Nonlinear fractional PI control of a class of fractional-order systems

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Abstract: This paper deals with the design of nonlinear PI control techniques for regulating a class of fractional-order dynamics governed by a commensurate-order model, possibly nonlinear, perturbed by an external disturbance. The suggested control algorithm is the combination between a fractional-order PI controller and a nonlinear robust version of it, namely a second-order sliding mode control algorithm called "super-twisting" controller in the literature. A key feature of the approach is the use of ad-hoc sliding manifolds whose construction involves fractional order derivatives. A constructive Lyapunov based synthesis is illustrated, which leads to simple tuning rules for the controller parameters guaranteeing the asymptotic rejection of the external disturbance under appropriate smoothness restrictions. Computer simulations illustrate the effectiveness of the proposed technique.

Keywords: Fractional order systems. Fractional order controllers. Nonlinear PI control. Sliding mode control.

1. INTRODUCTION

Fractional-order systems (FOSs), i.e. dynamical systems described using fractional (or, more precisely, non-integer) order derivative and integral operators, are studied with growing interest in recent years. It has been pointed out that a large number of physical phenomena can be modeled effectively by means of fractional-order models (see Sabatier et al. (2007)).

The long-range temporal or spatial dependence phenomena inherent to the FOS present unique and intriguing peculiarities, not supported by their integer-order counterpart, which raise numerous challenges and opportunities related to the development of control and estimation methodologies involving fractional order dynamics (see Vinagre et al. (2002); Ladaci et al. (2006); Podlubny (1999a)).

The pioneering applications of fractions calculus in control theory date back to the sixties (see Manabe (1961)). In the nineties, Oustaloup and his group proposed a non-integer robust control strategy named CRONE (Commande Robuste d’Ordre Non-Entier) (see Oustaloup et al. (1996)). Another well-known fractional control algorithm is the fractional-order PID (FPID, or $PI^\lambda D^\mu$) controller introduced by Podlubny (see Podlubny (1999a,b)). Recently, fractional calculus is penetrating other nonlinear control paradigms as well such as the model-reference adaptive control (see Vinagre et al. (2002); Ladaci et al. (2006)).

It is the task of this paper to study the properties of a control scheme for FOSs that combines together a fractional PI controller and a nonlinear version of it. The latter is a Sliding Mode Control (SMC) algorithm called "super-twisting" controller in the literature (see Levant (1993)).

Although fractional calculus has been previously combined with the sliding mode control methodology in the controller design for integer-order systems (see Efe et al. (2008); Caldero et al. (2006)), SMC techniques have been applied to fractional-order systems only recently, (see Si-Ammour et al. (2009); Efe (2009)). In Efe (2009), nonlinear single-input fractional-order dynamics expressed in a form that can be considered as a fractional-order version of the chain-of-integrators “Brunowsky” normal form were studied, which will be the class of reference in this work, too. Noticeably, sliding manifolds containing fractional-order derivatives were used in Si-Ammour et al. (2009) in combination with conventional relay control techniques. The same type of sliding manifolds has been later used, in combination with second-order sliding mode control methodologies, to address control, observation and fault detection tasks for certain classes of uncertain linear FOS (see Pisano et al. (2010, 2011)).

In this paper we consider a class of nonlinear FOSs expressed in the previously mentioned chain-of-integrators

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form and we investigate the stability properties of a control scheme combining the fractional PI controller and a fractional nonlinear PI algorithm. Unlike the standard stand-alone PI controller, which can asymptotically reject constant disturbances only, this combined scheme proves to be capable of asymptotically rejecting a nonvanishing disturbance of arbitrary shape and fulfilling the unique constraint that its time derivative as uniformly bounded by an a-priori known constant starting from some finite time instant on. Convergence to zero of the system variables, and asymptotic rejection of a class of matched disturbances, will be demonstrated by means of a Lyapunov approach.

