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Output Regulation for a Class of Parameter Uncertain Systems

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Abstract: The problem of robust output regulation is studied for a class of parameter uncertain systems under unity output feedback control. The objective is tracking of the desired reference trajectory in the presence of the disturbance, both generated by a common exosystem. Because of the anti-stability of the exosystem and potentially unbounded reference trajectory, the output tracking error is employed as the measurement signal and used as the input to the feedback controller. The method of p-copy of the internal model is utilized to augment the plant dynamics. Assuming that the output regulation condition is satisfied for all the parameter uncertainties, it is shown that the problem of robust output regulation is equivalent to the problem of robust output stabilization. Furthermore for quadratically bounded parameter uncertainties, an application of the notion of the quadratic stability leads to $\mathcal{H}_\infty$ based robust control, and the maximum allowable uncertainty bound can be computed, below which the robust output regulation can be achieved.

Keywords: Linear systems, output regulation, uncertain dynamic systems, quadratically bounded uncertainty, robust stabilizability

1. INTRODUCTION

Output regulation is concerned with output tracking of the reference trajectory in the presence of the disturbance, of which both the reference and disturbance are deterministic, and generated by a known exosystem. This is a well-studied subject, and it is well-documented in the research literature. The internal model principle Francis and Wonham (1976); Francis (1977) is now well understood for finite-dimensional linear time-invariant systems, and has been extended to nonlinear systems Huang (2004); Khalil (2001). In recent years this subject has received renewed interest, motivated by the consensus control for multi-agent control systems. However the robustness of the output regulation is much less studied and understood. Indeed this problem is first tackled in Davison (1976) via the expanded plant model and multi-loop feedback control to achieve robust stabilization for the parameter uncertain plant through design of the inner-loop controller, and output regulation by design of the outer-loop controller. This decoupling approach is followed in Scherer et al. (1997), but develops an LMI algorithm for design of the feedback controller to cope with the model uncertainty and achieve output regulation. See also the alternative filtering approach to the output regulation in Feng and Yagoubi (2016); Meinsma (1995). While the design procedures and numerical computation methods proposed in Davison (1976); Scherer et al. (1997) are effective, there are three potential problems. The first is the lack of quantitative results for robust output regulation, even though the equivalence to robust stabilization is shown in Davison (1976). The second is high complexity of the feedback controllers that involve addition of the internal model in a complicated way, and consisting interconnected controllers with increased number of input and output due to the expanded plant model proposed in Davison (1976). The third is the direct use of the plant output by the feedback controller, which is potentially unbounded. Recall that all modes of the exosystem are unstable. The problems of the controller complexity and processing potentially unbounded signals by the feedback controller are addressed in Knobloch et al. (1993) by using only the tracking error as the measurement signal and as the input to the feedback controller. The method of p-copy internal model is proposed, which is a lot simpler than that of Davison (1976); Scherer et al. (1997), and which is shown to achieve output regulation in a small neighborhood of the uncertain parameter set. Unfortunately it also lacks the quantitative results for the robust output regulation, similar to those reported in Davison (1976); Scherer et al. (1997). For more results on robust output regulation and the related researches, see Lu and Doyle (1997); Fan and Doyle (1991); Anderson et al. (2008); Nagy and Braatz (2003); Schmidt et al. (2012); Yu et al. (2016).

Motivated by the existing work in the robust output regulation, we study a class of parameter uncertain systems in which the state space realization involves parameter uncertainties. Our goal is to characterize the maximum radius of the parameter uncertainty set of the plant model below which the robust output regulation is achievable. In the same problem setting as in Knobloch et al. (1993), we have a unity output feedback control system in which the tracking error is the input to the feedback controller that avoids processing potentially unbounded signal. As it is widely known, the output regulation requires the solvability of the output regulation equation (ORE). This output regulation condition is assumed in Knobloch et al. (1993) for only the nominal model, which is the root for its lack of quantitative results on the robust output regulation. By assuming the satisfaction of the output regulation condition for all the uncertain parameters, and by employing the same p-copy internal model as in Knobloch et al. (1993), it will be shown that the robust output regulation is equivalent to the robust output stabilization. If in addition the realization of the augmented plant model (by the p-copy of the internal model) involves the quadratically bounded uncertainty, the notion of the quadratic stability Barmish (1985); Zhou and Khargonehr (1988) can be adopted to convert the robust output stabilization into $\mathcal{H}_\infty$ control, and to obtain the maximum radius of the parameter uncertainty set below which the robust output regulation is achievable. The known results from Glover and Doyle (1988); Green and Limebeer (1995); Zhou et al. (1996) can be used to synthesize the robust output regulation controller.

