SOLUTION FOR EXAM IN Process control (TKP4140). December 2022

The text in red is not required at the exam. It is mainly included to show that the theory works in practice.

Problem 1 (15%)

a) Approximate $\frac{3s+1}{2s+1}$ as $\frac{3}{2}$, according to Rule T1 (T0=3>tauo=2>tauc=1).
This leads to $G_2(s) = \frac{1.5e^{-0.5s}}{(7s+1)(0.8s+1)}$. Applying the half rule to obtain $G(s) = \frac{ke^{-\theta s}}{\tau s+1}$ leads to:

$$k = 1.5,\ \tau = 7 + \frac{0.8}{2} = 7.4,\ \theta = 0.5 + \frac{0.8}{2} = 0.9$$

b) With tauc=1 we get the following SIMC PI tunings:

$$K_c = \frac{1}{k}\frac{\tau}{\tau_c+\theta} = \frac{1}{1.5}\frac{7.4}{1+0.9} = 2.5965,\ \ \tau_I = \min(4 \times (1+0.9), 7.4) = \min(7.6, 7.4) = 7.4.$$

c) For this system, PI is probably ok, at least if we insist on using tauc=1.
From G2, we see that tau2=0.8 > theta=0.5, so PID will give some performance improvement, but then we need to reduce tauc, for example, to tauc=theta=0.5.

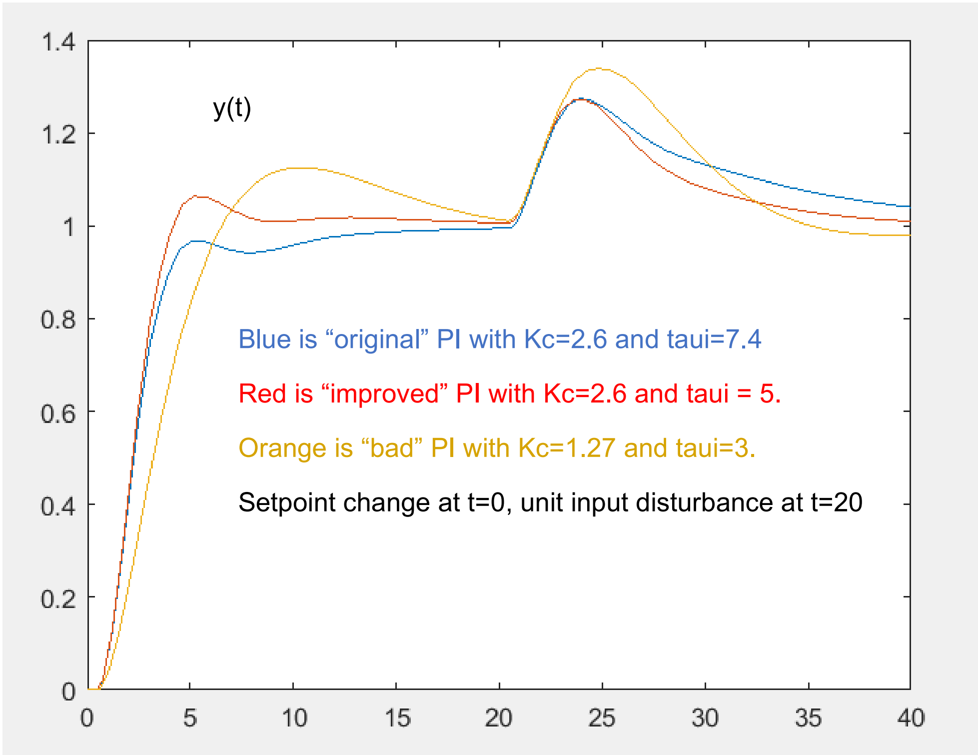
The suggested PI-tunings work OK (see blue response below), but are not necessarily the optimal. The "problem" is that the approximation of the zero using rules T1-T3 is not always "optimal"

Looking at response with the "original" PI-controller (see the blue response below), we see that the settling towards steady state is rather slow. Therefore, it seems that the integral time is too long. One approach would be select a lower tauc. This what I normally recommend if the model is good but this is not the best in this case, as it also increases Kc. A simpler approach is to just reduce taui and keep Kc unchanged. I tried reducing taui from 7.4 to 5. This works well: It gives a faster settling (see red curve below) and the robustness is still very good (with GM=4.4 and DM=2.37s). Note that the peak of the disturbance response is the same since Kc is unchanged,