The paper is structured as follows: in Section II the main definitions and properties of fractional order derivatives and integrals are recalled, with emphasis on their composition which plays an important role in the present developments. Section III states the control problem under investigation and presents the main result, namely the Lyapunov based stability analysis of the combined linear/nonlinear PI controller in question. Section IV presents some computer simulations, and the final Section V gives some concluding remarks and perspectives for next related research activities.

2. FRACTIONAL OPERATORS AND THEIR PROPERTIES

**Definition 1.** (Left) Riemann-Liouville fractional integral of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}d\tau,$$  \hspace{1cm} (1)

where $\Gamma(\cdot)$ denotes the Euler gamma function (see Kilbas et al. (2006)).

For integer values of $\alpha$, relation (1) reduces to the well-known Cauchy repeated integration formula (see Podlubny (1999a)). It can also be shown that when $\alpha$ approaches zero the fractional integral (1) reduces to the identity operator (see Saichev et al. (1996)). In the current paper, the fractional integral of order zero is taken to represent the identity operator by definition, i.e.

$$I^0 f(t) = f(t).$$  \hspace{1cm} (2)

**Definition 2.** (Left) Riemann-Liouville fractional derivative of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as the $n$th derivative of the left Riemann-Liouville fractional integral of order $n - \alpha$, where $n$ is the smallest integer greater than, or equal to, $\alpha$

$$RL D^\alpha f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t).$$  \hspace{1cm} (3)

**Definition 3.** (Left) Caputo fractional derivative of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as the left Riemann-Liouville fractional integral of order $n - \alpha$ of the $n$th derivative of $f$, where $n$ is the smallest integer greater than, or equal to, $\alpha$

$$C D^\alpha f(t) = I^{n-\alpha} \left(\frac{d}{dt}\right)^n f(t).$$  \hspace{1cm} (4)

For $\alpha \in (0, 1)$ the Riemann-Liouville and Caputo derivatives are related by the next equation, that will be used in the sequel

$$RL D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{f(0)}{t^\alpha} + C D^\alpha f(t)$$  \hspace{1cm} (5)

Since the fractional integral of order zero is the identity operator, in accordance with (2), it is obvious that both definitions of fractional derivative reduce to the classical derivative of order $n$ when $\alpha = n$. Particularly, when the differentiation order is zero, both definitions of fractional derivatives reduce to the identity operator. The next useful properties of the fractional integral and differential operators will be used in the sequel. The proofs can be found in a number of well-known textbooks (see e.g. (Kilbas et al. (2006)) and (Podlubny (1999a))).

**Lemma 1.** The left Riemann-Liouville fractional integral satisfies the semigroup property. Let $\alpha > 0$ and $\beta > 0$, then

$$I^\alpha \{I^\beta f(t)\} = I^{\alpha+\beta} f(t)$$  \hspace{1cm} (6)

**Lemma 2.** The left Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ is the left inverse of the left Riemann-Liouville fractional integral of the same order,

$$RL D^\alpha \{I^\alpha f(t)\} = f(t),$$  \hspace{1cm} (7)

for almost all $t \geq 0$. The opposite is, however, not true, since

$$I^\alpha \{RL D^\alpha f(t)\} = f(t) - \frac{f_{1-\alpha}(0)}{\Gamma(\alpha)} t^{\alpha-1},$$  \hspace{1cm} (8)

where $f_{1-\alpha}(0) = \lim_{t\to0} t^{1-\alpha} f(t)$.

**Lemma 3.** The following is true when applying left Riemann-Liouville fractional integral operator to the left Caputo fractional derivative of the same order of a signal $f(t)$

$$I^\alpha \{C D^\beta f(t)\} = f(t) - f(0).$$  \hspace{1cm} (9)

It is important to notice that, unlike the classical derivative, the fractional derivative operators do not commute. In general, in fact, one has that

$$RL D^\alpha \{RL D^\beta f(t)\} \neq RL D^{\alpha+\beta} f(t),$$

$$C D^\alpha \{C D^\beta f(t)\} \neq C D^{\alpha+\beta} f(t),$$

However, the following equalities hold true for all $\alpha > 0$ and $n \in \mathbb{N}$

$$\frac{d^n}{dt^n} \{RL D^\alpha f(t)\} = RL D^{n+\alpha} f(t),$$  \hspace{1cm} (10)

$$C D^\alpha \{\frac{d^n}{dt^n} f(t)\} = C D^{n+\alpha} f(t).$$  \hspace{1cm} (11)

The next Lemma, that will be instrumental in the present treatment, was proven in (Pisano et al. (2010)).

