Ch 10.4
Stability

Definition of Stability. An unconstrained linear system is said to be stable if the output response is bounded for all bounded inputs. Otherwise it is said to be unstable.

NOTE: Stability of a linear system is a SYSTEM property, that is, independent of the input signal (sine or step, etc.) and of where it enters the system (input or disturbance, etc.)

Linear system. Is g(s) stable?

• g(s) = n(s)/d(s). Poles are solutions s=p, to d(s)=0.

\[ g(s) = \frac{n(s)}{d(s)} \]
\[ d(s) = a_n s^n + \cdots + a_1 s + a_0 = a_n (s-p_1)(s-p_2) \cdots (s-p_n) \]

Step response. Partial fraction expansion:

\[ y(s) = g(s) \frac{1}{s} = \frac{A_0}{s} + \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots \]

Inverse Laplace:

\[ y(t) = A_0 + A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots \]

• So time response contains term \( e^{pt} \)
  - Example: \( G(s) = \frac{1}{(s-1)} \),\( p=1 \)
    - Step response: \( y(t) = -1(1-e^t) = e^t - 1 \)

• Conclusion:
  - Stability \( \iff \Re(p_i)<0 \) (all poles have negative real part)
  - \( \iff \) All poles in left half plane (LHP),
Pole in right half plane (RHP): UNSTABLE

\[ G(s) = \frac{n(s)}{d(s)} \text{ where} \]
\[ d(s) = (s-p_1)(s-p_2) \cdots \]

Real pole \( p \): Get term \( e^{pt} \).
For \( \text{Re}(p)>0 \) (RHP-pole):
Unstable since \( e^{pt} \rightarrow \infty \) (as \( t \rightarrow \infty \))

Complex pole pair \( (p_{12} = p \pm j\omega) \)
Gives oscillations:
\[ c_1 e^{p_{1}t} + c_2 e^{p_{2}t} = c e^{pt} \sin(\omega t + \phi) \]
Which are unstable if \( \text{Re}(p)>0 \)

![Graph showing step response](image)

Figure 15:20 Contributions of characteristic equation roots to closed loop response.
Poles = Eigenvalue of A-matrix

Linear system in deviation variables (state space form)

\[
\begin{align*}
\frac{dx}{dt} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

Laplace. Get transfer function from \( u \) to \( y \)

\[
G(s) = C(sI-A)^{-1}B + D = \frac{n(s)}{d(s)}
\]

From mathematics: \((sI-A)^{-1} = \frac{\text{adj}(sI-A)}{\det(sI-A)}, \text{ so}\)

\[
d(s) = \det(sI-A) = \text{pole polynomial}
\]

But \( \det(sI-A)=0 \) is also the formula for finding the eigenvalues of \( A \)

Conclusion: Solutions to \( d(s)=0 \) are the poles which are identical to the eigenvalues of \( A \)

\[
p_i = \text{eig}(A)
\]

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Stability of closed-loop systems

- Closed-loop transfer functions, \( G_{CL}(s) = \frac{n(s)}{d(s)} \)
  - \( n(s) = \text{direct path} \)
  - \( d(s) = 1 + \text{loop}(s) \)
    - where \( \text{loop}(s) = g(s)g_m(s)c(s) \)
- Same \( d(s) \) for any input/output!
  - Makes sense because stability is a system property
- Conclusion: Closed-loop poles are given by solutions to “closed-loop characteristic equation”
  - \( d(s) = 1 + \text{loop}(s) = 0 \)
How do we test for stability?

\[ T(s) = \frac{n(s)}{d(s)} \]

Method 1. Compute **poles** (eigenvalues) \( p = \lambda(A) \) and check if \( \text{Re}(p) < 0 \)
- OK numerically, but difficult to find poles \( p \) analytically

Method 2. Look at **coefficients** \( a_i \) in \( d(s) \),
- Good for analytical results. Don’t need to find poles \( p \)
**Test 1.** All a’s must have **same sign*** for stability (necessary condition)
**Test 2.** Routh array: Necessary and sufficient

