**Problem 1**

a) Balances: Energy balances over (1) heat exchanger (the 'box') and (2) the mixing point. For the heat exchanger we have assumed perfect mixing (simplified) and for the mixing point no mass (reasonable). Also assume no heat loss. \( m = \) mass inside heat exchanger [kg], which is assumed constant. \( c_p = \) heat capacity [J/kg/K], assumed constant (independent of \( T \)) and then we may also set \( T_{ref} = 0 \). Have also assumed \( c_V = c_P \) (which is Ok for liquid).

b) Assuming constant mass \( m \) and constant heat capacity \( c_p \) we can write the model as

\[
\frac{dT_1}{dt} = \frac{w_1}{m} (T_{in} - T_1) + \frac{Q}{m c_p}
\]

(1)

\[
0 = w_1 T_1 + w_2 T_{in} - (w_1 + w_2) T
\]

(2)

Introducing deviation variables and linearizing the model gives

\[
\frac{d\Delta T_1}{dt} = - \left( \frac{w_1}{m} \right)^* \Delta T_1 + \left( \frac{w_1}{m} \right)^* \Delta T_{in} + \left( \frac{T_{in} - T_1}{m} \right)^* \Delta w_1 + \frac{1}{m c_p} \Delta Q
\]

(3)

\[
0 = w_1^* \Delta T_1 + w_2^* \Delta T_{in} - (w_1 + w_2)^* \Delta T - (T_1 - T)^* \Delta w_1 + (T_{in} - T)^* \Delta w_2
\]

(4)

Setting \( dT_1/dt = 0 \) and solving for the steady state values we get

\[
w_1^* = 0.9 \quad [kg/s]
\]

\[
w_2^* = 0.1 \quad [kg/s]
\]

\[
Q^* = 36 \quad [kW]
\]

Taking Laplace transform and rearranging we have

\[
T_1(s) = \frac{1}{(\tau s + 1)} [k_1 w_1(s) + k_2 T_{in}(s) + k_3 Q(s)]
\]

(5)

\[
T(s) = k_4 w_1(s) + k_5 w_2(s) + k_6 T_{in}(s) + k_7 T_1(s)
\]

(6)
where

\[
\begin{align*}
\tau &= m/w_1 = 22.22 \\
k_1 &= (T_{in} - T_1)/w_1 = -11.11 \\
k_2 &= 1 \\
k_3 &= 1/(c_p w_1) = 0.28 \\
k_4 &= \left(\frac{T_1 - T}{w_1 + w_2}\right)^* = 1 \\
k_5 &= \left(\frac{T_{in} - T}{w_1 + w_2}\right)^* = -9 \\
k_6 &= w_2/(w_1 + w_2) = 0.1 \\
k_7 &= w_1/(w_1 + w_2) = 0.9
\end{align*}
\]

Thus, we have

\[
\begin{align*}
T_1(s) &= \frac{1}{(22.22s + 1)}[-11.11w_1(s) + 1 \cdot T_{in}(s) + 0.28Q(s)] \\
T(s) &= 1 \cdot w_1(s) - 9w_2(s) + 0.1T_{in}(s) + 0.9T_1(s)
\end{align*}
\]

(c) Insert \( w_1 = w - w_2 \). After some algebra we get

\[
\begin{align*}
g_1(s) &= k_4 + \frac{k_1 k_7}{\tau s + 1} = -0.405 \frac{-2.47s + 1}{22.22s + 1} \\
g_2(s) &= k_5 - k_4 - \frac{k_1 k_7}{\tau s + 1} = -222.22 \frac{s}{22.22s + 1} \\
g_3(s) &= k_6 + \frac{k_2 k_7}{\tau s + 1} = \frac{2.22s + 1}{22.22s + 1} \\
g_4(s) &= \frac{k_3 k_7}{\tau s + 1} = \frac{0.252}{22.22s + 1}
\end{align*}
\]

It is trivial to see that the steady state gain for \( g_2(s) \) is \( g_2(s = 0) = 0 \). This makes sense because the added heat \( Q \) and the total flow through the system \( w \) remain constant when changing \( w_2 \). To satisfy the mass and energy balances \( T \) must return to its original value at steady state.

Note also that there is an inverse response (RHP-zero in transfer function) from \( w_1 \) to \( T \) which limits the controllability.

The steady state gains are \( g_1(0) = -0.405 \) and \( g_3(0) = 1 \).
Problem 2

a) Using SIMC tuning rules we get

\[ K_c = \frac{1}{10 + 10} = 0.005 \]  \hspace{1cm} (13)

\[ \tau_I = \min(0.1, 4(10 + 10)) = 0.1 \]  \hspace{1cm} (14)

This is actually almost an integrating controller with \( K_I = K_c/\tau_I = 0.05 \).