Our paper is organized as follows. After the introduction section, the problem of robust output regulation will be described, and a preliminary result will be presented in Section II. The ORE will be studied in Section III: Under the augmentation of the p-copy of the internal model, it will be shown that the solution to the augmented ORE has a special form that helps to convert the robust output regulation into the robust output stabilization; A design procedure will be developed to synthesize the unity output feedback controller that achieves the output regulation in the presence of the uncertain parameters. In Section IV, a simulation example will be used to illustrate the proposed design procedure. The paper will be concluded in Section V.

The notation in this paper is fairly standard with $\mathbb{R}^N/\mathbb{C}^N$ for N-dimensional real/complex spaces. For a matrix A, $\overline{\sigma}(A)$ denotes its maximum singular value. For a transfer matrix $G(s)$, its $\mathcal{H}_\infty$ norm is defined by

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\mathrm{Re}[s]>0} \overline{\sigma}[G(s)].$$

Other notations will be made clear as we proceed.

2. PRELIMINARIES

We study the problem of robust output regulation for a linear parameter uncertain system described by

$$\begin{aligned} \dot{x}(t) &= A_\theta x(t) + B_\theta u(t) + d(t), \qquad d(t) = B_{0\theta} x_0(t), \\ \dot{x}_0(t) &= A_0 x_0(t), \qquad x_0(0) = x_{00} \neq 0, \\ e_y(t) &= C_\theta x(t) + D_\theta u(t) - C_0 x_0(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $x_0(t) \in \mathbb{R}^{n_0}$ is the reference state, and $u(t) \in \mathbb{R}^m$ is the control input. The system output and desired reference output are given by

$$y(t) = C_\theta x(t) + D_\theta u(t), \quad y_0(t) = C_0 x_0(t),$$

respectively. Thus the tracking error

$$e_y(t) = y(t) - y_0(t) \ \in \ \mathbb{R}^p.$$

Realization matrices $(A_\theta, B_\theta, B_{0\theta}, C_\theta, D_\theta)$ are functions of the parameter vector $\theta \in \mathcal{S}_\theta$ with $\theta = 0$ being the nominal one and $\mathcal{S}_\theta$ the given uncertainty set. For convenience we use

$$P_\theta(s) = \left[\begin{array}{c|c} A_\theta & B_\theta \\ \hline C_\theta & D_\theta \end{array}\right] := D_\theta + C_\theta (sI - A_\theta)^{-1} B_\theta \tag{2}$$

to represent the plant model. The block diagram of the feedback system is illustrated in Fig. 1 below.

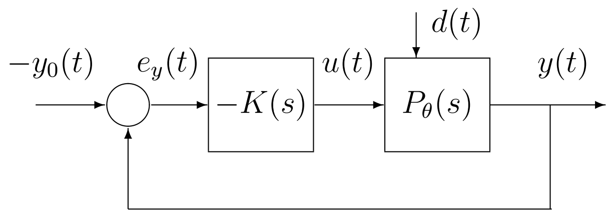

Fig. 1 Block diagram for output regulation

The objective in this paper is design of the feedback controller $K(s)$ that achieves not only the internal stability of the closed-loop system, but also zero tracking error asymptotically in the presence of the parameter uncertainties. This problem is termed robust output regulation, and is studied in Knobloch et al. (1993), assuming $D_\theta = 0$. It proves the robust output regulation in a small neighborhood of the parameter set without quantitative results. Under a rather different problem formulation, our paper is aimed at deriving a quantitative result for quadratically bounded parameter uncertain systems. Prior to tackling the robust output regulation in the next two sections, we provide a preliminary result that will be useful later.