**Lemma 4.** Consider an arbitrary signal $z(t) \in \mathbb{R}$. Let $\beta \in (0, 1)$. If there exists $T < \infty$ such that

$$\beta z(t) = 0 \quad \forall t \geq T$$  \hspace{1cm} (12)

then

$$\lim_{t\to\infty} z(t) = 0.$$  \hspace{1cm} (13)
3. NONLINEAR FRACTIONAL PI CONTROL FOR SISO FOS

We consider nonlinear uncertain commensurate-order fractional systems governed by the “chain of (fractional) integrators” dynamic model

\[
\begin{align*}
C D^{\alpha} x_1 &= x_2 \\
C D^{\alpha} x_2 &= x_3 \\
& \vdots \\
C D^{\alpha} x_{n-1} &= x_n \\
C D^{\alpha} x_n &= f(x, t) + u(t) + \psi(t).
\end{align*}
\] (14)

where \( \alpha \in (0, 1) \) is the commensurate order of differentiation of (14), vector \( x(t) = [x_1(t), x_2(t), ..., x_n(t)] \in \mathbb{R}^n \) collects the process internal variables (the notion of state variables is inappropriate and generally not used in the context of FOS), \( u(t) \in \mathbb{R} \) is the control input, \( \psi(t) \in \mathbb{R} \) is an exogenous disturbance, and \( f(x, t) : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R} \) is a nonlinear functions referred to as the “drift term”.

We refer to the Caputo definition of fractional derivatives as it allows to take into account a finite and physically reasonable initial condition \( x(0) \) for the process internal variables.

The external disturbance \( \psi(t) \) is supposed to fulfill the next restriction

**Assumption 1.** The exists an a-priori known constant \( M \) such that

\[
\left| \frac{d}{dt} \psi(t) \right| \leq M, \quad t \geq 0.
\] (15)

The aim is that of finding a control law capable of steering the variables of the closed loop process to the origin regardless of the presence of the unknown disturbance term \( \psi(t) \), satisfying the Assumption 1.

Consider the fractional order sliding variable

\[
\sigma(t) = I^{(1-\alpha)} \left[ x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t) \right],
\] (16)

where the constants \( c_1, c_2, ..., c_{n-1} \) are selected in such a way that all the roots \( p_i \) of the polynomial

\[
P(s) = s^{(n-1)} + \sum_{i=0}^{n-2} c_{i+1} s^i = \Pi_{i=1}^{n-1} (s - p_i)
\] (17)

satisfy the next relation

\[
\frac{\alpha \pi}{2} < \arg(p_i) \leq \pi.
\] (18)

The stability of system (14) once constrained to evolve along the sliding manifold \( \sigma(t) = 0 \) is analyzed in the next Lemma 5. A controller capable of steering the considered dynamics onto the sliding manifold in finite time will be illustrated later on.

**Lemma 5.** Consider system (14) and let the zeroing of the sliding variable (16) be fulfilled starting from the finite moment \( t_1 \), i.e. let

\[
\sigma(t) = 0, \quad t \geq t_1, \quad t_1 < \infty,
\] (19)

with the \( c_i \) parameters in (16) satisfying (17)-(18). Then, the next conditions hold

\[
\lim_{t \to \infty} x_i(t) = 0, \quad i = 1, 2, ..., n
\] (20)

**Proof of Lemma 5** Define the quantity

\[
\xi(t) = x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t).
\] (21)