• The closed-loop setpoint response is approximately first order with a delay of \( \theta = 10 \), \( y/y_s \approx e^{-10s}/(10s+1) \); see solid line (this is what we are aiming for when we derived the SIMC controller, but it actually requires a Smith Predictor controller; the actual response with PI-control is shown with the dashed line). There is no steady state offset because of the integral action in the controller.

• The gain margin can be computed as follows. Consider the frequency response of \( L(s) = gc = K_c e^{-\theta s}/\tau_I s = 0.05 e^{-10s}/s \).

\[ |L(j\omega)| = \frac{0.05}{\omega}; \quad \angle L(j\omega) = \frac{-\pi}{2} - 10\omega [\text{rad}] \]

To compute the gain margin we first need to find the frequency \( \omega_{180} \) for which the phase shift of \( L \) is \( -\pi/2 - \omega 10 = -\pi \). This gives \( \omega_{180} = 0.1571 \) rad/s. The loop gain at this frequency is \( |L(j\omega_{180})| = 0.3183 \). Therefore, the GM = \( 1/|L(j\omega_{180})| = 3.14 \).

To get the delay margin we first compute the phase margin. \( L \) crosses 1 at \( \omega_c = 0.05 \) rad/s. The phase margin is \( \text{PM} = \angle L(j\omega_c) + \pi = -10*0.05 + \pi/2 = 1.07 \) rad. The allowed time delay error = \( \text{PM}/\omega_c = 1.07 \text{ rad}/ 0.05 \text{ rad/s} = 21.4 \text{ s} \).

• Yes, this controller has good performance and is robust.

b) The loop transfer function is \( L = K_c e^{-10s}/(0.1s+1) \). The time constant of 0.1 can be neglected compared to the delay of 10, so the phase is approximately \(-10\omega \) [rad], and the phase crosses \(-\pi\) for \( \omega_{180} = -\pi/10 = 0.314 \). The loop gain at this frequency is approximately \( |L| = K_c \). In order to have GM = 3 we must choose \( K_c = 1/3 \).

• Computing the closed-loop transfer function from setpoint to output and making \( s = 0 \) we get

\[ K = \frac{K_c}{1 + K_c} = 0.25. \]  \hspace{1cm} (15)

Ideally, this gain should be 1. The steady state offset is therefore \( 1 - K = 0.75 \) (75%) which is very poor.

• Since \( |L| \) is always less than 1, we are stable with any delay (so we can tolerate infinite time delay error).

• This controller is not recommended because of the large offset.
Figure 1: Closed-loop setpoint response for the PI control
Figure 2: Closed-loop setpoint response for the P control
Problem 3

a) The four transfer functions are

\[
\begin{align*}
    h_1(s) &= \frac{c_2 g_2}{1 + c_2 g_2} \\
    h_2(s) &= \frac{g_1}{1 + c_2 g_2} \\
    h_3(s) &= \frac{c_2}{1 + c_2 g_2} \\
    h_4(s) &= \frac{-c_2 g_1}{1 + c_2 g_2}
\end{align*}
\]

(16) \quad (17) \quad (18) \quad (19)

b) \quad \frac{1}{1 + c_2 g_2} = \frac{5s+1}{6.5s+1}

\[
\begin{align*}
    h_1(s) &= \frac{1.5s}{6.5s + 1} \\
    h_2(s) &= \frac{2}{6.5s + 1} \\
    h_3(s) &= \frac{0.1 \cdot 5s + 1}{6.5s + 1} \\
    h_4(s) &= \frac{-0.2}{6.5s + 1}
\end{align*}
\]

(20) \quad (21) \quad (22) \quad (23)

c) (Comment: Almost no students were able to get this one right.) With a (small) delay of \( \theta \) added, the loop transfer function is \( L(s) = g_2 c_2 = \frac{1.5s}{5s+1} e^{-\theta s} \). At low frequency \((s \to 0)\), this transfer function is approximately \( L(s) \approx 1.5s \) (the gain increases with frequency and the phase is +90 degrees), and at high frequencies \((\omega > 1/5)\) we have \( L(s) \approx \frac{1.5}{5} e^{-\theta s} \). The phase of \( L \) at high frequency is \( \angle L = -\omega \theta \) [rad] (from the delay), and the gain is constant at 1.5/3. The phase crossover (where \( \angle L = -\pi \text{[rad]} = -180 \) degrees) will happen at a high frequency for a small delay and, as mentioned, at high frequency the gain \(|L|\) is 1.5/5 = 1/3.33 so the gain margin is GM=3.33, which is reasonable gain margin for a robust design. With GM=3.33 we cannot increase the controller gain by more than a factor of 3 before we get instability.

d) We have that \( c_2 = 2 \frac{5s+1}{5s} \), so \( c_2 g_2 = 6 \) and \( \frac{1}{1 + c_2 g_2} = \frac{1}{7} \).