Let $J_\ell(\lambda) \in \mathbb{R}^{\ell\times\ell}$ be the Jordan matrix with a single real eigenvalue λ. It has the form:

$$J_\ell(\lambda) = \begin{bmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix} \in \mathbb{R}^{\ell\times\ell}. \tag{3}$$

For a lower triangle and square Toeplitz matrix given by

$$T_{g_\ell} = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ g_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ g_\ell & \cdots & g_2 & g_1 \end{bmatrix} \in \mathbb{R}^{\ell\times\ell}, \tag{4}$$

there holds $J_\ell(\lambda)T_{g_\ell} = T_{g_\ell}J_\ell(\lambda)$ due to the commutative property of the lower triangle and square Toeplitz matrices. If the eigenvalue is complex, it has a block Jordon form:

$$J_\ell(\Lambda) = \begin{bmatrix} \Lambda & & & \\ I_2 & \ddots & & \\ & \ddots & \ddots & \\ & & I_2 & \Lambda \end{bmatrix} \in \mathbb{R}^{2\ell\times 2\ell},$$

$$\Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

In this case we can define a block Toeplitz matrix

$$T_{G_\ell} = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ G_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ G_\ell & \cdots & G_2 & G_1 \end{bmatrix} \in \mathbb{R}^{2\ell \times 2\ell}, \tag{5}$$

$$G_i = \begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Since $G_i\Lambda = \Lambda G_i$, there holds $T_{G_\ell}J_\ell(\Lambda) = J_\ell(\Lambda)T_{G_\ell}$. The following result is true and useful.

Lemma 1. (a) Consider $C_\ell = [\, c_1 \cdots c_\ell \,] \in \mathbb{R}^{1\times \ell}$ and $A_\ell = J_\ell(\lambda)$. The pair (C_ℓ, A_ℓ) is observable, if and only if $c_\ell \neq 0$. Suppose that $C_\ell \in \mathbb{R}^{1\times 2\ell}$ with $c_i' \in \mathbb{R}^2$ for each i, and $A_\ell = J_\ell(\Lambda)$ with $\beta \neq 0$. Then (C_ℓ, A_ℓ) is observable, if and only if $c_\ell \neq 0$.

(b) For any given $\{x_i\}_{i=1}^\ell$ and $\{y_i\}_{i=1}^\ell$, there exist $\{g_i\}_{i=1}^\ell$ satisfying

$$[\, y_1 \cdots y_\ell \,] = [\, x_1 \cdots x_\ell \,]\, T_{g_\ell},$$

if and only if $x_\ell \neq 0$. Suppose that $x_i', y_i' \in \mathbb{R}^2$ for each i. Then there exist $\{G_i\}_{i=1}^\ell$ with G_i in the same form as in (5) satisfying

$$[\, y_1 \cdots y_\ell \,] = [\, x_1 \cdots x_\ell \,]\, T_{G_\ell},$$

if and only if $x_\ell \neq 0$.

The proof is straightforward, and thus omitted.

3. AUGMENTATION WITH P-COPY OF INTERNAL MODEL

Denote the set of eigenvalues of A_0 by $\sigma(A_0)$. It is well known that the problem of robust output regulation is solvable, if and only if for each $\theta \in \mathcal{S}_\theta$, there holds

$$\text{rank}\left\{ \begin{bmatrix} \lambda I - A_\theta & B_\theta \\ C_\theta & D_\theta \end{bmatrix} \right\} = n + p \ \ \forall\, \lambda \in \sigma(A_0), \tag{6}$$

assuming $p \leq m$. This is in turn equivalent to that output regulation equation (ORE)

$$\begin{aligned} \text{(i)}\ & \Pi_\theta A_0 = A_\theta \Pi_\theta + B_\theta \Psi_\theta + B_{0\theta}, \\ \text{(ii)}\ & C_0 = C_\theta \Pi_\theta + D_\theta \Psi_\theta, \end{aligned} \tag{7}$$

admits a pair of solutions $(\Pi_\theta, \Psi_\theta)$ for each $\theta \in \mathcal{S}_\theta$. The following assumption is made:

Assumption a) The pair (A_θ, B_θ) is stabilizable, (C_θ, A_θ) is detectable, (C_0, A_0) is observable, and the ORE (7) admits a pair of solution $(\Pi_\theta, \Psi_\theta)$ for each $\theta \in \mathcal{S}_\theta$.

Remark 1. Assumption a) differs from Knobloch et al. (1993) that requires only the existence of a solution pair to ORE (7) for some nominal $\theta_0 \in \mathcal{S}_\theta$. However Assumption a) is not any stronger, because of the equivalence of the feasibility of ORE (7) to the feasibility of the output regulation at each $\theta \in \mathcal{S}_\theta$. While verification of Assumption a) is much harder than that in Knobloch et al. (1993), the main result of the next section will provide an easy way to check Assumption a) for quadratically bounded uncertainty sets by utilizing the notion of quadratic stability and stabilization, and the equivalence of the output regulation and feedback stabilization under the p-copy of the internal model. □

