Input-output Controllability Analysis

Idea: Find out how well the process can be controlled - without having to design a specific controller
Note: Some processes are impossible to control


Example: First-order with delay process
\[ g(s) = \frac{k e^{-\theta_i}}{1 + \tau_i s}; \quad G_d(s) = \frac{k_d e^{-\theta_d}}{1 + \tau_d s} \]
+ Measurement delays: \( \theta_{mi}, \theta_{md} \).

Problem: What values are desired for good controllability?

Qualitative results:

<table>
<thead>
<tr>
<th></th>
<th>Feedback control</th>
<th>Feedforward control</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>Large</td>
<td>Large</td>
</tr>
<tr>
<td>( \tau )</td>
<td>Small</td>
<td>Small</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Small</td>
<td>Small</td>
</tr>
<tr>
<td>( k_d )</td>
<td>Small</td>
<td>Small</td>
</tr>
<tr>
<td>( \tau_d )</td>
<td>Large</td>
<td>Large</td>
</tr>
<tr>
<td>( \theta_{mi} )</td>
<td>No effect</td>
<td>Large</td>
</tr>
<tr>
<td>( \theta_{md} )</td>
<td>Small</td>
<td>No effect</td>
</tr>
</tbody>
</table>

WANT TO QUANTIFY!
Rules

• Rules 1-3: speed of response
  – Rule 1: Fast response required to reject large disturbance
  – BUT (rule 2): Response time is limited by effective time delay
  – Rule 3: Fast response needed for stabilization

• Rule 4: Input constraints
  – Large disturbances may give input saturation

Rules for speed of response (assuming control with integral action)

• Define $\omega_c = 1/\tau_c$ = closed-loop bandwidth = where $|L|$ is approx. 1
• Define $\omega_d$ as frequency where $|g_d| = 1$ (scaled model, frequency where $|y| = 1$ for $|d| = 1$)

• Rule 1: Fast response required to reject large disturbance
  – Need $\omega_c > \omega_d$ ($\tau_c < 1/\omega_d$
    – Rule 1 is for typical case where $|g_d|$ is highest at low frequencies
    – The more exact rule is: We need $|Sg_d| < 1$, or approximately: $|L| > |g_d|$ at frequencies where $|g_d| > 1$

• Rule 2: Response time is limited by effective time delay
  – Need $\omega_c < 1/\theta$ ($\tau_c > \theta$. SIMC-rule)
    – Where $\theta$ is effective time delay

• Rule 3: Fast response needed for stabilization
  – Need $\omega_c > p$ ($\tau_c < 1/p$)
    – Where $p$ is unstable pole, $g(s) = k/(s-p)$...

• Rule 4: Input constraints: Large disturbances may give input saturation
  – With scaled model: Need $|G| > |G_d|$ at frequencies where $|G_d| > 1$

This situation is OK according to rules 1-3:
Comment: Ideal controller inverts the plant

- \( y = g(s)u + d \)
- **Ideal controller inverts the plant** \( g(s) \):
  - Think feedforward, \( u = c(s)(y - d) \)
  - **Perfect control**: want \( y = y_s \rightarrow c = 1/g(s) = g^{-1} \)

**Limitations on perfect control**: Inverse cannot always be realized:
1. Input saturation, \( |u| > |u_{\text{max}}| \)
2. Time delay, \( g = e^{-\theta s} \).
   - Solution: Omit
3. Inverse response, \( g = -Ts + 1 \).
   - Solution: Omit
4. More poles than zeros, \( g = 1/(Ts+1) \).
   - Solution: Replace by: \( (Ts + 1)/(f_s + 1) \) where \( f_s < s \)
   - Example. \( g(s) = e^{-s} \)

- **So we know what limits us from having perfect control**
  - Same limitations apply to feedback control
- **Controllability analysis**: Want to find out what these limitations imply in terms of “acceptable control”, \( |y - y_s| < y_{\text{max}} \)

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**We use scaled model**

**Rules 1 and 4**

Original model:

\[
y' = g'u' + g'd'
\]

Scaled model:

\[
\frac{y'}{y_{\text{max}}} = g'u_{\text{max}} + g'd_{\text{max}}
\]

or

\[
y = gu + gd
\]

where \( |y| < 1, |u| < 1, |d| < 1 \).
Closed-loop frequency response (S)

Recall:

No control (c = 0): $e_{OL} = y_a - y = y_a - g_d d$
With control: $e = y_a - y = Sy_a - Sg_d d = Se_{OL}$

$S = \frac{1}{1+L}$ - sensitivity function — effect of control
$L = gc$ - loop transfer function

Low $\omega$ where $|S| < 1$: Control reduces error
Intermediate $\omega$ where $|S| > 1$: Control increases error
High $\omega$ where $S = 1$ ($L \rightarrow 0$): Control has no effect

$M_s = \text{peak of } |S|$
Typical requirement: $M_s < 2$
At stability limit: $M_s \rightarrow \infty$

Disturbances and Loop gain $L$

- $S = 1/(1+L)$ where $L = gc$
- No control («open-loop»): $y = g u + g_d d$
- With control: $y = S g_d d$
- Scaled variables: Want $|S g_d| < 1$ at all $\omega$
- Approximation at low frequencies where $|L|$ is large: $S = 1/L$
- So want (in scaled variables): $|L| > |g_d|$
  - Up to about frequency $\omega_c$ where $|L|=1$
Time domain

- Consider response to step disturbance, $g_d = k_d/(T_d s+1)$
  - Output reaches $\Delta y = (k_d \theta /T_d) \Delta d$ at time $\theta$ (approximately)
  - If this is larger than acceptable then we are in trouble
  - If $\Delta d=1$ and requirement is $|\Delta y|<1$ then we must require $k_d/T_d < 1/\theta$
    (combined rule 1+2)

- Easier to generalize in frequency domain
  - Consider disturbance $d(t)=\sin\omega t$

Scaled model

**MAIN REASON FOR CONTROL: DISTURBANCES!**

1. **DISTURBANCES (speed of response)**

Need control up to frequency $w_d$
where $|G_d|=1$ -> Need $w_c > w_d$
($w_c$ is frequency where $|L|=1$)

**Note:** Have $\omega_c=1/T_c$
2. INPUT CONSTRAINTS

Process model

\[ y = G_u + G_d d \]

1. Worst-case disturbance: \(|d| = 1\). To achieve perfect control \((e = 0)\) with \(|u| < 1\) we must require

**Rule 4:**

\[ |G| > |G_d| \] at frequencies where \(|G_d| > 1\)

(3)

2. Worst-case reference: \(|r| = R_{max}\). To achieve perfect control \((y = r)\) with \(|e| < 1\) we must require

\[ |G| > |R_{max}| \quad \forall \omega \leq \omega_r \]

(4)

**Rules for speed of response (assuming control with integral action)**

- Define \(\omega_c = 1/\tau_c\) = closed-loop bandwidth = where \(|L|\) is approx. 1
- Define \(\omega_d\) as frequency where \(|g_d| = 1\) (scaled model, frequency where \(|y| = 1\) for \(|d| = 1\))

- **Rule 1:** Fast response required to reject large disturbance
  - Need \(\omega_c > \omega_d\) \((\tau_c < 1/\omega_d)\)
    - Rule 1 is for typical case where \(|g_d|\) is highest at low frequencies
    - The more exact rule is: We need \(|Sg_d| < 1\), or approximately: \(|L| > |g_d|\) at frequencies where \(|g_d| > 1\).

- **Rule 2:** Response time is limited by effective time delay
  - Need \(\omega_c < 1/\theta\) \((\tau_c > \theta)\) \(\text{SIMC-rule!}\)
    - Where \(\theta\) is effective time delay

- **Rule 3:** Fast response needed for stabilization
  - Need \(\omega_c > p\) \((\tau_c < 1/p)\)
    - Where \(p\) is unstable pole, \(g(s) = k/(s-p)\)

- **Rule 4:** Input constraints: Large disturbances may give input saturation
  - With scaled model: Need \(|G| > |G_d|\) at frequencies where \(|G_d| > 1\)

This situation is OK according to rules 1-3:
Example, $g = \frac{500}{(50s+1)(10s+1)}$

$gd = \frac{9}{10s+1}$

$w = \text{logspace}(-3, 1, 1000);$

$[\text{mag}, \text{phase}] = \text{bode}(g, w);$

$[\text{magd}, \text{phase}] = \text{bode}(gd, w);$

$\loglog(w, \text{mag}(:,\text{')}, 'blue', w, \text{magd}(:,\text{')}, 'red', w, 1, 'black'); \text{ grid on}$