By taking into account Lemma 4 specialized with \( \beta = 1-\alpha \) and \( z(t) = \xi(t) \), it yields that the finite time zeroing of \( \sigma(t) \) guarantees that signal \( \xi(t) \) decays asymptotically to zero. We then simply derive from (21) that

\[
x_n(t) = -\sum_{i=1}^{n-1} c_i x_i(t) + \xi(t)
\] (22)

and notice that (24) form a reduced-order (as compared to (14)) fractional order system with an asymptotically decaying input term \( \xi(t) \). It readily follows from (17)-(18) that system (24) is Mittag-Leffler stable when \( \xi(t) = 0 \) (see Podlubny (1999a)), thereby the input decay property (23) implies the same for the process variables \( x_i(t) \) with \( i = 1, 2, ..., n-1 \). We now conclude from (22) that \( x_n(t) \) asymptotically decays, too. Lemma 5 is proved. \( \square \)

It is worth to remark that the enforcement of conditions (21), (23) actually “cancels” the last equation of (14) as

\[
\begin{align*}
C D^{\alpha} x_1 &= x_2 \\
C D^{\alpha} x_2 &= x_3 \\
& \vdots \\
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The stability of system (14) once constrained to evolve along the sliding manifold \( \sigma(t) = 0 \) is analyzed in the next Lemma 5. A controller capable of steering the considered dynamics onto the sliding manifold in finite time will be illustrated later on.

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It is worth to remark that the enforcement of conditions (21), (23) actually “cancels” the last equation of (14) by making the system to behave as the reduced order one (24). We shall treat the sliding variable \( \sigma(t) \) in (16) as the error variable, and we seek for a control law expressed in the form

\[
u(t) = w^q(t) + u^q(t) + u^eq(t)
\] (25)

where \( w^q(t) \) and \( u^q(t) \) are, respectively, nonlinear versions of the proportional and integral control actions taking the form

\[
u^q(t) = -k_1 \sigma - k_3 |\sigma|^{1/2} \text{sign} \sigma
\] (26)

\[
u^eq(t) = -k_3 \sigma - k_4 \text{sign} \sigma, \quad u^eq(0) = 0
\] (27)

and \( w^eq(t) \) is a control component that will be specified later on. By setting constants \( k_3 \) and \( k_4 \) to zero then the sum of the control components (26)-(27) reduces to the standard PI controller. On the other hand, by setting \( k_1 \) and \( k_3 \) to zero one obtains the well-known “super-twisting” controller (see Levant (1993)), which belongs to the family of second-order sliding mode controllers. The similarity
Theorem 1. Consider system (14) along with the sliding variable (16)-(18), and let Assumption 1 be in force. Then, the control law (25)-(27), specified with variable (16)-(18), and let Assumption 1 be in force. Then, Theorem 1.

We are now in position to state the next main result.

Theorem 1. Consider system (14) along with the sliding variable (16)-(18), and let Assumption 1 be in force. Then, the control law (25)-(27), specified with variable (16)-(18), and let Assumption 1 be in force. Then, Theorem 1.

where $\rho > M$, (31) provides the asymptotic decay of the state $x(t)$.

Proof of Theorem 1 By virtue of Definition 2, specified with $n = 1$ and $f(t) = x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t)$, and exploiting as well the linearity of the fractional derivative operator, one can easily derive that

$$
\frac{d}{dt}\sigma(t) = RL D^\alpha x_n(t) + \sum_{i=1}^{n-1} c_i RL D^\alpha x_i(t) = RL D^\alpha x_n(t) + \sum_{i=1}^{n-1} c_i RL D^\alpha x_i(t)
$$

(32)

In light of relation (5), eq. (32) can be rewritten in terms of Caputo derivatives as follows

$$
\frac{d}{dt}\sigma(t) = C D^\alpha x_n(t) + \sum_{i=1}^{n-1} c_i C D^\alpha x_i(t) + \varphi(t)
$$

(33)

where

$$
\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{x_n(t)}{t^{\alpha}} + \sum_{i=1}^{n-1} c_i x_i(t) = \frac{K_0}{t^{\alpha}}
$$

(34)

with implicitly defined constant $K_0 = \frac{x_n(0) + \sum_{i=1}^{n-1} c_i x_i(0)}{\Gamma(1-\alpha)}$.