A known approach to the robust output regulation is the augmentation with p-copy of the internal model. Specifically let the internal model be represented by its transfer function

$$w(s) = \frac{\beta(s)}{\alpha(s)} := \beta_0 + \frac{\beta_1 s^{n_0-1} + \cdots + \beta_{n_0}}{s^{n_0} + \alpha_1 s^{n_0-1} + \cdots + \alpha_{n_0}}$$

where $\alpha(s) = \det(sI - A_0)$, and $\beta(s)$ is a Hurwitz polynomial. We assume that $\beta(s)$ and $\alpha(s)$ are coprime, and thus there exist $\phi_0', \xi_0 \in \mathbb{R}^{n_0}$ such that

$$w(s) = \frac{\beta(s)}{\alpha(s)} = \beta_0 + \phi_0(sI - A_0)^{-1}\xi_0 \tag{8}$$

in which (A_0, ξ_0) is controllable, and (ϕ_0, A_0) is observable.

Let $W_k(s) = w(s)I_k$ for $k = p$ or $k = m$. The controller $K(s)$ in Fig. 1 is set as

$$K(s) = K_{\rm a}(s)W_p(s) = W_m(s)K_{\rm a}(s).$$

The loop transfer matrix is now given by

$$\begin{aligned} P_\theta(s)K(s) &= P_{{\rm a}_\theta}(s)K_{\rm a}(s), \\ P_{{\rm a}_\theta}(s) &= P_\theta(s)W_m(s) = W_p(s)P_\theta(s). \end{aligned}$$

Since $p \leq m$, we regard the above augmentation with $W_p(s)$ as p-copy of the internal model. Design of $K(s)$ is now converted equivalently into design of $K_{\rm a}(s)$ based on the augmented plant model $P_{{\rm a}_\theta}(s)$, and $K(s) = K_{\rm a}(s)W_p(s)$ will be implemented in the feedback system in Fig. 1.

Let $S_i \in \mathbb{R}^{n_0\times n_0}$ be nonsingular, and

$$A_{0i} = S_i A_0 S_i^{-1}, \qquad \xi_{0i} = S_i\xi_0, \qquad \phi_{0i} = \phi_0 S_i^{-1} \tag{9}$$

for $1 \leq i \leq m$. Then $w(s) = \beta_0 + \phi_{0i}(sI - A_{0i})^{-1}\xi_{0i}$. Denote $D_0 = I_m \otimes \beta_0$, and realization matrices of $W_m(s)$ as

$$\begin{aligned} \underline{A}_0 &= \text{diag}(A_{01}, \cdots, A_{0m}), \\ \Phi &= \text{diag}(\phi_{01}, \cdots, \phi_{0m}), \\ \Xi &= \text{diag}(\xi_{01}, \cdots, \xi_{0m}). \end{aligned} \tag{10}$$

Although $W_m(s) = w(s)I_m$, the rows of Φ may differ from each other; So do the columns of Ξ. Let $(A_{{\rm a}_\theta}, B_{{\rm a}_\theta}, C_{{\rm a}_\theta}, D_{{\rm a}_\theta})$ be the realization of

$$P_{{\rm a}_\theta}(s) = P_\theta(s)W_m(s).$$

Then

$$\begin{aligned} A_{{\rm a}_\theta} &= \begin{bmatrix} A_\theta & B_\theta\Phi \\ 0 & \underline{A}_0 \end{bmatrix}, \quad B_{{\rm a}_\theta} = \begin{bmatrix} B_\theta D_0 \\ \Xi \end{bmatrix}, \\ C_{{\rm a}_\theta} &= [\, C_\theta \ \ D_\theta\Phi \,], \quad D_{{\rm a}_\theta} = D_\theta D_0. \end{aligned} \tag{11}$$

Normally $\beta_0 = 0$ and thus $D_0 = 0$ is taken Knobloch et al. (1993). The following result is instrumental.