PI (original, tauc=1)	Kc=2.60, taui=7.4	(GM=13.3 dB = 4.6, PM=71o (1.24 rad) at wc=0.433 rad/s, DM=2.86 s)
PI with smaller taui:	Kc=2.60, taui=5	(GM=12.8 dB = 4.4, PM=62o (1.08 rad) at wc=0.455 rad/s, DM=2.37 s)

Comment: It doesn't help deriving PI-tunings using the alternative approximation (3s+1)/(7s+1) ~= 0.714/(2s+1) according to Rule T3. With tauc=1 it gives Kc=1.27 and taui=3. The resulting response (orange curve) is not very good (but it is very robust with GM=7.9 and DM=3.57s).

The simulations below compare the three PI-controllers when applied to the original process G0(s).

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%The setpoint response and the robustness margins can be generated using sisotool
s=tf('s')
g = (3*s+1)*exp(-0.5*s)/((7*s+1)*(2*s+1)*(0.8*s+1))
Kc=2.6, taui=7.4
c=Kc*(1+1/(taui*s))
sisotool(g,c)
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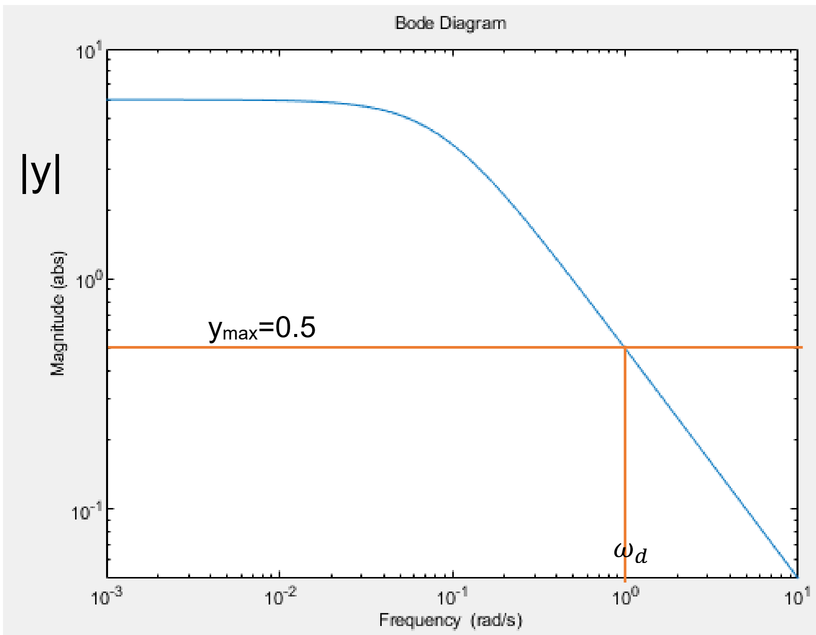
Problem 2 (20%)

a) Without control we have y=G_d d. The amplitude of the output for a sinusoidal disturbance is |y|=|G_d(jω)| |d|.

Analytical expression with |d|=2: (time delay is irrelevant here):
$|y| = |G_d(j\omega)|\, 2 = 6/\sqrt{(12\omega)^2+1}$

The corresponding Bode magnitude plot of |y|=2 G_d(jω) is shown below



b) To keep |y|<ymax=0.5, we need control up the frequency ω_d where

$$|G_d\,(j\omega_d)| = y_{max}/d_{max} = 0.5/2 = 0.25.$$

We find:

$$\frac{3}{\sqrt{(12\omega_d)^2+1}} = 0.25 \rightarrow (12\omega_d)^2 = 143 \rightarrow \omega_d \approx 1 \rightarrow \tau_c = 1/\omega_d = 1$$

Let $\omega_c = \frac{1}{\tau_c}$ where τ_c is the closed-loop time constant. Then we must require that $\omega_c > \omega_d$ or $\tau_{\{c,max\}} = 1/\omega_d$. Thus, we select $\tau_c = 1/\omega_d = 1$.