The system equations (14) can be now substituted into (33), yielding the simplified expression

$$
\dot{\sigma}(t) = f(x, t) + u(t) + \psi(t) + \sum_{i=1}^{n-1} c_i x_{i+1}(t) + \varphi(t)
$$

(35)

Although the disturbance (34) and all its time derivatives are unbounded at $t = 0$, one has that the first-order time derivative

$$
\frac{d}{dt}\sigma(t) = -\frac{\alpha K_0}{t^{\alpha+1}}
$$

(36)

is bounded, in magnitude, along any time interval $t \in [t_1, \infty)$, $t_1 > 0$, according to

$$
\frac{d}{dt}\sigma(t) \leq \frac{\alpha K_1}{t^{\alpha+1}} \equiv \Psi_1, \quad K_1 = \left|x_n(0) + \sum_{i=1}^{n-1} c_i x_i(0)\right|
$$

(37)

We now substitute the control (25)-(28) into (35), yielding

$$
\frac{d}{dt}\sigma = -k_1 \sigma - k_2 |\sigma|^{1/2} \text{sign} (\sigma) + u'(t) + \psi(t) + \alpha(t)
$$

(38)

Define

$$
z(t) = u'(t) + \psi(t) + \alpha(t)
$$

(39)

and rewrite (38)-(39) as

$$
\frac{d}{dt}\sigma = -k_1 \sigma - k_2 |\sigma|^{1/2} \text{sign} (\sigma) + z(t)
$$

(40)

$$
\frac{d}{dt}z = -k_3 \sigma - k_4 \text{sign}(\sigma) + \frac{d}{dt}\psi(t) + \frac{d}{dt}\alpha(t)
$$

(41)

Notice that, by Assumption 1 and by relation (37), the perturbation terms in (39) fulfill the next estimation

$$
\left|\frac{d}{dt}\psi(t) + \frac{d}{dt}\varphi(t)\right| \leq M + \Psi_1, \quad t \geq t_1 > 0
$$

(42)

with arbitrary $t_1 > 0$. Thus, by setting $\rho$ as in (31) it readily follows that there exist a finite moment $t_2 > 0$ such that $|\frac{d}{dt}\psi(t) + \frac{d}{dt}\varphi(t)| \leq \rho$ at every $t \geq t_2$.

Stability of a class of systems including the dynamics (41)-(43) above was already investigated in the literature (cfr. Moreno et al. (2008), Th. 5), where, particularly, the global finite time stability of the uncertain system trajectories was demonstrated by means of a positive definite and radially-unbounded non-smooth Lyapunov function which specifies as follows in the present context

$$
V = \xi^T \Pi \xi, \quad \xi = \begin{bmatrix} |\sigma|^{1/2} \text{sign}(\sigma) \\ \sigma \\ z \end{bmatrix},
$$

(43)

$$
\Pi = \frac{1}{2} \begin{bmatrix} 4k_1 k_2 & k_1 k_2 - k_2 & -k_2 \\ k_1 k_2 & 2k_3 + k_4^2 - k_1 & -k_1 \\ -k_2 & -k_1 & 2 \end{bmatrix}.
$$

(44)

It turns out after the appropriate computations (cfr. Moreno et al. (2008), Proof of Th. 5) that the tuning conditions (29)-(31) imply the existence of a positive constant $\gamma_1$ such that

$$
\frac{d}{dt}V \leq -\gamma_1 \sqrt{V}, \quad t \geq t_1.
$$

(45)

Inequality (46) guarantees the global finite time convergence of $V$ to zero, and, hence, the same property...
for the $\sigma(t)$ and $z(t)$ variables. By (38), the finite time convergence to zero of $\frac{d}{dt}\sigma(t)$ can be easily concluded, too. The asymptotic decay of $x(t)$, thus, readily follows from Lemma 4. Theorem 1 is proven. □.