Theorem 1. Suppose that $p \leq m$ and the ORE in (7) admits a pair of solutions $(\Pi_\theta, \Psi_\theta)$ for each $\theta \in \mathcal{S}_\theta$. Then the ORE for the augmented plant model given by

$$\begin{aligned} \text{(ia)}\ \ & \Pi_{{\rm a}_\theta} A_0 = A_{{\rm a}_\theta}\Pi_{{\rm a}_\theta} + B_{{\rm a}_\theta}\Psi_{{\rm a}_\theta} + B_{0{\rm a}_\theta}, \\ \text{(iia)}\ \ & C_0 = C_{{\rm a}_\theta}\Pi_{{\rm a}_\theta} + D_{{\rm a}_\theta}\Psi_{{\rm a}_\theta}, \end{aligned} \tag{12}$$

admits a pair of solutions $(\Pi_{{\rm a}_\theta}, \Psi_{{\rm a}_\theta})$ satisfying $\Psi_{{\rm a}_\theta} = 0$ for each $\theta \in \mathcal{S}_\theta$ where $B_{0{\rm a}_\theta} = \left[\, B_{0\theta}' \ \ 0 \,\right]' \in \mathbb{R}^{(n+mn_0)\times n_0}$.

The observability of (C_0, A_0) is not assumed in Knobloch et al. (1993). As such, $\alpha(s) = \det(sI - A_0)$ in (8) cannot be used anymore; instead, $\alpha(s)$ needs to be the minimal polynomial of A_0 in order to ensure internal stability of the feedback system. On the other hand, the observability of (C_0, A_0) can always be assumed without loss of generality, because the Kalman canonical decomposition can be employed to remove the unobservable modes of (C_0, A_0). The next result converts the problem of robust output regulation into an equivalent robust stabilization problem.

Theorem 2. Suppose that $p \leq m$ and ORE in (7) admits a pair of solutions $(\Pi_\theta, \Psi_\theta)$, and thus the augmented ORE in (12) admits a pair of solutions $(\Pi_{a_\theta}, \Psi_{a_\theta})$ satisfying $\Psi_{a_\theta} = 0$ for each $\theta \in \mathcal{S}_\theta$. Then the feedback system in Fig. 1 achieves the output regulation for all $\theta \in \mathcal{S}_\theta$, if and only if it is robustly stabilized for all $\theta \in \mathcal{S}_\theta$.

Because of the page limit, the proofs of the two theorems are omitted. Interested readers can contact authors for the full version of this paper.

4. ROBUST OUTPUT REGULATION

The output regulation problem has been studied in the control community when the plant model involves parameter uncertainties, and the method based on the the p-copy of the internal model has been shown to be robust in the sense that the output regulation can be achieved for all the plant models whose realization matrices are in a small neighborhood of the nominal ones Huang (2004); Knobloch et al. (1993). However there lacks quantitative analysis regarding the size of the neighborhood. In this paper, we take $\theta = 0$ as the nominal value, and assume that the parameter set $\mathcal{S}_\theta$ is described by realization matrices

$$\begin{bmatrix} A_{a_\theta} & B_{a_\theta} \\ C_{a_\theta} & D_{a_\theta} \end{bmatrix} = \begin{bmatrix} A_{a_0} & B_{a_0} \\ C_{a_0} & D_{a_0} \end{bmatrix} + \begin{bmatrix} B_{1a} \\ D_{21a} \end{bmatrix} \Delta_\theta \, [\, C_{1a} \;\; D_{12a} \,], \tag{13}$$

with $\mathcal{S}_\theta = \mathcal{D}_\delta := \left\{\Delta_\theta : \ \Delta_\theta' \Delta_\theta \leq \delta^2 I\right\}$ as the uncertain set. The above gives rise to the following state space description:

$$\begin{aligned} \dot{\tilde{x}}_a(t) &= A_{a_0}\tilde{x}_a(t) + B_{1a}\tilde{d}(t) + B_{a_0}u_a(t), \\ z(t) &= C_{1a}\tilde{x}_a(t) + D_{12a}u_a(t), \quad \tilde{d}(t) = \Delta_\theta z(t), \\ e_y(t) &= C_{a_0}\tilde{x}_a(t) + D_{21a}\tilde{d}(t) + D_{a_0}u_a(t). \end{aligned} \tag{14}$$

Define 2×2 block transfer matrix $G_a(s)$ by

$$G_a(s) := \begin{bmatrix} G_{a11}(s) & G_{a12}(s) \\ G_{a21}(s) & G_{a22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_{a_0} & B_{1a} & B_{a_0} \\ \hline C_{1a} & 0 & D_{12a} \\ C_{a_0} & D_{21a} & D_{a_0} \end{array}\right].$$

In light of Theorem 2, the robust output regulation is now equivalent to the robust output stabilization for the feedback control system shown in Fig. 2.