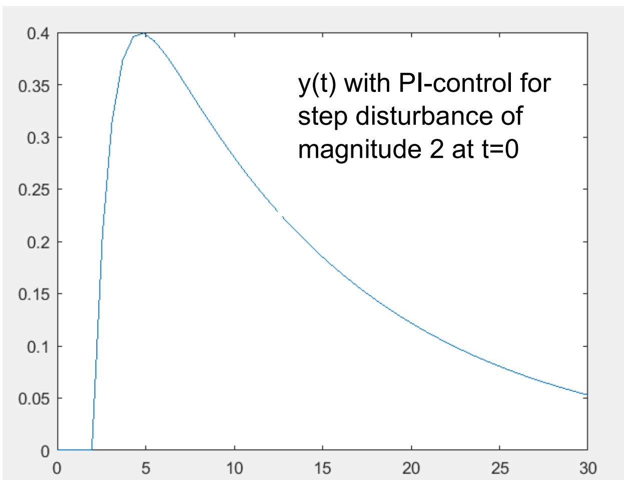
Comment (An alternative more exact analysis). The same result could be obtained from analyzing $|S\,G_d|$, where $S(s) = \tau_c s/(\tau_c s+1)$ provided we use SIMC-tunings with taul=tau1=2 (which we will show is satisfied). The requirement is to have |y| = |S Gd d| < ymax=0.5 at all frequencies. If we plot |S Gd| as a function of frequency, then we see that has it has a flat peak region between the disturbance break frequency at 1/12 and the S break frequency at 1/τ_c. In this region, $|S\,G_d| = \tau_c\,3/12$, so to get |S Gd d| < ymax we must require require that $\tau_c 3/12 < 0.5/2$ → $\tau_c < 1$. This is the slowest control we can accept for acceptable disturbance rejection, so tauc-max = 1.

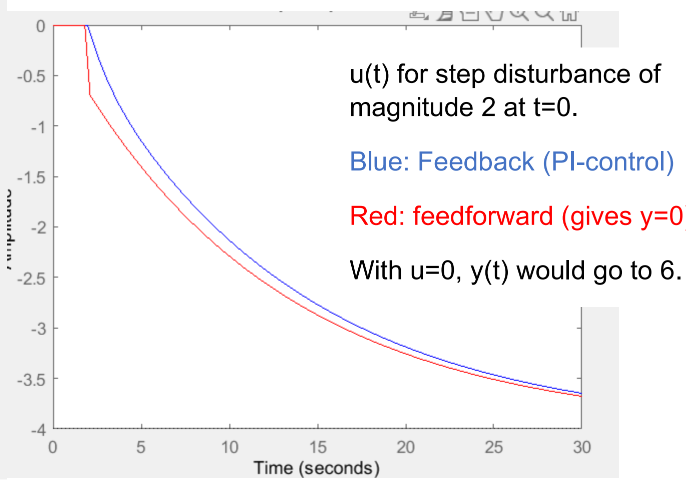
For τ_c=1, the SIMC tuning rules give:

$$K_c = \frac{1}{1.5}\frac{2}{\tau_c} = 1.33, \qquad \tau_I = \min(4\,\tau_c, 2) = 2$$

Comment: The simulation to a step disturbance of magnitude 2 is shown below. It confirms the above analysis and design as we see that y(t) peaks at 0.4 and thus stays just below ymax=0.5. (Yes, I know that it is stated that we should consider a sinusoid, but from the simulation we see that the frequency analysis is also useful also for step responses.)

For comparison is also shown the input u(t), both for the PI-controller and for the ideal feedforward controller (designed in part c). Note that the PI-controller will give about the same y(t) even with model error, but this is not the case with the feedforward controller. For example, if the gain of Gd is changed from 3 to 2.5 (not a large change), then the feedforward controller will overreact and y(t) will go to (2.5-3)*2 = -1 at steady-state.