Method 3. Closed-loop system. Frequency analysis (see later)
- Consider loop transfer function, \( L = GC \)
- Bode stability test for stability: \( |L| < 1 \) at frequency \( \omega_{180} \)
- Easy to include time delay

* Necessary and sufficient for 2nd order system

**Example 2**

For which \( K_c \) is the closed-loop system stable?

\[ g(s) = \frac{1}{s+1} \]
\[ g_m(s) = \frac{-s+1}{s+1} \]
\[ c(s) = K_c \]

Method 1. Analytic solution using poles. Much more work!*

Matlab commands:
- \text{sym} s Kc
- \text{g} = 1/(s+1)
- \text{gm} = (-s+1)/(s+1)
- \text{clpoles = solve(1+Kc*g*gm==0)}
- \text{solve(real(clpoles(1))==0)}
- \text{solve(real(clpoles(2))==0)}

Solution:
- \( g = \frac{1}{6s + 1} \)
- \( g_m = \frac{-s + 1}{s + 1} \)
- \( c = K_c \)
- \( \text{clpoles} = [\frac{K_c}{12} + \frac{(K_c^2 - 38*K_c + 25)^{1/2}}{12} - \frac{7}{12}, \frac{K_c}{12} - \frac{(K_c^2 - 38*K_c + 25)^{1/2}}{12} - \frac{7}{12}] \)
- \( \text{ans} = -1.0 \)
- \( \text{ans} = 7.0 \)

Method 2. Coefficients
**Test 1.** Check signs of Char. Eq.:
- \( d(s) = 1 + \text{loop} = 1 + \frac{K_c*(-s+1)/(s+1)(6s+1)}{0} \)
- \( 6s^2 + (7-Kc)s + (1+Kc) = 0 \)
- Stable -> all coefficients positive -> \( K_c > 1 \) (lower limit for positive feedback)
- \( K_c < 7 \) (upper limit because of RHP-zero)
(necessary and sufficient for 2nd order system)

* Almost impossible for systems of order 4 or higher
Test 2. ROUTH array: Find location (RHP/LHP) of poles without actually having to find the “Advanced version of looking for negative sign in d(s)-polynomial”

Example 3. \( g(s) = \frac{1}{(5s+1)(2s+1)} \). \( g_m = \frac{1}{s+1} \), P-control

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How do the closed-loop poles depend on the controller gain \( K_c \)?

Example 4. \( g(s) = K_c \)

10.5 ROOT LOCUS DIAGRAMS

In the previous section we have seen that the roots of the characteristic equation play a crucial role in determining system stability and the nature of the closed-loop responses. In the design and analysis of control systems, it is instructive to know how the roots of the characteristic equation change when a particular system parameter, such as a controller gain changes. A root locus diagram provides a convenient graphical display of this information, as indicated in the following example.

EXAMPLE 10.13

Consider a feedback control system that has the open-loop transfer function:

\[
G_O(s) = \frac{4}{s(s+1)(s+2)(s+3)}
\]

Plot the root locus diagram for \( K \leq K_c \leq 20 \).

Comment:
- Bold：closed-loop poles
- Thin：open-loop poles
- \( K_c = 0 \): Starts oscillating
- \( K_c = 15 \): Goes unstable
- Step response for \( K_c = 1.62 \)

SOLUTION

The characteristic equation is

\[
1 + KG_m = 0
\]

The root locus diagram in Fig. 10.27 shows how the three roots of this characteristic equation vary with \( K_c \). When \( K_c = 0 \), the roots are exactly the poles of the open-loop transfer function, \(-1, -2, \) and \(-3\). These are designated by an \( * \) symbol in Fig. 10.27. As \( K_c \) increases, the root at \(-3\) decreases monotonically. The other two roots converge and then form a complex conjugate pair when \( K_c = 0.1 \). When \( K_c = K_c = 15 \), the complex roots cross the imaginary axis and enter the unstable region. This illustrates why the substitution of \( s = \sigma + j \omega \) (Section 10.3) determines the unstable controller gain. Thus, the root locus diagram indicates that the closed-loop system is unstable for \( K_c > 15 \). It also indicates that the closed-loop response will be nonoscillatory for \( K_c < 0.1 \).