Let $T_c(s) = \mathcal{F}[G_a(s), K_a(s)]$ be the linear fractional transform (LFT) defined by

$$\mathcal{F}[G_a, K_a] := G_{a11} - G_{a12}K_a[I + G_{a22}K_a]^{-1}G_{a21}.$$

Then $T_c(s)$ is the transfer matrix from $\tilde{d}(t)$ to $z(t)$, and we thus have an equivalent feedback system consisting of $T_c(s)$ in the forward path, and Δ_θ in the feedback path. The feedback controller $K(s)$, i.e., $K_a(s)$ needs to be designed to achieve not only internal stability of $T_c(s)$, but also robust stability for all $\Delta_\theta \in \mathcal{D}_\delta$, i.e., $\Delta_\theta'\Delta_\theta \leq \delta^2 I$.
A commonly adopted approach to the robust stabilization as described in (14) and illustrated in Fig. 2 is the quadratic stabilization, extended from the quadratic stability Barmish (1985). Specifically suppose that $K_a(s)$ is strictly proper, which is indeed the case to be shown

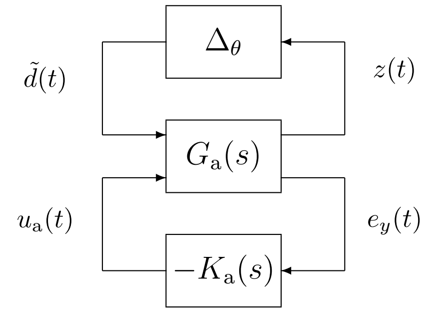

Fig. 2 Block diagram for equivalent robust stabilization

in Theorem 3. Then $T_c(s) = \mathcal{F}[G_a(s), K_a(s)]$ is strictly proper. In addition $T_c(s)$ is internally stable, by the stability requirement in the case of $\Delta_\theta = 0$. Let $\{A_c, B_c, C_c\}$ be realization of $T_c(s)$ with state vector $x_c(t)$. Then the closed-loop system in Fig. 2 can be described by

$$\dot{x}_c(t) = (A_c + B_c\Delta_\theta C_c)x_c(t), \qquad x_c(0) \neq 0,$$

where $\Delta_\theta \in \mathcal{D}_\delta$ and A_c is a Hurwitz stability matrix. The above parameter uncertainty system is quadratically stable, if there exists a single Lyapunov function

$$V[x_c(t)] = x_c(t)'X_c x_c(t)$$

that is positive definite, and its derivative along the solution trajectory is negative definite for all $\Delta_\theta \in \mathcal{D}_\delta$. It is well known that quadratic stability for the above system is equivalent to the following $\mathcal{H}_\infty$-norm condition Zhou and Khargonehr (1988):

$$\|C_c(sI - A_c)^{-1}B_c\|_{\mathcal{H}_\infty} = \|T_c\|_{\mathcal{H}_\infty} < \delta^{-1}.$$

That is, the feedback system in Fig. 2 is quadratically stabilized for all $\Delta_\theta \in \mathcal{D}_\delta$ by strictly proper $K_a(s)$, if and only if the small gain condition $\|\mathcal{F}[G_a(s), K_a(s)]\|_{\mathcal{H}_\infty} < \delta^{-1}$ holds. The following result is true in light of Glover and Doyle (1988); Green and Limebeer (1995); Zhou et al. (1996).

Theorem 3. Assume that $R_a = D_{12a}'D_{12a} > 0$ and $\tilde{R}_a = D_{21a}D_{21a}' > 0$. Denote $A_{a_R} = A_{a_0} - B_{a_0}R_a^{-1}D_{12a}'C_{1a}$ and $A_{a_{\tilde{R}}} = A_{a_0} - B_{1a}D_{21a}'\tilde{R}_a^{-1}C_{a_0}$. The feedback system as in Fig. 2 is quadratically stabilizable for all $\Delta_\delta \in \mathcal{D}_\delta$, if and only if there exist the stabilizing solutions $X \geq 0$ and $Y \geq 0$ to algebraic Riccati equations (AREs)

$$A_{a_R}'X + XA_{a_R} + C_{1a}'(I - D_{12a}R_a^{-1}D_{12a}')C_{1a} - X(B_{a_0}R_a^{-1}B_{a_0}' - \delta^2 B_{1a}B_{1a}')X = 0, \tag{15}$$

$$A_{a_{\tilde{R}}}Y + YA_{a_{\tilde{R}}}' + B_{1a}(I - D_{21a}'\tilde{R}_a^{-1}D_{21a})B_{1a}' - Y(C_{a_0}'\tilde{R}_a^{-1}C_{a_0}' - \delta^2 C_{1a}'C_{1a})Y = 0, \tag{16}$$