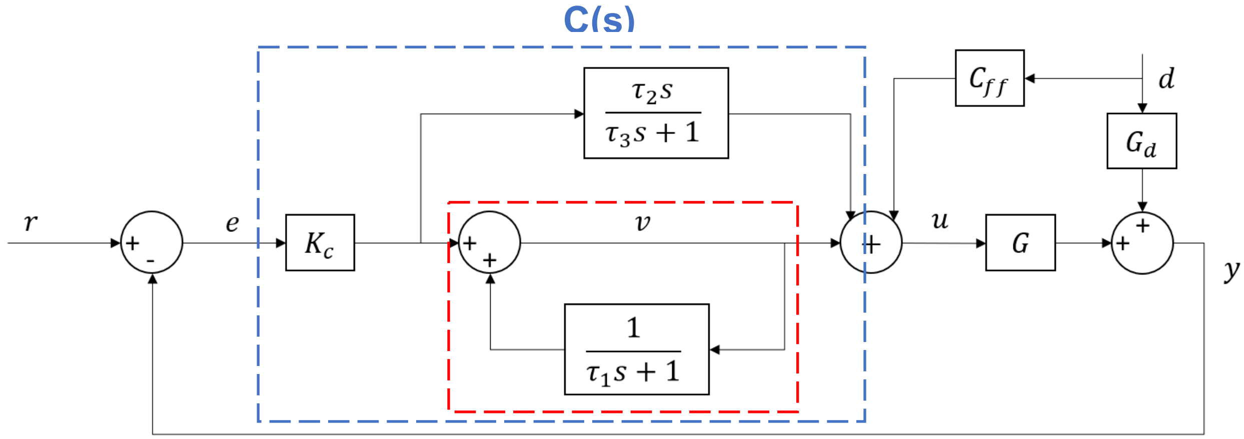
c) Ideal feedforward controller (u = cFF d):

$$c_{FF} = -\frac{G_d}{G} = -\frac{\frac{3e^{-2s}}{12s+1}}{\frac{1.5}{2s+1}} = \frac{-2\,e^{-2s}\,(2s+1)}{12s+1}$$

This controller is realizable, and no further simplification is necessary. Therefore, if the model (G, Gd) and the disturbance measurement is perfect, we get $y(t) = 0$.

Also note that there is no problem with input saturation since to reject the disturbance d=2 at steady-state (which requires the largest input) we need |u|= 4, which is less than umax=10 (see simulation above). However, note that feedforward control is always sensitive to model uncertainty.

Problem 3 (20%)



a) Consider first the red box:

$$v = K_c\, e + \frac{1}{\tau_1 s + 1} v \;\Rightarrow\; \left(1 - \frac{1}{\tau_1 s + 1}\right) v = \frac{\tau_1 s}{\tau_1 s + 1} v = K_c\, e \;\Rightarrow\; v = K_c \left(1 + \frac{1}{\tau_1 s}\right) e$$

We then get:

$$u = v + K_c \frac{\tau_2 s}{\tau_3 s + 1} e + C_{ff} d = K_c \left(1 + \frac{1}{\tau_1 s} + \frac{\tau_2 s}{\tau_3 s + 1}\right) e + C_{ff} d$$

$$\Rightarrow\; C(s) = K_c \left(1 + \frac{1}{\tau_1 s} + \frac{\tau_2 s}{\tau_3 s + 1}\right)$$

τ_1: integral action time constant
τ_2: derivative action time constant
τ_3: time constant for filter in derivative action

b) Using algebra:

$$y = G\, u + G_d\, d = G\left(C\, e + C_{ff}\, d\right) + G_d\, d = G\, C\,(r - y) + \left(G\, C_{ff} + G_d\right) d$$

$$\Rightarrow\; (1 + G\, C)\, y = G\, C\, r + \left(G_d + G\, C_{ff}\right) d \Rightarrow\; y = \frac{G\, C}{1 + G\, C} r + \frac{G_d + G\, C_{ff}}{1 + G\, C} d$$

$$\Rightarrow T(s) = \frac{G\, C}{1 + G\, C}, \qquad T_d(s) = \frac{G_d + G\, C_{ff}}{1 + G\, C}$$

Alternatively, we can derive T(s) and Td(s) directly using the rule “direct/(1+loop)”.

c) In general, the requirement of perfect control (no offset) at steady-state requires that $T(s = 0) = 1$ and $T_d(s = 0) = 0$. This will be satisfied in our case because C(s) has integral action.