The closed-loop transfer functions with PI control are

\[
\begin{align*}
    h_1(s) &= \frac{6}{\frac{7}{5}} = 0.8571 \\
    h_2(s) &= \frac{2/7}{\frac{5}{5s + 1}} = \frac{0.2857}{5s + 1} \\
    h_3(s) &= \frac{2 \cdot 5s + 1}{35 s} = \frac{0.0575}{s}
\end{align*}
\]

(24) \quad (25) \quad (26)
\[ h_4(s) = \frac{-4/35}{s} = \frac{-0.1143}{s} \]  \hspace{1cm} (27)

\( h_3 \) and \( h_4 \) are unstable because of the integrator. This instability is due to the integrator in the controller which cancels the zero in \( g_2 \) so that \( c_2 g_2 \) contains no integrator (this is sometimes called internal instability). \textbf{Comment: To stabilize the system, we may control \( u_2 \) using \( u_1 \) in an “outer” loop, which is what is done with “input resetting”.

Which controller is recommended, P or PI? The gain for a P-controller needs to be very low, so this controller is not very effective. The PI-controller gives much better closed-loop control of \( y \) (the setpoint transfer function is \( h_1 = 0.8571 \), which is not quite the ideal value of 1, but it means that most of the control is done almost immediately with the use of \( u_2 \)). However, for stability it requires that we control \( u_2 \) in an outer loop as just mentioned (and with \( u_2 \) controlled, we will also find that we have no steady-state offset in \( y \).

\textbf{Comment 1:} Actually, the transfer functions given in this problem are similar to the heat exchanger in Problem 1, if we set \( y = T, u_2 = w_2, u_1 = w_1 \). To get tight control of \( y \) we use \( u_2 \) (the bypass), and then we use \( u_1 \) (the total flow) in a slower outer loop to reset \( u_1 \) to a desired set value (e.g., 10\% of the total flow). As we just noted, this outer loop is also required to stabilize the system if the controller \( c_2 \) has integral action.

\textbf{Comment 2:} It seems that from a control point of view, we should move the control valve on the main flow from \( w \) to \( w_1 \). Then we no longer have zero steady-state gain in the transfer function \( g_2 \) from \( u_2 = w_2 \) to \( y = T \), so we avoid the internal stability problem (and we get \( h_1(0) = 1 \) with PI-control, but this is not so critical because the outer loop is needed in any case because \( u_2 \) is too small to handle large disturbances).

\textbf{Problem 4}

\textbf{a)} With a P controller, \( c = K_c \), the system will always oscillate. The closed loop poles are given by \( 1 + gc = 1 + \frac{K_c}{s^2} = 0 \Rightarrow s^2 = -K_c \Rightarrow s = \pm j\sqrt{K_c} \), that is, purely complex poles which correspond to oscillations that never die out. Conclusion: No, we cannot stabilize the system with a P-controller.

\textbf{b)} The phase from the process itself is -180 degrees at all frequencies. We need to add D-action to add positive phase and stabilize the process. Conclusion: we can use PD-controller or PID-controller.

\textbf{Problem 5}

\textbf{a)} Without cascade:

With cascade:
Figure 3: Process flow diagram without cascade control

Figure 4: Block diagram without cascade control

Figure 5: Process flow diagram with cascade control
Figure 6: Block diagram with cascade control
b) Without cascade:

\[ h = g_v g_p \]  

(28)

with cascade:

\[ h = \frac{c_2 g_v}{1 + c_2 g_v} g_p \]  

(29)

Assuming that \( c_2 \) is fast, \( \frac{c_2 g_v}{1 + c_2 g_v} \approx 1 \) and \( h = g_p \). The cascade controller removes the effect of \( g_v \) on \( h \) as long as the inner loop is sufficiently fast.

c) • The exponential valve is recommended because the gain \( k \) varies the least with this valve.

• With the linear valve, the gain increase from high load to low load is a factor \( 3.5/0.2 = 17.5 \), so we would need a gain margin at high load of more than 17.5 to have stability at low load. This means that a controller tuned for high loads with a gain margin of 3 will be **unstable** at low load.

Comment: A large gain margin at high load (about 35 to have GM=2 at low load), will give a small controller gain and result in poor control performance at high load (with a large closed-loop time constant). This clearly shows that we should either (1) use the exponential valve which has only a small gain variation or (2) use the linear valve with cascade control using a flow controller as explained in part b.