satisfying $\delta^2\rho(XY) < 1$. If such two stabilizing solutions exist, then a central $\mathcal{H}_\infty$ controller is given by strictly proper

$$K_a(s) = F(sI - A_K)^{-1}ZL, \quad Z = (I - \delta^2 XY)^{-1},$$

where $A_K = A_{a_0} + \delta^2 B_{1a}B_{1a}'X + B_{a_0}F + ZLC_{a_0} + ZLD_{a_0}F$ with $F = -R_a^{-1}(B_{a_0}'X + D_{12a}'C_{1a})$ and $L = -(YC_{a_0}' + B_{1a}D_{21a}')\tilde{R}_a^{-1}$.

The supremum of $\delta = \sqrt{\rho(XY)^{-1}}$ over all the stabilizing solutions $X \geq 0$ and $Y \geq 0$ to AREs (15) and (16), respectively, is exactly the same as the stability margin $\delta_{\max}$. It is exactly the maximum allowable uncertainty bound below which the problem of robust output regulation is solvable.

Hence Theorem 3 provides a quantitative result for the robust output regulation. The existence of the stabilizing solution $X \geq 0$ to ARE (15) requires stabilizability of (A_{a_0}, B_{a_0}), and

$$\text{rank}\left\{\begin{bmatrix} A_{a_0} - j\omega I & B_{a_0} \\ C_{1a} & D_{12a} \end{bmatrix}\right\} = n + m \ \ \forall\, \omega \in \mathbb{R}. \quad (17)$$

Moreover there exists $\delta_{\rm csup} \geq \delta_{\max}$ such that ARE (15) admits the stabilizing solution only if $\delta < \delta_{\rm csup}$. Similarly the existence of the stabilizing solution $Y \geq 0$ to ARE (16) requires detectability of (C_{a_0}, A_{a_0}), and

$$\text{rank}\left\{\begin{bmatrix} A_{a_0} - j\omega I & B_{1a} \\ C_{a_0} & D_{21a} \end{bmatrix}\right\} = n + p \ \ \forall\, \omega \in \mathbb{R}. \quad (18)$$

Furthermore there exists $\delta_{\rm fsup} \geq \delta_{\max}$ such that ARE (16) admits the stabilizing solution only if $\delta < \delta_{\rm fsup}$. Finally the existence of the robust output regulation controller require the coupling condition $\delta^2\rho(XY) < 1$ to hold.

Remark 2. If the uncertainties are modeled by

$$\begin{bmatrix} A_\theta & B_\theta \\ C_\theta & D_\theta \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \Delta_\theta \begin{bmatrix} C_1 & D_{12} \end{bmatrix},$$

before augmentation of the p-copy of the internal model, then the realization of the augmented system can have the same form as in (13) by taking $D_{a_0} = DD_0$, $D_{12a} = D_{12}D_0$, and

$$A_{a_0} = \begin{bmatrix} A & B\Phi \\ 0 & \underline{A}_0 \end{bmatrix}, \quad B_{a_0} = \begin{bmatrix} BD_0 \\ \Xi \end{bmatrix}, \quad B_{1a} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$
$$C_{a_0} = \begin{bmatrix} C & D\Phi \end{bmatrix}, \quad C_{1a} = \begin{bmatrix} C_1 & D_{12}\Phi \end{bmatrix}, \quad D_{21a} = D_{21}.$$

However the quadratic stabilizability condition as in Theorem 3 is only sufficient, but not necessary anymore. In addition the rank conditions in (17) and (18) may fail after augmentation, if $P_\theta(s)$ is strictly proper. In this case, a small perturbation can be added to D_{12a}, D_{21a}, B_{1a}, or C_{1a} in order to satisfy the full rank condition, and to solve the stabilizing solutions X, Y to AREs (15), and (16), respectively.