Problem 4 (20%)

The poles are the roots of the denominator polynomials. We get:

$0.04\, s^2 + 0.12\, s + 1$:

$$\Delta = 0.12^2 - 4 \cdot 0.04 = -0.1456 \quad \rightarrow \quad p_{1,2} = \frac{-0.12 \pm i\,\sqrt{0.1456}}{2 \cdot 0.04} = -1.5 \pm 4.77\, i$$

$0.24\, s^2 + \ s + 1$:

$$\Delta = 1^2 - 4 \cdot 0.24 = 0.04 \quad \rightarrow \quad p_{1,2} = \frac{-1 \pm \sqrt{0.04}}{2 \ \cdot \ 0.24} \rightarrow \quad p_1 = -1.667, \qquad p_2 = -2.5$$

$0.6\, s^2 + 1.6\, s + 1$:

$$\Delta = 1.6^2 - 4 \cdot 0.6 = 0.16 \quad \rightarrow \quad p_{1,2} = \frac{-1.6 \pm \sqrt{0.16}}{2 \ \cdot \ 0.6} \rightarrow \quad p_1 = -1, \qquad p_2 = -1.667$$

TF	Poles	Zeros	Steady-state gain	Initial slope	Conclusion
$G_1(s)$	$-1.5 \pm 4.77i$ (oscillates)	-5	1	5	D
$G_2(s)$	$-1.667, -2.5$	-0.625	1	6.667	C
$G_3(s)$	$-1, -1.667$	–	1	0	F
$G_4(s)$	$-1.5 \pm 4.77i$ (oscillates)	8	1.6	-5	A
$G_5(s)$	$-1.667, -2.5$	5	1	-0.833	E
$G_6(s)$	$-1, -1.667$	-2	1.6	1.333	B

Problem 5 (25%)

a) Model equations and assumptions.

(1) is the mass balance for the pipeline section [kg/s]

(2) is the ideal gas equation on mass basis with the temperature T is assumed constant.

(3) and (4) are the assumed valve equations. Note that we have assumed a linear valve characteristic.

Variables:

F_1: inlet flow
F_2: outlet flow
z_1: inlet valve opening
z_2: outlet valve opening
C_1: inlet valve constant
C_2: outlet valve constant
m: mass of gas in the pipeline
p: pressure of gas in the pipeline
p_1: pressure of gas at the inlet
p_2: pressure of gas at the outlet
V: volume of pipeline
T: temperature of the system
R: ideal gas constant
M_w: molar mass of gas

b) At steady state, $F_1 = F_2$, and therefore:

$$C_1 = \frac{F_1}{z_1\sqrt{p_1 - p}} = \frac{1}{0.5 \times \sqrt{2 - 1.88}} = 5.773\ kg/s \cdot bar^{1/2}$$

$$C_2 = \frac{F_2}{z_2\sqrt{p - p_2}} = \frac{1}{0.5 \times \sqrt{1.88 - 1.8}} = 7.071\ kg/s \cdot bar^{1/2}$$

$$k_p = \frac{VM_w}{RT} = \frac{130 \times 18 \times 10^{-3}}{8.31 \times 300}\frac{m^3 \times \frac{kg}{mol}}{\frac{J}{mol\ K}\ K} = 9.386 \times 10^{-4}\ kg/Pa = 93.86\ kg/bar$$

$$m = k_p\, p = 93.86 \times 1.88 = 176.457\ kg$$

Residence time: $t_R = m/F_1 = 176.457\ s$

c) Linearizing the model. First linearize the two static valve equations (3) and (4):

$$y_1 = \Delta F_1 = \left(C_1\sqrt{p_1 - p}\right)\Big|_* \Delta z_1 + \left(-\frac{C_1 z_1}{2\sqrt{p_1 - p}}\right)\Bigg|_* \Delta p = 2\,\Delta z_1 - 4.166\,\Delta p$$

$$\Delta F_2 = \left(C_2\sqrt{p - p_2}\right)\Big|_* \Delta z_2 + \left(\frac{C_1 z_2}{2\sqrt{p - p_2}}\right)\Bigg|_* \Delta p = 2\,\Delta z_2 + 6.250\,\Delta p$$

From (2) the mass balance (1) becomes k_p dp/dt = $F_1 - F_2$ which gives the linearized model for $y_2 = \Delta p$:

$$k_p \frac{d\Delta p}{dt} = \Delta F_1 - \Delta F_2 = 2\,\Delta z_1 - 2\,\Delta z_2 - 10.416\,\Delta p$$

$$\Rightarrow 93.86\frac{d\Delta p}{dt} + 10.416\,\Delta p = 2\,\Delta z_1 - 2\,\Delta z_2$$

$$\Rightarrow 9.011\frac{d\Delta p}{dt} + \Delta p = 0.192\,\Delta z_1 - 0.192\,\Delta z_2$$