3.1 Uncertain drift term

Now assume that the uncertain drift term $f(x, t)$ is imprecisely known by means of a certain estimate $\hat{f}(x, t)$. We shall devise sufficient condition guaranteeing that the previously presented control, with the estimate $\hat{f}(x, t)$ used in (28) in place of the actual function $f(x, t)$, guarantees the same robust performance demonstrated in Theorem 1. Denote

$$\epsilon(x, t) = f(x, t) - \hat{f}(x, t)$$

and assume what follows

**Assumption 2** There is an a-priori fixed constant $W$ such that

$$\left| \frac{d}{dt}\epsilon(x, t) \right| \leq W$$

Under Assumption 2, it can be developed a synthesis procedure similar to that in Theorem 1, which leads to the same controller and tuning inequalities (29)-(30) for the procedure similar to that in Theorem 1, which leads to the same controller and tuning inequalities (29)-(30) for the uncertain drift term $f(x, t)$ used in (28) in place of the actual function $f(x, t)$, guarantees the same robust performance demonstrated in Theorem 1.

4. SIMULATION RESULTS

Consider system

$$C D^{0.5} x_1 = x_2$$

$$C D^{0.5} x_2 = x_3$$

with nonlinear drift term function $f(x, t) = -0.5x_1 - 0.5x_2^2 - 0.5x_3^2$ and a sinusoidal time dependent matched uncertainty. This system was considered in the related publication (Efe (2009)). Constant $M$ upperbounding the sinusoidal uncertainty time derivative according to Assumption 1 can be evaluated as $M = \pi$. The initial conditions are $x_1(0) = x_2(0) = x_3(0) = 2$. Let us bear in mind that the Caputo definition of the fractional derivative in (49) allows to take into account finite and physically meaningful initial conditions of the process variables, in opposition to what happens with the RL definition which brings infinite values for the initial conditions.

The fractional order sliding variable is defined according to (16) as

$$\sigma(t) = f^{0.5} [x_3(t) + c_2x_2(t) + c_1x_1(t)]$$

An effective choice for the $c_1$ and $c_2$ constants which guarantees conditions (17)-(18) is as follows

$$c_1 = \lambda^2, \quad c_2 = 2\lambda, \quad \lambda > 0$$

and, particularly, is such that $p_1 = p_2 = -\lambda$.

The perfect knowledge of the drift term function $f(x, t)$ is assumed in the first TEST 1, and control (25)-(31), (50)-(51) has been implemented with the parameter values $\lambda = 10$, $k_1 = 10$, $k_2 = 10$, $k_3 = 15$, $k_4 = 15$.

Figure 2 shows the time evolutions of the sliding variable $\sigma(t)$ and of signal $\xi(t) = x_3(t) + 2\lambda x_2(t) + \lambda^2 x_1(t)$. Particularly, the left plots show the entire history of the signals while the right plots display a zoom of the corresponding steady-state behaviour. The upper plots of Figure 2 then confirm the finite-time convergence to zero of the chosen sliding variable, while the lower plots show that, according to Lemma 4, signal $\xi(t)$ goes to zero once the system motion is constrained along the sliding manifold $\sigma(t) = 0$ featuring the slow “creeping” behaviour exhibited by fractional order dynamics (see Podlubny (1999a)).

Figure 3 shows the time evolutions of the process variables, and Figure 4 displays the control variable which, as expected, is a continuous, although non smooth function of time.

![Fig. 2. TEST 1. Fractional sliding variable $\sigma(t)$ and signal $\xi(t)$.](image)
A nonlinear PI control technique has been suggested and analyzed in the framework of the regulation problem for a class of nonlinear fractional-order processes. The proposed methodology is capable of asymptotically rejecting a class of arbitrarily shaped external disturbances with uniformly bounded time derivative. Distinctive issue of the approach is a fractional order sliding surface tailored to the considered class of systems. More complex tracking control problems and more general classes of plants will be investigated in the future. Additionally, new types of sliding surfaces will be sought to speed up the convergence of the process variables, which actually suffer of the “creeping” effect, namely an extremely slow convergence to the desired operating point.

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