Effective delay $8$ limits $\omega_c$

PI-control: $\omega_c < 1/8 = 1/5 = 0.2$

PID-control: $\omega_c < 1/8 = 1/0 = \infty$

Check controllability analysis with simulations
**Scaled Model**

**PI control not acceptable**

- As expected since need $\omega_c > \omega_d = 0.9$, but can only achieve $\omega_c < 1/\theta = 1/5 = 0.2

\[
s = \text{tf}(s)
g = \frac{500}{(50s+1)(10s+1)} \\
gd = \frac{9}{(10s+1)} \\
\% \text{SIMC-PI with } tauc=theta=5 \\
Kc=(1/500)*(55/(5+5)); \text{ taui}=55; \text{ taud}=0;
\]

**Scaled Model**

**PID control acceptable: y and u are within ±1**

\[
g = \frac{500}{(50s+1)(10s+1)} \\
gd = \frac{9}{(10s+1)} \\
\% \text{SIMC-PID (cascade form) with } tauc=wd=1: \\
Kc=(1/500)*(50/(1+0)); \text{ taui}=50; \text{ taud}=10;
\]
If process is not controllable: Need to change the design

• For example, dampen disturbance by adding buffer tank:

Level control unimportant, but need good mixing

Integral action is not recommended for averaging level control

Problem 1

\[ G(s) = \frac{2}{s+1}, \quad G_d(s) = \frac{3}{5s+1} \]

Figure 3: Magnitude of \( G \) and \( G_d \).
Problem 2

\[ G(s) = \frac{3}{5s + 1} \quad G_d(s) = \frac{2}{s + 1} \]

Problem 3

\[ G(s) = \frac{3}{5s + 1} \quad G_d(s) = \frac{7.5}{(s + 0.2)(s + 20)} - 0.8 \]
Problem 4

\[ G(s) = \frac{200}{(20s+1)(10s+1)(s+1)} \quad G_d(s) = \frac{4}{(3s+1)((s+1)^3} \]

\[ g = \frac{200}{((20s+1)(10s+1)(s+1))} \]
\[ g_d = \frac{4}{((3s+1)((s+1)^3)} \]
\[ K_c = \frac{1}{200} \cdot \frac{20}{\tau_{ai}} = \frac{20}{\tau_{ai} \cdot \tau_d} = 10.5 \]

Problem 5

\[ G(s) = \frac{2.5 e^{-0.1s} (1 - 0.5s)}{(3s+1)((s+1)^3)} \quad G_d(s) = \frac{2}{s+1} \]
Problem 6

\[ y = c_{\text{H}} - c_{\text{OH}} \text{(want=0 \pm 10^{-6} mol/l, pH=7 \pm 1)} \]
\[ u = q_{\text{base}} \text{ (c_{\text{OH}}=10 mol/l, pH=15)} \]
\[ d = q_{\text{acid}} \text{ (c_{\text{H}}=10 mol/l, pH=-1)} \]

Using tanks in series, Acid and base in tank 1.

Scaled model: \( k_d = 2.5 \times 10^6 \)
Each tank: \( \tau = 1000 \text{s} \)
Control: \( \theta = 10 \text{s} \) (meas. delay for pH)

Problem: How many tanks?

Control system

- 3 tanks: Neutralization (base addition) only in tank 1 gives large effective delay (>> 10s) because of tank dynamics in \( g(s) \)
- Suggested solution is to add (a little) base also in the other tanks:

Conclusion

• Use controllability analysis
  – To avoid spending time on impossible control problem
  – To help design the process (e.g., size buffer tanks)
• Also useful for tuning.
  – $T_c$ = SIMC tuning parameter = $1/\omega_c$
  – Must for acceptable controllability have:
    \[
    \omega_d \leq \omega_c \leq \frac{1}{\theta} \iff \theta \leq T_c \leq \frac{1}{\omega_d}
    \]
  • Agrees with SIMC-rules
    – Tight control: $T_c = \theta$
    – “Smooth” control: $T_c = 1/\omega_d$

Exam

• Wednesday 06 December 2017 from 09:00 to 13:00
• The test (questions) is in English but you may answer in Norwegian or English.
• Permitted examination support material:
  – One (1) A4 double-sided piece of paper with your handwritten notes (it does not need to be approved or stamped prior to the exam).
  – No other written material.
  – Standard calculator.
• Note: Remember to state clearly all assumptions you make.

Q&A session

Alternative: Monday 04 Dec. 12-14