Applying the Laplace transform to the last expression gives the transfer function for $y_2 = \Delta p$:

$$\Delta p(s) = \frac{0.1925}{9.011\,s+1}\Delta z_1 - \frac{0.1925}{9.011\,s+1}\Delta z_2$$

The expression for $y_1 = \Delta F_1$ then becomes

$$\Delta F_1 = \left(2 - 4.166 \times \frac{0.1925}{9.011\,s+1}\right)\Delta z_1 - 4.166 \times \left(\frac{-0.1925}{9.011\,s+1}\right)\Delta z_2$$

$$= \left(\frac{2\times(9.011\,s+1) - 4.166\times 0.1925}{9.011\,s+1}\right)\Delta z_1 + \frac{0.800}{9.011\,s+1}\Delta z_2$$

$$= \left(\frac{18.022\,s + 1.200}{9.011\,s+1}\right)\Delta z_1 + \frac{0.8}{9.011\,s+1}\Delta z_2 = 1.2\left(\frac{15.018\,s+1}{9.011\,s+1}\right)\Delta z_1 + \frac{0.8}{9.011\,s+1}\Delta z_2$$

Conclusion

$$\begin{bmatrix}\Delta p\\ \Delta F_1\end{bmatrix} = G(s)\begin{bmatrix}\Delta z_1\\ \Delta z_2\end{bmatrix}, \qquad G(s) = \begin{bmatrix}\dfrac{0.1925}{9.011\,s+1} & \dfrac{-0.1925}{9.011\,s+1}\\ 1.2\left(\dfrac{15.018\,s+1}{9.011\,s+1}\right) & \dfrac{0.8}{9.011\,s+1}\end{bmatrix}$$

