawn by sweeping over \( \omega \) and \( S_\gamma \) were found accordingly for \( \gamma = 1.6, 2, 4 \) and 8. In Fig. 11 we observed that \( S_\gamma \) were contained in the stabilizing set \( S \) and \( S_\gamma \subset S_\gamma \) if \( \gamma_1 < \gamma_2 \). So, \( S_\gamma \) for \( \gamma \in [1, \infty) \) is the telescoping series of sets. If \( k_p, k_i \) were chosen from sets \( S_\gamma \), the Nyquist plot must stay outside of the critical point \(-1 + j0\) with the minimum distance of \(1/\gamma\). We chose some boundary points in \( S_\gamma \) that were inside the stabilizing set \( S \) and draw the Nyquist plots in Fig. 12. Each Nyquist plot was at least 0.5 away from the critical point.
Fig. 12. Nyquist plots with $k_p$, $k_i$ along the curve of $\gamma = 2$.

Following Theorem 1, the guaranteed gain margin was

$$ \left[ \frac{\gamma}{\gamma + 1} \frac{\gamma}{\gamma - 1} \right] = \left[ \frac{2}{3} \frac{1}{2} \right]. $$

and the guaranteed phase margin $\phi$ was

$$ \phi = 2 \sin^{-1}\left( \frac{1}{2\gamma} \right) = 28.955^\circ $$

for $\gamma = 2$. Fig. 13 shows the guaranteed gain and phase margins when we choose $k_p$ and $k_i$ from $S_\gamma$ for $\gamma = 2$. For all controllers achieving the same $H_\infty$ norm at the boundary of $S_\gamma$, there is a trade off between gain and phase margins. When higher gain margin is desired, one should sacrifice some phase margin and vice versa. Nevertheless with the $H_\infty$ norm we get the guaranteed gain and phase margins calculated in Eqs. (37) and (38).

**Example 2.** Consider a rational transfer function given in Blanchini et al. (2004) and the PID controller:

$$ P(s) = \frac{10s^3 + 9s^2 + 362.4s + 36.16}{2s^4 + 2.7255s^3 + 138.4292s^2 + 156.471s^2 + 637.6472s + 360.1779} $$

$$ C(s) = k_p + \frac{k_i}{s} + k_ds. $$

The stabilizing set was computed using the signature method as shown in Fig. 14. We adopted $k_d = 9$ as in Krajewski and Viaro (2012) and computed $S_\gamma$ for $\gamma = 1$ in $k_p$, $k_i$, $k_d$ space using the signature method.

Fig. 14. The stabilizing set in $k_p$, $k_i$, $k_d$ space using the signature method.

We observed that the stabilizing set with $k_d = 9$ was unbounded in $k_p$, $k_i$ plane. However, the $S_\gamma$ for $\gamma = 1$ in the same plane was bounded. For the high values of $\omega$ the major and minor axes of the ellipses grow as the centers $c_1$ and $c_2$ in (26) go away from the origin. So, we suggest that the family of ellipses be computed for high enough values of $\omega$ to get the exact set $S_\gamma$.

Clearly in this case, $S_\gamma$ is not empty and the $H_\infty$ norm condition less than $\gamma = 1$ provides very good robustness, namely $[0.5, \infty]$ gain margin and $60^\circ$ phase margin. Thus, all of the points in $S_\gamma$ guarantee such good robustness.

In fact, since the open loop transfer function $P(s)C(s)$ is strictly proper, the Nyquist plot of $P(j\omega)C(j\omega)$ goes to 0 as $\omega \to \infty$ and so every point in $S_\gamma$ achieves the same $H_\infty$ norm.

**Time response considerations**

So far we have discussed stability and robustness. However, the design of a controller should pay attention to the time response considerations. In order to demonstrate, we chose the following three design points:
The Nyquist plots in Fig. 16 confirm that all three design points satisfy the robustness condition. The step responses in Fig. 17 shows that the three controller designs result in different time responses in terms of overshoot and settling time. While $C_1(s)$ and $C_2(s)$ have highest and intermediate integral gains, $C_3(s)$ provides much shorter settling time and lesser overshoot than the other two controllers do.

The integrator in the controller provided zero steady state error and we found all stabilizing controllers achieving prescribed $H_\infty$ norm of the error transfer function. While the robustness and zero steady state error could be achieved by the proposed method, one should also consider the quality of the transient response when tuning the PID parameters within the set $S_\gamma$. Thus, the PID controller design for better transient response within the same degree of robustness is an important area of research.

6. CONCLUDING REMARKS

The results of this paper could be extended to discrete time and time-delay systems. Another important area of research is the extension of these results to multivariable systems.

REFERENCES