5. AN ILLUSTRATIVE EXAMPLE

We consider output regulation for the uncertain state space model described in Remark 1 with realization matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix},$$
$$C = \begin{bmatrix} 2 & 0 & -2 \end{bmatrix}, \qquad D = 0, \qquad C_1 = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 1 \end{bmatrix},$$

and $D_{21} = D_{12}' = \begin{bmatrix} -1 & 0 \end{bmatrix}$. So $\Delta_\theta \in \mathbb{R}^{2\times 2}$. For simplicity $B_0 = 0$ is taken. The reference trajectories consist of sinusoidal signals with A_0 and C_0 specified by

$$A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}.$$

It follows that $\alpha(s) = \det(sI - A_0) = (s^2 + 1)$. Setting $\beta(s) = s + 3$ yields realization of $w(s) = \beta(s)/\alpha(s)$ in (8) as

$$\phi_0 = \begin{bmatrix} 1 & 3 \end{bmatrix}, \quad \theta_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}'.$$

Since $m = p = 1$, we need to make only 1-copy of the internal model. The augmented realization matrices as in (13) can be easily obtained but their numerical values are skipped. It can be verified that (A_{a_0}, B_{a_0}) is stabilizable, and (C_{a_0}, A_{a_0}) is detectable. By adding a small perturbation (10^{-3} in the Frobenius norm) to D_{12a} and to the zero sub-matrix of B_{1a}, both conditions in (17) and (18) can be made true, and D_{12a} has the full row rank as well. Our calculations show that $\delta_{\max} \approx 0.6125$. The feedback controller $K_a(s)$ for $\delta = 0.61 < \delta_{\max}$ is computed. So is $K(s) = K_a(s)\beta(s)/\alpha(s)$. However their numerical values are skipped. It is interesting to observe that the zero eigenvalue of A can be perturbed to either left or right half plane of the complex plane by the bounded parameter uncertainties. That is, the uncertain plant model does not have a fixed number of unstable poles.

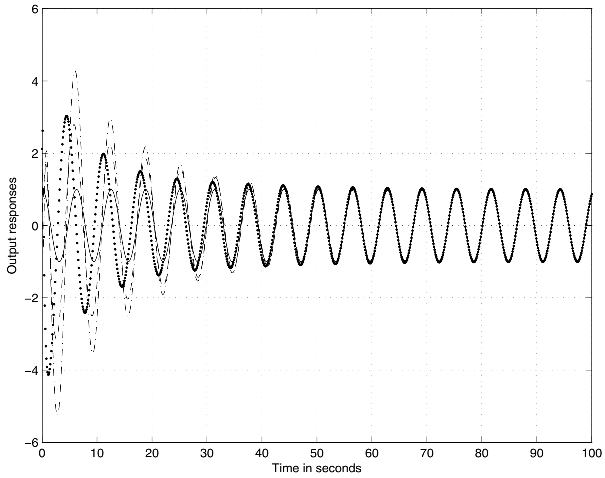

Fig. 3 Output responses of the plant model

The desired reference trajectory for the uncertain system to track is specified by $y_0(t) = \cos(t)$, plotted in solid line in Fig. 3. The output responses for three different $\Delta \in \mathbb{R}^{2\times 2}$ are computed with Matlab, and are plotted in Fig. 3 with dashed, dotted, and dot-dashed curves. The parameter uncertainties $\Delta \in \mathbb{R}^{2\times 2}$ are randomly generated with their maximum singular values scaled to $\delta = 0.55 < \delta_{\max}$. The associated initial conditions of the plant model are also randomly generated. The simulation results show that the output responses track the reference trajectory asymptotically.

Before ending this section, it needs to be mentioned that the δ value cannot be too close to $\delta_{\max}$, which may result in closed-loop poles close to the imaginary axis, and thus lead to long settling time. It is tricky to design a robust controller that takes the transient responses into consideration, because of the presence of the parameter uncertainties.

6. CONCLUSION

In this paper the problem of robust output regulation has been studied, assuming that the plant model involves bounded parameter uncertainties. By employing the p-copy of the internal model, it is shown that the robust output regulation can be converted into the robust output stabilization. More importantly by assuming quadratically bounded parameter uncertainties for the realization of the augmented plant model, and by applying the notion of the quadratic stability, the well-known $\mathcal{H}_\infty$ control design method can be used to synthesize the output feedback control system. A design procedure is developed to design the controller that not only stabilizes the closed-loop system internally, but also enables tracking of the reference trajectories asymptotically. A numerical example is worked out to illustrate our proposed design method.

While our work provides a quantitative result for the robust output regulation, complementing the known qualitative results in Davison (1976); Knobloch et al. (1993); Scherer et al. (1997), performance objectives are not considered for feedback control design in this paper. Inclusion of additional performance objectives may lead to nonlinear optimization, especially when multi-objective performance indexes are involved. This is a very challenging problem. Application of the robust output regulation to cooperative output regulation for multi-agent systems is another important research topic that has received great attention. Both are currently under our study, and will be reported in the future.

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