Note that the time constant of 9s is much smaller than the residence time of 176s. This is typical for gas systems. Also note that u1=z1 has a direct effect on y2=F1(as expected from physics:; see also element g21 in step response below which has an overshoot because of the zero).

```
s=tf('s')
g11=0.1925/(9*s+1); g12=-g11;
g21=1.2*(15*s+1)/(9*s+1);
g22=0.8/(9*s+1);
G=[g11 g12; g21 g22];
step(G*exp(-10*s)) % To make plot
clearer I put in a delay so that step is
at t=10
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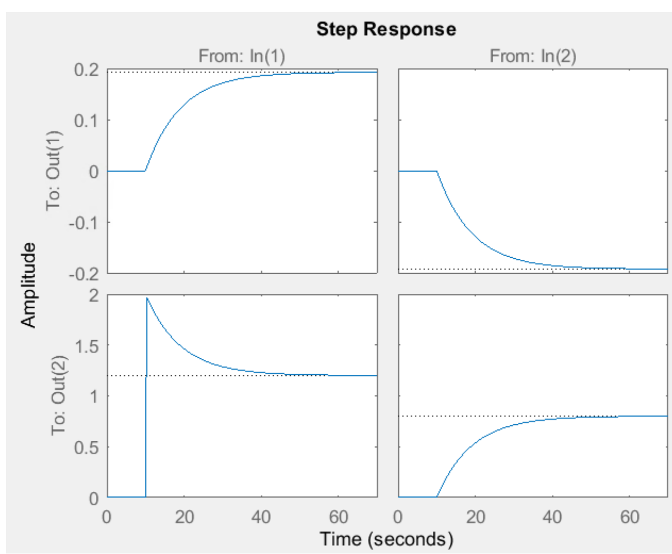
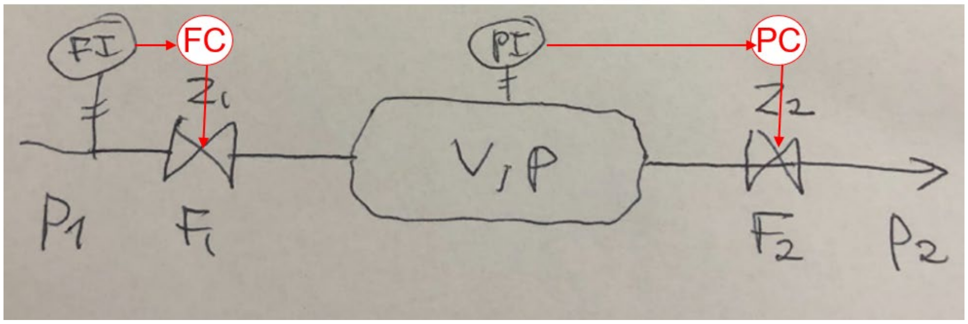


d)

Steady-state gain matrix: $G(0) = \begin{bmatrix} 0.1925 & -0.1925 \\ 1.2 & 0.8 \end{bmatrix} \rightarrow$

steady-state RGA matrix: $\Lambda = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}$ $where\ \lambda = \frac{0.1925\times 0.8}{0.1925\times 0.8+0.1925\times 1.2} = 0.4.$

From the steady-state RGA, the recommended pairing is then the off-diagonal pairing, that is, $F_1 - z_1$ and $p - z_2$. This happens to coincide with the intuitive pairing ("pair-close rule") since z1 has a direct effect on F1. It also agrees with what we get from the RGA if we consider the initial response (high frequency).



However, high steady-state interaction is to be expected, since Λ is far from the ideal case (identity matrix). Possible solutions are the implementation of a decoupler (probably steady-state decoupler is OK), or separating the timescales of the two loops.

Since the flow control has a direct effect from z1 to F1, this should probably be the fast loop, and then the pressure loop can be about 5 times slower. But if both loops should be equally fast, a decoupler is preferred.

What about the tuning of the flow loop? What model should we use? We have that

$$G_0(s) = 1.2\left(\frac{15s+1}{9\,s+1}\right)$$

Note that T0=15 > tau0=9. How should we approximate this as a first-order with delay model? It will depend on the value for tauc. If we apply the LHP-zero approximation rules then we get.

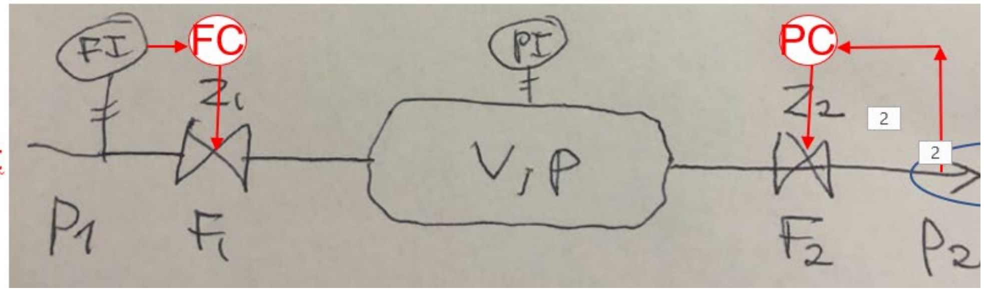
Small tauc (tauc<9): (15s+1)/(9s+1) ≈ 15/9 (Rule T1) ⇒ G(s)=1.2*15/9 = 2
Intermediate tauc (9<taux<15): (15s+1)/(9s+1) ≈ 15/tauc (Rule T1a) ⇒ G(s)=18/tauc
Large tauc (tauc>15) . (15s+1)/(9s+1) ≈ 1 (Rule T1b) ⇒ G(s)=1.2

In all these three case the SIMC PI-controller becomes a pure I-controller C(s)=KI/s with KI = 1/(k*tauc). Note that for the intermediate tauc we get KI=1/18 (independent of Kc).

e) This is a trick question, because it will not work. This control strategy would not be consistent, as we can see that that is does not follow the radiation rule. In general, the control of pressures that are external to the process is equivalent to a flow specification (TPM), which in this case would conflict with the specification of F_1.

NO!

Not consistent



Comment: One may think that it does not work since p is left uncontrolled, which may lead to blow-up of the pipeline. However, the fact that p is uncontrolled does not by itself mean that we have a problem. In most cases, pressures are left uncontrolled as we rely on self-regulation, for example, valve 2 may be kept fully open to minimize pressure drop. Thus, the problem is not that p is left uncontrolled, but rather that p2 is controlled.

If we try implementing the proposed unworkable control structure, then we will see that one of the valves, z1 or z2 goes fully open and we loose control of either flow or pressure. We will loose control of the variable (CV) for which the setpoint is asking for the largest flow.